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ON THE CAYLEY GRAPHS OF SYMMETRIC GROUP S_4

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Abstract. Let S_n be the symmetric group of degree n. In this paper, we classify non-isomorphic Cayley graphs of S_4 of valency 3. Moreover, we verify that there are exactly 10 non-isomorphic Cayley graphs of S_4 with valency 3. Also, we classify the valency 3 CI-graphs of Cayley graphs of S_4 and we prove that S_4 is not a CI-group and does not possess the 3-CI-property. We show that there are at least 10 non-isomorphic Cayley graphs of the symmetric group S_n with valency 3.

Keywords: Cayley graph, Symmetric group, Isomorphism.
2020 MSC: 05C50, 05A05.

1. Introduction

Arthur Cayley introduced Cayley graphs in 1878. Cayley graphs serve as a bridge between group theory in abstract algebra and graph theory in combinatorics. These graphs elegantly encode group elements and their interactions [7, 12]. We focus on symmetric groups, denoted by S_n, for a natural number n. The symmetric groups are key objects in group theory and are fundamental in various areas of mathematics, such as combinatorics and cryptography. The Cayley graphs of the symmetric group S_n and the alternating group A_n have been studied in several papers.

In [11] Gu and Li showed that there are exactly 22 non-isomorphic Cayley graphs of A_4. The number of undirected Cayley graphs of S_n and A_n has been determined by Adiga and Ariamanesh in [1]. They also showed that there are only 8 Cayley graphs of S_3 and 4 Cayley graphs of S_4 of valency 2, up to isomorphism. Recently, Fadzil, Sarmin and Erfanian [5] studied the eigenvalues and the energy of the Cayley graphs of S_4 with valency up to 2. Moreover, in [6] they examined the Cayley graphs of alternating groups with valency up to 2, computing their eigenvalues and energy. The classification of finite CI-groups is an interesting problem related to CI-graphs. In [11] it has been shown that the group A_4 is a CI-graph, and all disconnected Cayley graphs of A_5 are also CI-graphs.

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In this paper, we investigate the Cayley graphs of symmetric groups, focusing particularly on the valency 3 Cayley graphs of S_4. We also study the CI-graphs of the Cayley graphs of S_4 with valency 3. Additionally, we establish a relationship between the Cayley graph of S_{n+1} and the Cayley graph of S_n for a fixed subset S.

2. Preliminaries

Here we introduce some definitions and preliminary results from graph theory and group theory. Readers can refer to the book [9], for terminology and notations which have not been defined in this paper.

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E. Suppose that $x, y \in V$. The vertices x and y are adjacent if $xy \in E$. For a vertex $x \in V$ the set of all neighbors of x is defined as $N_\Gamma(x) = N(x) = \{y \in V : xy \in E\}$. The degree of x in the graph Γ is defined as $d(x) = |N_\Gamma(x)|$. The graph Γ is an m-regular graph if, for each $x \in V$, $d(x) = m$. A graph Γ is said to be connected if, for each $x, y \in V$, there is a walk between x and y in Γ. Let K_n be a complete graph on n vertices and C_n be an undirected cycle on n vertices.

The disjoint union of the graphs $\Gamma = (V, E)$ and $\Lambda = (U, F)$, for $V \cap U = \varnothing$, is a graph with the vertex set $V \cup U$ and edge set $E \cup F$; we denote it by $\Gamma \cup \Lambda$.

The Cartesian product of the graphs $\Gamma = (V, E)$ and $\Lambda = (U, F)$ is a graph with vertex set $V \times U$ where two vertices (v_1, u_1) and (v_2, u_2) are adjacent if and only if $[v_1 = v_2$ and $u_1u_2 \in F]$ or $[u_1 = u_2$ and $v_1v_2 \in E]$; we denote it by $\Gamma \square \Lambda$.

Moreover, if there exists a bijection $g : V \to U$ such that $xy \in E$ if and only if $x^g y^g \in F$, where $x^g = g(x)$, then the graphs Γ and Λ are called *isomorphic*. Such a bijection between Γ and Λ is referred as an *isomorphism*.

Suppose that G is a finite group with the identity 1_G. Let S be a subset of the group G such that $1_G \notin S$ and S is an inverse-closed subset, meaning $S = S^{-1}$, where $S^{-1} := \{s^{-1} | s \in S\}$. The associated *Cayley graph*, denoted by $\mathrm{Cay}(G, S)$, is a graph where the vertices are the elements of G, and two distinct vertices $g, h \in G$ are adjacent if and only if $gh^{-1} \in S$. The number $|S|$ is called the *valency* of the Cayley graph. Since S is an inverse-closed subset of G, $\mathrm{Cay}(G, S)$ is an undirected graph.

Proposition 2.1. *[2] Suppose that S is a subset of the group G such that $1_G \notin S$, $S = S^{-1}$, and $|S| = m$. Then* $\mathrm{Cay}(G, S)$ *is an m-regular graph.*

Thus, any Cayley graph of valency m is an m-regular graph. It is easy to see that a Cayley graph $\mathrm{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$, meaning S is a generating set of the group G. Therefore, if $G \neq \langle S \rangle$, then $\mathrm{Cay}(G, S)$ is disconnected. In this case, studying its connected components becomes important.

