Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

PROPERTY OF THE CURVATURES OF INTEGRABLE POLY-NORDEN MANIFOLDS AND THEIR SUBMANIFOLDS

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Article type: Research Article
(Received: 06 December 2024, Received in revised form 08 April 2025)
(Accepted: 13 May 2025, Published Online: 14 May 2025)

Abstract. In the present paper, almost poly-Norden and locally almost poly-Norden manifolds are investigated. Ricci tensor and Riemannian curvature of integrable poly-Norden manifolds are studied. Geometric properties of submanifolds of these types of manifolds are studied. Moreover, slant submanifolds of almost poly-Norden manifolds are characterized and illustrated by non-trivial examples.

Keywords: Almost contact manifolds, Integrable poly-Norden manifold, Slant submanifold, Riemannian curvature.
2020 MSC: 50D45, 53C15, 53C25.

1. Introduction

One of the important structures that has been investigated and studied on an odd-dimensional Riemannian manifold is the almost contact structure. The cosymplectic, Sasakian and Kenmotsu manifolds are three well-known and important classes among the almost contact manifolds. Most of these structures have many applications in various sciences, especially in physics and in the fields of cosmology, relativity and gravity [2, 7, 12].

Poly-Norden structures have been introduced and studied by Sahin [11] in 2018. After than, in [9], the fundamental and geometrical properties of their hypersurfaces and submanifolds were investigated. Poly-Norden structures have interesting relations with almost contact metric structures. In an special case, by using an almost contact metric structure one can obtain an induction poly-Norden structure.

On the other hand, since the notion of slant submanifold is a generalization of the concepts such as invariant and anti-invariant submanifolds [7–9], many authors have studied special models of slant submanifolds in various structures. For example, hemi-slant, invariant, slant, anti-invariant and semi-slant submanifolds of metallic manifolds have been analyzed in [1, 4, 6]. Furthermore, in [3, 7] slant submanifolds of contact 3-structures, golden manifolds and

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https://doi.org/10.22103/jmmr.2025.24475.1728

Publisher: Shahid Bahonar University of Kerman
How to cite: M. Tofighi, M.B. Kazemi Balgeshir, *Property of the curvatures of integrable poly-Norden manifolds and their submanifolds*, J. Mahani Math. Res. 2025; 14(2): 329-341.

locally conformal Kaehler manifolds have been investigated and the slant lightlike submanifolds of cosymplectic structures have been introduced in [9].

So, motivated by the above works and by using the approaches and proofing techniques of the papers [9,11], in this paper, we investigate slant submanifolds of almost poly-Norden manifolds. We first review the notion of an almost poly-Norden manifold and investigate an example for that in Section 2. In Section 3, we study ψ-invariant and ψ-anti-invariant submanifolds of almost poly-Norden manifolds, and finally in Section 4 we consider slant submanifolds and give an example of this type of submanifolds in a poly-Norden manifold.

2. Locally poly-Norden Riemannian manifolds

Definition 2.1. [11] On a smooth Riemannian manifold $(\overline{B}, \overline{\mathcal{G}})$, an almost poly-Norden structure is a (1,1)-tensor ψ which satisfies

$$\psi^2 = m\psi - I, \tag{1}$$

where I is the identity tensor on $\overline{B}$ and $m \in \mathbb{R} - \{0\}$. In this case $(\overline{B}, \psi)$ is called an almost poly-Norden manifold.

The Riemannian metric $\overline{\mathcal{G}}$ is said to be ψ-compatible if

$$\overline{\mathcal{G}}(\psi E, \psi F) = m\overline{\mathcal{G}}(\psi E, F) - \overline{\mathcal{G}}(E, F), \tag{2}$$

for any $E, F \in \mathcal{T}(\overline{B})$. From this it follows ψ has symmetric property with respect to $\overline{\mathcal{G}}$, which means

$$\overline{\mathcal{G}}(\psi E, F) = \overline{\mathcal{G}}(E, \psi F). \tag{3}$$

In this paper, we denote the Levi-Civita connection with respect to the Riemannian metric $\overline{\mathcal{G}}$, by $\overline{\mathfrak{D}}$.

Definition 2.2. [11] An almost poly-Norden structure $(\overline{B}, \psi)$ is called a locally almost poly-Norden manifold (or integrable), if its Nijenhuis tensor field N_ψ is equal to zero, i.e.

$$N_\psi(E, F) = \psi^2[E, F] + [\psi E, \psi F] - \psi[\psi E, F] - \psi[E, \psi F] = 0.$$

Note that $N_\psi = 0$ is equivalent to $\overline{\mathfrak{D}}\psi = 0$ ([11]). This means ψ is parallel with respect to the Levi-Civita connection associated to $\overline{\mathcal{G}}$.