Proposition 2.2. *[2] Suppose that S is a subset of the group G such that $1_G \notin S$ and $S = S^{-1}$. Then all components of* $\mathrm{Cay}(G, S)$ *are isomorphic to* $\mathrm{Cay}(\langle S\rangle, S)$.

Thus, for a disconnected Cayley graph, all of its connected components are isomorphic to each other.

Here, we explain the definition of CI-graphs [11]. Let S and S' be two subsets of the group G such that $1_G \notin S$, $1_G \notin S'$, $S = S^{-1}$, and $S' = S'^{-1}$. Suppose that $\sigma \in \mathrm{Aut}(G)$; then σ acts naturally on G. If $S' = S^\sigma$, then σ induces an isomorphism from $\mathrm{Cay}(G; S)$ to $\mathrm{Cay}(G; S')$, which is called a *Cayley isomorphism*. The graph $\mathrm{Cay}(G, S)$ is called a CI-graph (where CI represents Cayley isomorphic) of the group G if $\mathrm{Cay}(G, S)$ is isomorphic to $\mathrm{Cay}(G, S')$ implies that there exists a permutation $\sigma \in \mathrm{Aut}(G)$ such that $S' = S^\sigma$. Moreover, for a natural number k, the group G has the k-CI-property, if all of the Cayley graphs of G of valency k are CI-graphs. A group G is called a CI-group if every Cayley graph of G is a CI-graph.

Let n be a natural number. Consider the group of all permutations on the set $\{1, ..., n\}$ with function composition as its group operation. This group is called the symmetric group, denoted S_n. The group S_n has $n!$ elements and it is not abelian for $n \geq 3$. The subgroup of all even permutations in S_n is denoted by A_n and it is called the alternating group. In this paper, we focus on the symmetric group S_4. The elements of S_4 with their cycle types are:

{(1), (12), (13), (14), (23), (24), (34), (123), (132), (124), (142), (134),(143), (234), (243), (1234), (1432), (1243), (1342), (1324), (1423), (12)(34), (13)(24), (14)(23)}.

3. Main Results

Our purpose here is to classify Cayley graphs of the group S_4 of valency 3. For different subsets S of the group S_4 of valency 3 such that $(1) \notin S$ and $S = S^{-1}$, we consider the graph $\mathrm{Cay}(S_4, S)$. Thus, we first determine the different classes of the subset S that satisfy these necessary conditions.

Lemma 3.1. *Let S be a subset of S_4 such that $|S| = 3$, $(1) \notin S$ and $S = S^{-1}$. Then, all possible subsets S up to isomorphism are in Table 1, where i, j, l and k are distinct elements in $\{1, 2, 3, 4\}$.*

Proof. Since for each distinct element i, j, l, and k in $\{1, 2, 3, 4\}$, we have $(ij)^{-1} = (ij)$, $(ij)(lk)^{-1} = (ij)(lk)$, $(ijl)^{-1} = (ilj)$ and $(ijlk)^{-1} = (iklj)$. Thus, it is clear that each subset S in Table 1 has the required properties. Moreover, it is straightforward to check that every subset of S_n with the desired conditions is isomorphic to one of the cases mentioned in Table 1.

□

TABLE 1. Possible subsets S for S_4.

No.	*The subset S up to isomorphism*
1	$\{(ij),(il),(jl)\}$
2	$\{(12)(34),(14)(23),(13)(24)\}$
3	$\{(ij),(lk),(ij)(lk)\}$
4	$\{(ij)(lk),(iljk),(ikjl)\}$
5	$\{(ij),(il)(kj),(ik)(lj)\}$
6	$\{(ij),(iljk),(ikjl)\}$
7	$\{(ij)(lk),(ijlk),(iklj)\}$
8	$\{(ij),(ijl),(ilj)\}$
9	$\{(ij),(ij)(lk),(il)(jk)\}$
10	$\{(ij)(lk),(ijl),(ilj)\}$
11	$\{(ij),(il),(jk)\}$
12	$\{(ij),(ijlk),(iklj)\}$
13	$\{(ij),(ilk),(ikl)\}$
14	$\{(ij),(il),(il)(jk)\}$
15	$\{(ij),(jl),(ij)(lk)\}$
16	$\{(ij),(il),(ik)\}$

In the next part of this section, we will determine the graph $\mathrm{Cay}(S_4, S)$ up to isomorphism for each type of the set S listed in Table 1.

Proposition 3.2. *Let S be a subset of S_4 such that $S = \{(ij),(il),(jl)\}$, where i, j, and l are distinct elements in $\{1,2,3,4\}$. Then, $\mathrm{Cay}(S_4, S)$ is isomorphic to $\cup_{t=1}^{4} K_{3,3}$.*

Proof. Since $S = \{(ij),(il),(jl)\}$, we have:

$$\langle S\rangle = \{(1),(ilj),(ijl),(jl),(il),(ij)\}.$$

The graph $\mathrm{Cay}(\langle S\rangle, S)$ is shown in Figure 1, which is isomorphic to the graph $K_{3,3}$.