Lemma 2.3. *Let $(\overline{B}, \psi)$ be an almost poly-Norden Riemannian manifold then $(\overline{\mathfrak{D}}_E\psi)\psi F = (mI - \psi)(\overline{\mathfrak{D}}_E\psi)F$, for any $E, F \in \mathcal{T}(\overline{B})$.*

Proof. For any $E, F \in \mathcal{T}(\overline{B})$, from definition of almost poly-Norden manifolds, we have

$$\begin{aligned}(\overline{\mathfrak{D}}_E\psi)\psi F &= \overline{\mathfrak{D}}_E\psi^2 F - \psi\overline{\mathfrak{D}}_E\psi F\\ &= m\overline{\mathfrak{D}}_E\psi F - \overline{\mathfrak{D}}_E F - \psi\overline{\mathfrak{D}}_E\psi F\\ &= (mI-\psi)(\overline{\mathfrak{D}}_E\psi)F.\end{aligned}$$

□

Lemma 2.4. *If $(\overline{B}, \psi, \overline{\mathcal{G}})$ is an almost poly-Norden Riemannian manifold then $\overline{\mathcal{G}}((\overline{\mathfrak{D}}_E\psi)F, W) = \overline{\mathcal{G}}(F, (\overline{\mathfrak{D}}_E\psi)W)$ for any $E, F, W \in \mathcal{T}(\overline{B})$.*

Proof. By using equation (3) we have for any $E, F, W \in \mathcal{T}(\overline{B})$:

$$\begin{aligned}\overline{\mathcal{G}}((\overline{\mathfrak{D}}_E\psi)F, W) &= \overline{\mathcal{G}}(\overline{\mathfrak{D}}_E\psi F, W) - \overline{\mathcal{G}}(\psi\overline{\mathfrak{D}}_E F, W)\\ &= E\overline{\mathcal{G}}(\psi F, W) - \overline{\mathcal{G}}(\psi F, \overline{\mathfrak{D}}_E W) - E\overline{\mathcal{G}}(F, \psi W) + \overline{\mathcal{G}}(F, \overline{\mathfrak{D}}_E\psi W)\\ &= \overline{\mathcal{G}}(F, (\overline{\mathfrak{D}}_E\psi)W).\end{aligned}$$

□

The fundamental 2-form Ψ on almost poly-Norden manifold $(\overline{B}, \psi, \overline{\mathcal{G}})$ is defined as follows:

$$\Psi(E, F) = \overline{\mathcal{G}}(E, \psi F),$$

for any $E, F \in \mathcal{T}(\overline{B})$.
According to the above definition we, get that Ψ is symmetric because for any $E, F \in \mathcal{T}(\overline{B})$:

$$\Psi(E, F) = \overline{\mathcal{G}}(E, \psi F) = \overline{\mathcal{G}}(\psi E, F) = \Psi(F, E).$$

Lemma 2.5. *Let $(\overline{B}, \psi, \overline{\mathcal{G}})$ be an almost poly-Norden Riemannian manifold then*
$(\overline{\mathfrak{D}}_E\Psi)(F, W) = \overline{\mathcal{G}}(F, (\overline{\mathfrak{D}}_E\psi)W)$ for all $E, F \in \mathcal{T}(\overline{B})$.

Proof. According to the definition of Ψ, directly we conclude

$$\begin{aligned}(\overline{\mathfrak{D}}_E\Psi)(F, W) &= E\Psi(F, W) - \Psi(\overline{\mathfrak{D}}_E F, W) - \Psi(F, \overline{\mathfrak{D}}_E W)\\ &= E\overline{\mathcal{G}}(F, \psi W) - \overline{\mathcal{G}}(\overline{\mathfrak{D}}_E F, \psi W) - \overline{\mathcal{G}}(F, \psi\overline{\mathfrak{D}}_E W)\\ &= \overline{\mathcal{G}}(\overline{\mathfrak{D}}_E F, \psi W) + \overline{\mathcal{G}}(F, \overline{\mathfrak{D}}_E\psi W) - \overline{\mathcal{G}}(\overline{\mathfrak{D}}_E F, \psi W) - \overline{\mathcal{G}}(F, \psi\overline{\mathfrak{D}}_E W)\\ &= \overline{\mathcal{G}}(F, (\overline{\mathfrak{D}}_E\psi)W),\end{aligned}$$

for all $E, F, W \in \mathcal{T}(\overline{B})$. □

Theorem 2.6. *If $(\overline{B}, \psi, \overline{\mathcal{G}})$ is a locally almost poly-Norden Riemannian manifold then for $E, F, W \in \mathcal{T}(\overline{B})$, we get*
$d\Psi(E, F, W) - d\Psi(E, \psi F, \psi W) = 2\{\Psi(F, \overline{\mathfrak{D}}_E W) + \Psi(E, [W, F])$
$- \Psi(\psi F, \overline{\mathfrak{D}}_E\psi W) + \Psi(E, [\psi F, \psi W])\}$