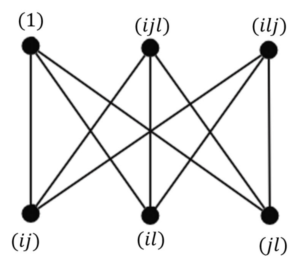

FIGURE 1. The Graph $\mathrm{Cay}(\langle S\rangle, S) \simeq K_{3,3}$

By Proposition 2.2, all components of the graph $\mathrm{Cay}(S_4, S)$ are isomorphic to $\mathrm{Cay}(\langle S\rangle, S)$. Since $\langle S\rangle$ has 4 cosets in S_4, thus $\mathrm{Cay}(S_4, S)$ is isomorphic to the disjoint union of 4 copies of $\mathrm{Cay}(\langle S\rangle, S)$. So,

$$\mathrm{Cay}(S_4, S) \simeq \cup_{t=1}^{4} K_{3,3}.$$

□

Proposition 3.3. *Let S, S' and S'' be subsets of S_4, such that:*

$$S = \{(12)(34), (14)(23), (13)(24)\},$$

$$S' = \{(ij), (lk), (ij)(lk)\},$$

and

$$S'' = \{(ij)(lk), (iljk), (ikjl)\},$$

where i, j, l, and k are distinct elements in $\{1, 2, 3, 4\}$. Then, $\mathrm{Cay}(S_4, S)$, $\mathrm{Cay}(S_4, S')$ *and* $\mathrm{Cay}(S_4, S'')$ *are isomorphic to the graph $\cup_{t=1}^{6} K_4$.*

Proof. Since $S = \{(12)(34), (14)(23), (13)(24)\}$, we have:

$$\langle S\rangle = \{(1), (12)(34), (14)(23), (13)(24)\}.$$

The graph $\mathrm{Cay}(\langle S\rangle, S)$ is shown in Figure 2A, which is isomorphic to the graph K_4.

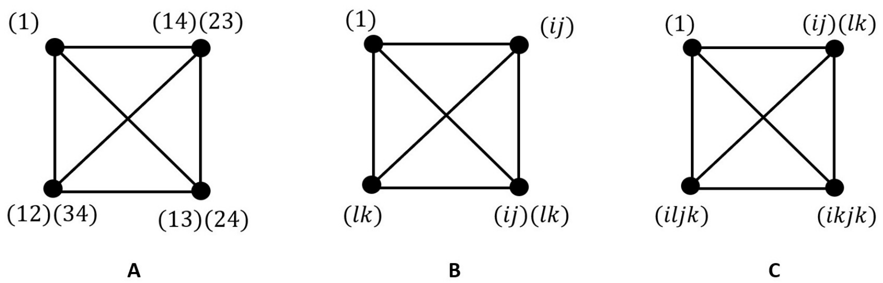

FIGURE 2. The Graphs $\mathrm{Cay}(\langle S\rangle, S), \mathrm{Cay}(\langle S'\rangle, S'), \mathrm{Cay}(\langle S''\rangle, S'')$

By Proposition 2.2, all components of the graph $\mathrm{Cay}(S_4, S)$ are isomorphic to $\mathrm{Cay}(\langle S\rangle, S)$. Thus, $\mathrm{Cay}(S_4, S)$ is isomorphic to the disjoint union of 6 copies of $\mathrm{Cay}(\langle S\rangle, S)$,

$$\mathrm{Cay}(S_4, S) \simeq \cup_{t=1}^{6} K_4.$$

Now consider the subset S'. Since $S' = \{(ij), (lk), (ij)(lk)\}$, we have:

$$\langle S'\rangle = \{(1), (ij), (lk), (ij)(lk)\}.$$

The graph $\mathrm{Cay}(\langle S'\rangle, S')$ is shown in Figure 2B, which is also isomorphic to the graph K_4.

Again, by Proposition 2.2, all components of the graph $\mathrm{Cay}(S_4, S')$ is isomorphic to $\mathrm{Cay}(\langle S'\rangle, S')$. Thus,

$$\mathrm{Cay}(S_4, S') \simeq \cup_{t=1}^{6} K_4.$$

For the third case, we have $S'' = \{(ij)(lk), (iljk), (ikjl)\}$, so

$$\langle S''\rangle = \{(1), (ij)(lk), (iljk), (ikjl)\}.$$

The graph $\mathrm{Cay}(\langle S''\rangle, S'')$ is shown in Figure 2C, and it is isomorphic to the graph K_4.

Similar to the previous case, we have

$$\mathrm{Cay}(S_4, S'') \simeq \cup_{t=1}^{6} K_4.$$

□

Proposition 3.4. *Let S, S' and S'' be subsets of S_4 such that:*

$$S = \{(ij), (il)(kj), (ik)(lj)\},$$
$$S' = \{(ij), (iljk), (ikjl)\},$$

and

$$S'' = \{(ij)(lk), (ijlk), (iklj)\},$$

where i, j, l and k are distinct elements in $\{1, 2, 3, 4\}$. Then, $\mathrm{Cay}(S_4, S)$, $\mathrm{Cay}(S_4, S')$ *and* $\mathrm{Cay}(S_4, S'')$ *are isomorphic to the graph $\cup_{t=1}^{3} C_4 \Box K_2$.*

Proof. Since $S = \{(ij), (il)(kj), (ik)(lj)\}$, we have

$$\langle S\rangle = \{(1), (ij), (il)(kj), (ik)(lj), (ij)(lk), (lk), (iljk), (ikjl)\}.$$

The graph $\mathrm{Cay}(\langle S\rangle, S)$ is shown in Figure 3A, which is isomorphic to the graph $C_4 \Box K_2$.