Proof. Before proving the proposition, we obtain the following equation $d\Psi(E,F,W) = 2\{\Psi(F,\overline{\mathfrak{D}}_E W) + \Psi(E,[W,F])\}$.
By direct computations and since $\overline{\mathfrak{D}}\psi = 0$, we have

$$d\Psi(E,F,W) = E\Psi(F,W) - F\Psi(E,W) + W\Psi(E,F) - \Psi([E,F],W) + \Psi([E,W],F)$$
$$- \Psi([F,W],E) = (\overline{\mathfrak{D}}_E\Psi)(F,W) + \Psi(F,\overline{\mathfrak{D}}_E W) - (\overline{\mathfrak{D}}_F\Psi)(E,W) - \Psi(E,\overline{\mathfrak{D}}_F W)$$
$$+ (\overline{\mathfrak{D}}_W\Psi)(E,F) + \Psi(E,\overline{\mathfrak{D}}_W F) + \Psi(\overline{\mathfrak{D}}_E W,F) - \Psi(\overline{\mathfrak{D}}_F W,E) + \Psi(\overline{\mathfrak{D}}_W F,E)$$
$$= \overline{\mathcal{G}}(F,(\overline{\mathfrak{D}}_E\psi)W) - \overline{\mathcal{G}}(E,(\overline{\mathfrak{D}}_F\psi)W) + \overline{\mathcal{G}}(E,(\overline{\mathfrak{D}}_W\psi)F) + 2\{\Psi(F,\overline{\mathfrak{D}}_E W)$$
$$+ \Psi(E,[W,F])\} = 2\{\Psi(F,\overline{\mathfrak{D}}_E W) + \Psi(E,[W,F])\}.$$

Now, by using the above equation and previous lemmas, for all $E,F,W \in \mathcal{T}(\overline{B})$, we obtain

$$d\Psi(E,F,W) - d\Psi(E,\psi F,\psi W) = 2\{\Psi(F,\overline{\mathfrak{D}}_E W) + \Psi(E,[W,F]) - \Psi(\psi F,\overline{\mathfrak{D}}_E \psi W)$$
$$+ \Psi(E,[\psi F,\psi W])\}.$$

□

Definition 2.7. A Matrix A is called an almost poly-Norden matrix if there exists a real number $m \neq 0$ such that $A^2 = mA - I$, where I denotes the identity matrix and we show it as a pair of (A,m).

Example 2.8. *The pairs $(A,2)$, $(B,\frac{10}{3})$ and $(C,\frac{2}{3})$ in which A,B and C are defined as follows are almost poly-Norden matrices.*

$$A = \begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 2 & \frac{1}{3} & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & \frac{5}{3} & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{8}{9} \\ -\frac{3}{5} & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} \end{pmatrix}$$

3. Curvatures and submanifolds of integrable poly-Norden manifolds

Suppose $(B,\mathcal{G})$ is a submanifold of an almost poly-Norden structure $(\overline{B},\overline{\mathcal{G}},\psi)$, where $\mathcal{G}$ is the induced metric on B.
We denote the Levi-Civita connection on the submanifold B by the notation $\mathfrak{D}$. So, the Gauss and Weingarten formulas can be written as follows

$$\overline{\mathfrak{D}}_E F = \mathfrak{D}_E F - h(E,F), \tag{4}$$
$$\overline{\mathfrak{D}}_E C = -A_C E + \mathfrak{D}^{\perp}_E C. \tag{5}$$

In which h is the second fundamental form, A is the shape tensor and we have $\mathcal{G}(h(E,F),C) = \mathcal{G}(A_C E, F)$.

Lemma 3.1. *Let $(\overline{B}, \psi)$ be a locally almost poly-Norden Riemannian manifold and let σ be a (1,1)-tensor on a submanifold B of $\overline{B}$ such that $\sigma = \psi\mid_B$. Then, (B,σ) is a locally almost poly-Norden submanifold if and only if $h(E,\sigma F) = \sigma h(E,F)$.*