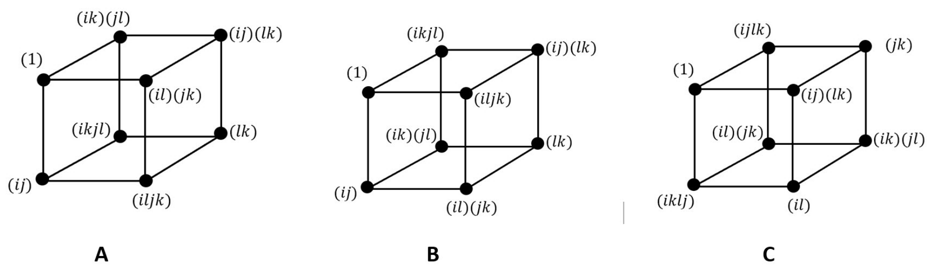

FIGURE 3. The Graphs $\mathrm{Cay}(\langle S\rangle, S)$, $\mathrm{Cay}(\langle S'\rangle, S')$, $\mathrm{Cay}(\langle S''\rangle, S'')$

Now, by Proposition 2.2, all components of the graph $\mathrm{Cay}(S_4, S)$ are isomorphic to $\mathrm{Cay}(\langle S\rangle, S)$. Thus, $\mathrm{Cay}(S_4, S)$ is isomorphic to the disjoint union of 3 copies of $\mathrm{Cay}(\langle S\rangle, S)$:

$$\mathrm{Cay}(S_4, S) \simeq \cup_{t=1}^{3} C_4 \Box K_2.$$

For the subset S', since $S' = \{(ij), (iljk), (ikjl)\}$, we have

$$\langle S' \rangle = \{(1), (ij), (lk), (il)(kj), (ik)(lj), (ij)(lk), (iljk), (ikjl)\}.$$

The graph $\mathrm{Cay}(\langle S' \rangle, S')$ is shown in Figure 3B, and it is isomorphic to the graph $C_4 \square K_2$.

Again, by Proposition 2.2, we have

$$\mathrm{Cay}(S_4, S') \simeq \cup_{t=1}^{3} C_4 \square K_2.$$

For the third case, we have $S'' = \{(ij)(lk), (ijlk), (iklj)\}$, so

$$\langle S'' \rangle = \{(1), (ij), (lk), (il)(kj), (ik)(lj), (ij)(lk), (iljk), (ikjl)\}.$$

The graph $\mathrm{Cay}(\langle S'' \rangle, S'')$ is shown in Figure 3C, which is isomorphic to the graph K_4. Similar to the previous case, we have

$$\mathrm{Cay}(S_4, S'') \simeq \cup_{t=1}^{3} C_4 \square K_2.$$

□

Proposition 3.5. *Let S be a subset of S_4, such that*

$$S = \{(ij), (ijl), (ilj)\},$$

where i, j, and l are distinct elements in $\{1, 2, 3, 4\}$. Then, $\mathrm{Cay}(S_4, S)$ *is isomorphic to the graph $\cup_{t=1}^{4} C_3 \square K_2$.*

Proof. Since $S = \{(ij), (ijl), (ilj)\}$, we have

$$\langle S \rangle = \{(1), (ij), (il), (jl), (ijl), (ilj)\}.$$

The graph $\mathrm{Cay}(\langle S \rangle, S)$ is shown in Figure 4, which is isomorphic to the graph $C_3 \square K_2$.

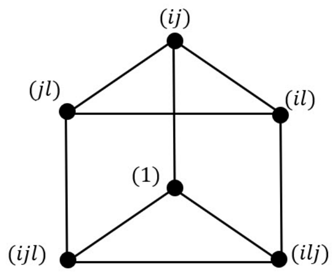

FIGURE 4. The Graph $\mathrm{Cay}(\langle S \rangle, S) \simeq C_3 \square K_2$

Now, by Proposition 2.2, we have 4 components of the graph $\mathrm{Cay}(S_4, S)$ which are all isomorphic to $C_3 \square K_2$. Thus,

$$\mathrm{Cay}(S_4, S) \simeq \cup_{t=1}^{4} C_3 \square K_2.$$

□

Proposition 3.6. *Let S be a subset of S_4, such that*

$$S = \{(ij), (ij)(lk), (il)(jk)\},$$

where $i, j, l,$ and k are distinct elements in $\{1, 2, 3, 4\}$. Then, $\mathrm{Cay}(S_4, S)$ *is isomorphic to $\cup_{t=1}^{3}\Gamma_1$, where Γ_1 is isomorphic to the graph in Figure 5A.*

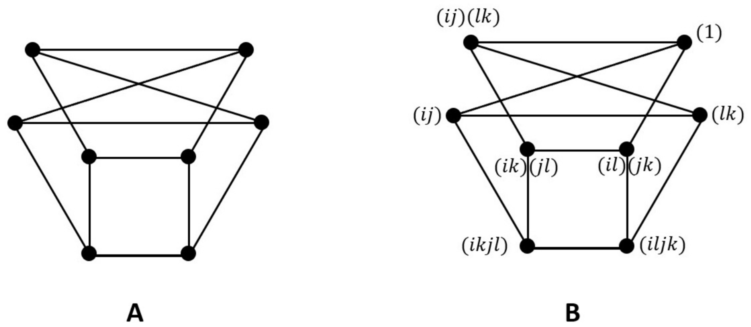

FIGURE 5. A:The Graph Γ_1 B:The Graph $\mathrm{Cay}(\langle S\rangle, S)$

Proof. Since $S = \{(ij), (ij)(lk), (il)(jk)\}$, we have

$$\langle S\rangle = \{(1), (ij)(lk), (il)(jk), (ij), (lk), (ik)(jl), (iljk), (ikjl)\}.$$

The graph $\mathrm{Cay}(\langle S\rangle, S)$ is shown in Figure 5B.

Now, by Proposition 2.2, we have 3 components of the graph $\mathrm{Cay}(S_4, S)$ which are all isomorphic to Γ_1. Thus,

$$\mathrm{Cay}(S_4, S) \simeq \cup_{t=1}^{3}\Gamma_1.$$

□

Proposition 3.7. *Let S be a subset of S_4, such that*

$$S = \{(ij)(lk), (ijl), (ilj)\},$$

where $i, j, l,$ and k are distinct elements in $\{1, 2, 3, 4\}$. Then, $\mathrm{Cay}(S_4, S)$ *is isomorphic to $\cup_{t=1}^{2}\Gamma_2$, where Γ_2 is isomorphic to the graph in Figure 6A.*

Proof. Since $S = \{(ij)(lk), (ijl), (ilj)\}$, we have

$$\langle S\rangle = \{(1), (ij)(lk), (ijl), (ilj), (ikl), (ikj), (ik)(jl), (il)(jk), (ilk), (jlk), (ijk), (jkl)\}.$$

The graph $\mathrm{Cay}(\langle S\rangle, S)$ is shown in Figure 6B. Now, by Proposition 2.2, we have 2 components of the graph $\mathrm{Cay}(S_4, S)$ which are all isomorphic to Γ_2. Thus,

$$\mathrm{Cay}(S_4, S) \simeq \cup_{t=1}^{2}\Gamma_2.$$

□

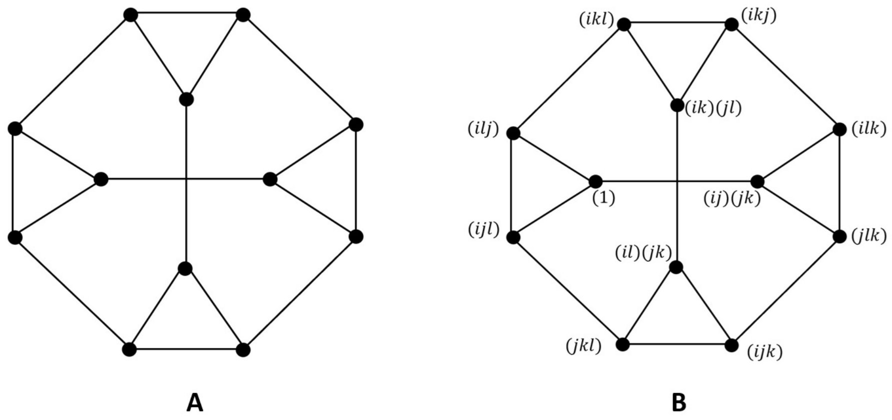

FIGURE 6. A:The Graph Γ_2 B:The Graph $\mathrm{Cay}(\langle S\rangle, S)$

Proposition 3.8. *Let S and S' be subsets of S_4, such that*

$$S = \{(ij), (il), (jk)\}$$

and

$$S' = \{(ij), (ijlk), (iklj)\},$$

where i, j, l, and k are distinct elements in $\{1, 2, 3, 4\}$. Then, $\mathrm{Cay}(S_4, S)$ *and* $\mathrm{Cay}(S_4, S')$ *are isomorphic to Γ_3, where Γ_3 is the graph in Figure 7A.*

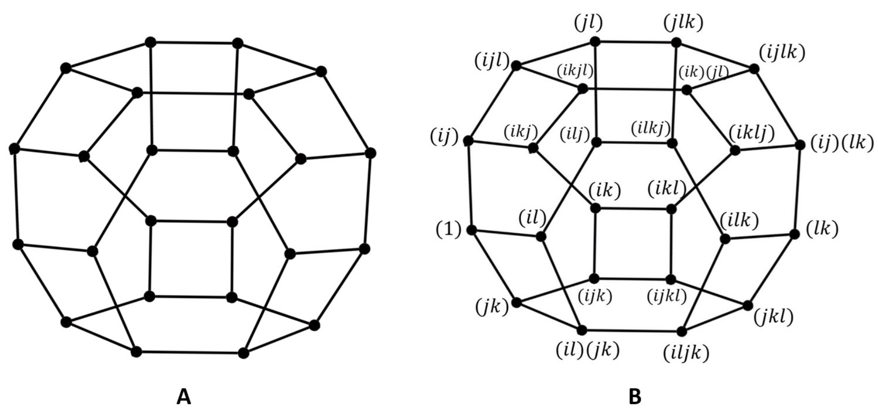

FIGURE 7. A:The Graph Γ_3 B:The Graph $\mathrm{Cay}(S_n, S)$

Proof. Since $S = \{(ij), (il), (jk)\}$, we have

$$\langle S\rangle = S_4.$$

The graph $\mathrm{Cay}(S_4, S)$ is shown in Figure 7B. Thus, it is isomorphic to Γ_3 in Figure 7A.

For the subset S', since $S' = \{(ij), (ijlk), (iklj)\}$, we have

$$\langle S' \rangle = S_4.$$

Now, define a bijection $g : S_n \to S_n$ such that:

$$(jlk)^g = (ilj), (ilj)^g = (jlk), (ikl)^g = (ijk), (ijk)^g = (ikl),$$
$$(jkl)^g = (ijl), (ilk)^g = (ikl), (ikj)^g = (ilk), (ijl)^g = (jkl),$$
$$(jl)^g = (ijkl), (ilkj)^g = (ik), (ik)^g = (ilkj), (ijkl)^g = (jl),$$
$$(jk)^g = (iklj), (il)^g = (ijlk), (iklj)^g = (jk), (ijlk)^g = (il).$$

Any other elements of S_n are mapped to themselves by g. It is easy to see that g is an isomorphism between $\mathrm{Cay}(S_4, S)$ and $\mathrm{Cay}(S_4, S')$. Thus, the graph $\mathrm{Cay}(S_4, S')$ is isomorphic to Γ_3 in Figure 7A. □

Proposition 3.9. *Let S be a subset of S_4, such that*

$$S = \{(ij), (ilk), (ikl)\},$$

where i, j, l, and k are distinct elements in $\{1, 2, 3, 4\}$. Then, $\mathrm{Cay}(S_4, S)$ *is isomorphic to Γ_4, where Γ_4 is the graph in Figure 8A.*

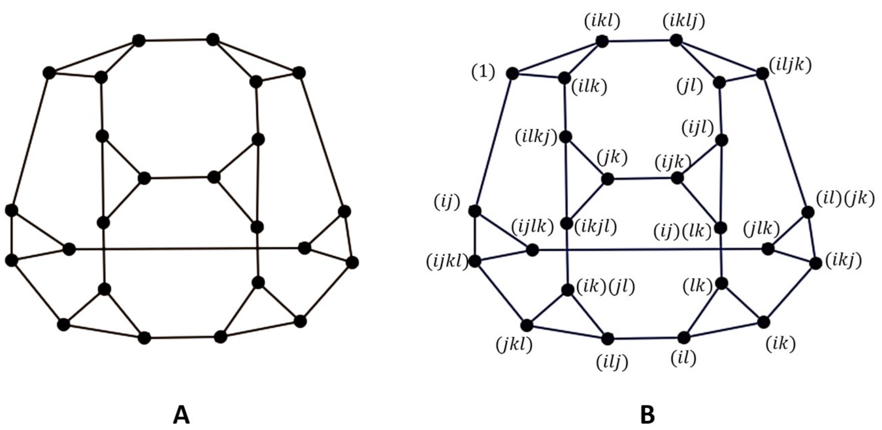

FIGURE 8. A:The Graph Γ_4 B:The Graph $\mathrm{Cay}(S_n, S)$

Proof. Since $S = \{(ij), (ilk), (ikl)\}$, we have

$$\langle S \rangle = S_4.$$

The graph $\mathrm{Cay}(S_4, S)$ is shown in Figure 8B. Thus, it is isomorphic to Γ_4 in Figure 8A. □

Proposition 3.10. *Let S and S' be subsets of S_4, such that*

$$S = \{(ij), (il), (il)(jk)\}$$

and

$$S' = \{(ij), (jl), (ij)(lk)\},$$

where $i, j, l,$ and k are distinct elements in $\{1, 2, 3, 4\}$. Then, $\mathrm{Cay}(S_4, S)$ *and* $\mathrm{Cay}(S_4, S')$ *are isomorphic to Γ_5, where Γ_5 is the graph in Figure 9A.*

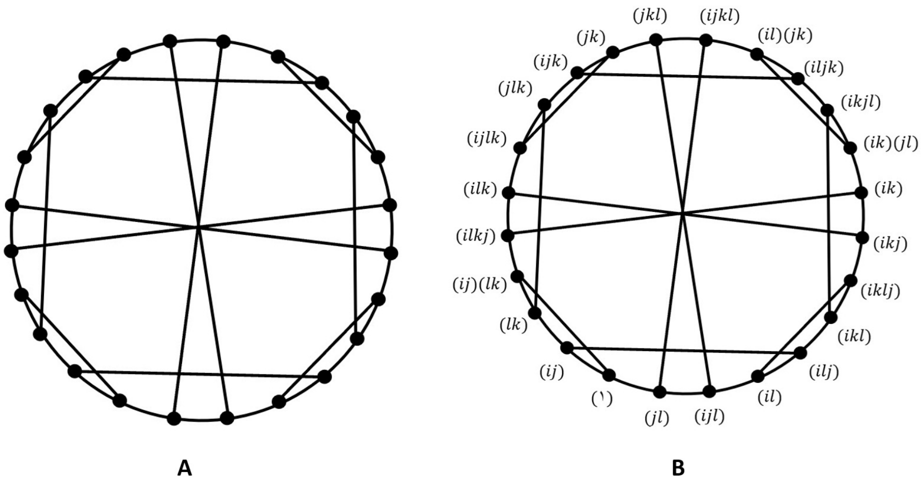

FIGURE 9. A:The Graph Γ_5 B:The Graph $\mathrm{Cay}(S_n, S)$

Proof. Since $S = \{(ij), (il), (il)(jk)\}$, we have

$$\langle S \rangle = S_4.$$

The graph $\mathrm{Cay}(S_4, S)$ is shown in Figure 9B. Thus, it is isomorphic to Γ_5 in Figure 9A.

For the subset S', since $S' = \{(ij), (jl), (ij)(lk)\}$, we have

$$\langle S' \rangle = S_4.$$

Now, define a bijection $g : S_n \to S_n$ such that

$$(il)(jk)^g = (ij)(lk), (ik)(jl)^g = (il)(jk), (ij)(lk)^g = (ik)(jl), (jlk)^g = (ikl),$$
$$(ikl)^g = (ijk), (ijk)^g = (jlk), (jkl)^g = (ilk), (ilk)^g = (ikj),$$
$$(ikj)^g = (jkl), (lk)^g = (ik), (ikjl)^g = (ijkl), (iljk)^g = (ilkj),$$
$$(ij)^g = (jl), (jl)^g = (il), (ilkj)^g = (iklj), (ik)^g = (jk), (jk)^g = (lk),$$
$$(ijkl)^g = (ijlk), (il)^g = (ij), (iklj)^g = (iljk), (ijlk)^g = (ikjl).$$

Any other elements of S_n are mapped to themselves by g. It is easy to see that g is an isomorphism between $\mathrm{Cay}(S_4, S)$ and $\mathrm{Cay}(S_4, S')$. Thus, the graph $\mathrm{Cay}(S_4, S')$ is isomorphic to Γ_5 in Figure 9A.

□

Proposition 3.11. *Let S be a subset of S_4, such that*

$$S = \{(ij), (il), (ik)\},$$

where i, j, l, and k are distinct elements in $\{1, 2, 3, 4\}$. Then, $\mathrm{Cay}(S_4, S)$ is isomorphic to Γ_6, where Γ_6 is the graph in Figure 10A.

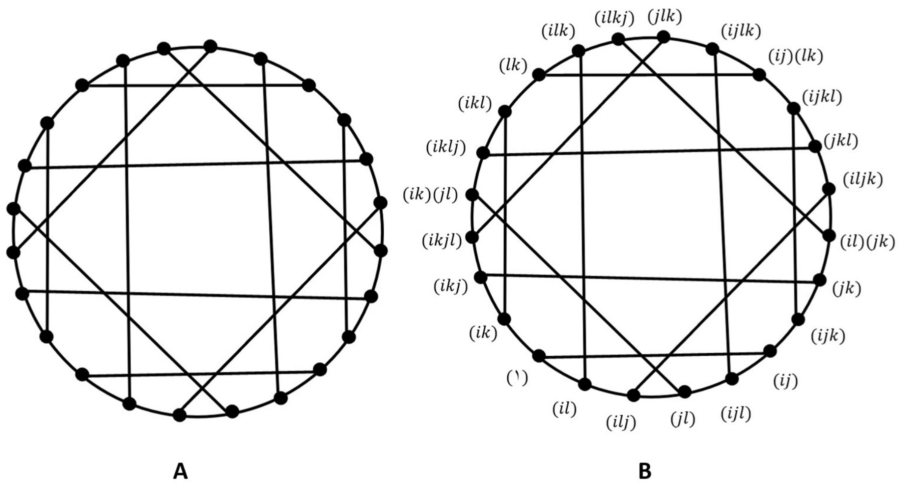

FIGURE 10. A:The Graph Γ_6 B:The Graph $\mathrm{Cay}(S_n, S)$

Proof. Since $S = \{(ij), (il), (ik)\}$, we have

$$\langle S \rangle = S_4.$$

The graph $\mathrm{Cay}(S_4, S)$ is shown in Figure 10B. Thus, it is isomorphic to Γ_6 in Figure 10.

□

In the next theorem, we classify non-isomorphic Cayley graphs of symmetric group S_4 of valency 3, by using Propositions 3.2 to 3.11.

Theorem 3.12. *Up to isomorphism, there are exactly 10 Cayley graphs of the symmetric group S_4 of valency 3, and they are given in Table 2.*

Remember that, the graph $\mathrm{Cay}(G, S)$ is called a CI-graph of the group G, if $\mathrm{Cay}(G, S)$ is isomorphic to $\mathrm{Cay}(G, S')$ implies that there is a permutation $\sigma \in \mathrm{Aut}(G)$ such that $S' = S^\sigma$. So, we have the following corollaries.

TABLE 2. Cayley graphs of S_4 of valency 3.

No.	S	$\mathrm{Cay}(S_4, S)$
1	$\{(ij),(il),(jl)\}$	$\cup_{t=1}^{4} K_{3,3}$
2	$\{(12)(34),(14)(23),(13)(24)\}$ $\{(ij),(lk),(ij)(lk)\}$ $\{(ij)(lk),(iljk),(ikjl)\}$	$\cup_{t=1}^{6} K_4$
3	$\{(ij),(il)(kj),(ik)(lj)\}$ $\{(ij),(iljk),(ikjl)\}$ $\{(ij)(lk),(ijlk),(iklj)\}$	$\cup_{t=1}^{3} C_4 \square K_2$
4	$\{(ij),(ijl),(ilj)\}$	$\cup_{t=1}^{4} C_3 \square K_2$
5	$\{(ij),(ij)(lk),(il)(jk)\}$	$\cup_{t=1}^{3} \Gamma_1$
6	$\{(ij)(lk),(ijl),(ilj)\}$	$\cup_{t=1}^{2} \Gamma_2$
7	$\{(ij),(il),(jk)\}$ $\{(ij),(ijlk),(iklj)\}$	Γ_3
8	$\{(ij),(ilk),(ikl)\}$	Γ_4
9	$\{(ij),(il),(il)(jk)\}$ $\{(ij),(jl),(ij)(lk)\}$	Γ_5
10	$\{(ij),(il),(ik)\}$	Γ_6

Corollary 3.13. *The graphs* $\cup_{t=1}^{4} K_{3,3}$, $\cup_{t=1}^{4} C_3 \square K_2$, $\cup_{t=1}^{3} \Gamma_1$, $\cup_{t=1}^{2} \Gamma_2$, Γ_4 *and* Γ_6 *are the CI-graphs for the group* S_4.

Corollary 3.14. *The graphs* $\cup_{t=1}^{6} K_4$, $\cup_{t=1}^{3} C_4 \square K_2$, Γ_3 *and* Γ_5 *are not CI-graphs for the group* S_4.

Corollary 3.15. S_4 *is not a CI-group. Moreover, it does not have the* 3*-CI-property.*

Proof. By Corollary 3.14. □

In the final part of this section, we investigate the relation between the Cayley graph of S_{n+1} and the Cayley graph of S_n for a set S, such that S is a subset of S_n and we can consider it as a subset of S_{n+1}.

Theorem 3.16. *Suppose that* S *is a subset of the group* S_n*, such that* $1_G \notin S$ *and* $S = S^{-1}$*. Then* $\mathrm{Cay}(S_{n+1}, S)$ *is the disjoint union of* $n+1$ *copies of* $\mathrm{Cay}(S_n, S)$.

Proof. By Proposition 2.2, all components of $\mathrm{Cay}(S_n, S)$ are isomorphic to $\mathrm{Cay}(\langle S\rangle, S)$. Thus, the graph $\mathrm{Cay}(S_n, S)$ is isomorphic to the disjoint union of $n!/k$ copies of $\mathrm{Cay}(\langle S\rangle, S)$, where $|\langle S\rangle| = k$.

We consider S_n as a subgroup of S_{n+1}, such that for each $\rho \in S_n$ we have $(n+1)^\rho = n+1$ and so $\rho \in S_{n+1}$. As well, suppose that S is a subset of S_{n+1}. Since, $S_n \leq S_{n+1}$ and $|S_n| = n!$ we have $[S_{n+1} : S_n] = n+1$. Thus, $[S_{n+1} : \langle S\rangle] = k(n+1)$. Again by Proposition 2.2, $\mathrm{Cay}(S_{n+1}, S)$ is isomorphic to the disjoint union of $(n+1)!/k$ copies of $\mathrm{Cay}(\langle S\rangle, S)$. So, $\mathrm{Cay}(S_{n+1}, S)$ is isomorphic to the disjoint union of $n+1$ copies of $\mathrm{Cay}(S_n, S)$.

□

Now, we have the following corollary.

Corollary 3.17. *Up to isomorphism, there are at least 10 Cayley graphs of the symmetric group S_n of valency 3.*

Proof. By Theorem 3.16, we have

$$\mathrm{Cay}(S_n, S) = \bigcup_{x=1}^{n!/4!} (\mathrm{Cay}(S_4, S)).$$

Therefore, each isomorphic class of $\mathrm{Cay}(S_4, S)$ creates an isomorphic class of the graph $\mathrm{Cay}(S_n, S)$, and we are down. □

4. GAP Code for Computing Cayley Graphs

In this section, we explore the computation of Cayley graphs for permutation groups, specifically the symmetric group S_4. Using a programmatic approach, we can generate these graphs in various computational algebra systems. For instance, the following GAP code, (GAP: a system for computational discrete algebra) snippet demonstrates how to construct the Cayley graph of the symmetric group S_4 using specific generators. The GAP programming language provides tools for working with groups and can efficiently compute Cayley graphs [8].

Below is a GAP program that constructs the Cayley graph for the symmetric group S_4 using the GRAPH package [10] and for specified generators:

```gap
# Load the necessary package
LoadPackage("Graph");

# Define the symmetric group S4
G := SymmetricGroup(4);

# Define the generators
S := [ (3,1), (3,2,4), (3,4,2) ];

# Create the Cayley graph
C := CayleyGraph(G, S);

# Display the Cayley graph
Display(C);
```

This GAP code initializes the symmetric group S_4, defines the desired generators, constructs the Cayley graph, and finally displays it. By running this code, researchers can visualize the structure of the group and its relationships as represented by the Cayley graph.

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