Proof. First, we show that (B,σ) is an almost poly-Norden Riemannian submanifold and then by using equation (4) we prove that it is locally almost poly-Norden. We have

$$\sigma^2 E = \psi^2 \mid_B E = m\psi \mid_B E - E = m\sigma E - E,$$

and

$$\begin{aligned}
&(\overline{\mathfrak{D}}_E \psi \mid_B)F = 0,\\
&\overline{\mathfrak{D}}_E \psi \mid_B F - \psi \mid_B \overline{\mathfrak{D}}_E F = 0,\\
&\mathfrak{D}_E \sigma F + h(E,\sigma F) - \sigma \mathfrak{D}_E F - \sigma h(E,F) = 0,\\
&(\mathfrak{D}_E \sigma)F + h(E,\sigma F) - \sigma h(E,F) = 0.
\end{aligned}$$

So (B,σ) is a locally almost poly-Norden if and only if $h(E,\sigma F) = \sigma h(E,F)$. □

Definition 3.2. Let B be a submanifold of $\overline{B}$, B is called the ψ-invariant submanifold of $\overline{B}$ if $\psi(TB) \subset TB$ and B is ψ-anti-invariant of $\overline{B}$ if $\psi(TB) \subset TB^{\perp}$([9]).

For any $E \in \Gamma(B)$ and $C \in \Gamma(B^{\perp})$ we put

$$\psi E = TE + NE, \tag{6}$$

$$\psi C = tC + nC, \tag{7}$$

where these projection maps act as $T : \Gamma(B) \longrightarrow \Gamma(B)$, $N : \Gamma(B) \longrightarrow \Gamma(B^{\perp})$ and
$t : \Gamma(B^{\perp}) \longrightarrow \Gamma(B)$, $n : \Gamma(B^{\perp}) \longrightarrow \Gamma(B^{\perp})$.
From (6) and (7) we can easily get the following equations [4]

$$\mathcal{G}(TE,F) = \mathcal{G}(E,TF), \tag{8}$$

$$\mathcal{G}(nU,C) = \mathcal{G}(U,nC), \tag{9}$$

$$\mathcal{G}(nE,C) = \mathcal{G}(E,tC), \tag{10}$$

for any $E, F \in \Gamma(B)$ and $U, C \in \Gamma(B^{\perp})$.

Lemma 3.3. *For any $E, F \in \Gamma(B)$ and $U, C \in \Gamma(B^{\perp})$, the following relations hold.*

$$T^2 = mT - I - tN, \tag{11}$$

$$N = \frac{1}{m}(NT + nN), \tag{12}$$

$$t = \frac{1}{m}(Tt + tn), \tag{13}$$

$$n^2 = mn - I - Nt. \tag{14}$$

Proof. Apply the ψ to the (6), then we have

$$\psi^2 E = \psi TE + \psi NE,$$

now using the (1),(6) and (7) we get

$$m\psi E - E = T^2E + NTE + tNE + nNE,$$

$$mTE + mNE - E = T^2E + NTE + tNE + nNE, \qquad \forall E \in \Gamma(TB).$$

Now, we separate the tangential and normal components of the above statement, then (11) and (12) obtained.
Similar to the previous case, apply the ψ to the (7), then we have

$$\psi^2 C = \psi tC + \psi nC,$$

now using the (1), (6) and (7) we obtain

$$m\psi C - C = TtC + NtC + tnC + n^2C,$$

$$mtC + mnC - C = TtC + NtC + tnC + n^2C, \qquad \forall C \in \Gamma(TB^{\perp}),$$

by separating the tangential and normal components, we get (13) and (14). □

Note if B is an invariant submanifold, it implies $N = 0$, so from (11) and (14) we have

$$T^2 = mT - I, \qquad n^2 = mn - I,$$

hence $(T, \mathcal{G})$ and $(n, \mathcal{G})$ is an almost poly-Norden structure on B.

Moreover, by taking covariant derivative on the project operators we get

$$(\mathfrak{D}_E T)F = \mathfrak{D}_E TF - T\mathfrak{D}_E F, \tag{15}$$

$$(\overline{\mathfrak{D}}_E N)F = \mathfrak{D}^{\perp}_E NF - N\mathfrak{D}_E F, \tag{16}$$

$$(\mathfrak{D}_E t)C = \mathfrak{D}_E tC - t\mathfrak{D}_E C, \tag{17}$$

$$(\overline{\mathfrak{D}}_E n)C = \mathfrak{D}^{\perp}_E nC - n\mathfrak{D}^{\perp}_E C. \tag{18}$$

for any $E, F \in \Gamma(B)$ and $C \in \Gamma(B^{\perp})$.

Lemma 3.4. *If B is a submanifold in $\overline{B}$ and almost poly-Norden structure ψ on almost poly-Norden manifold $(\overline{B}, g, \psi)$ is integrable, then we obtain*

$$\mathcal{G}((\overline{\mathfrak{D}}_E N)F, C) = \mathcal{G}((\mathfrak{D}_E t)C, F),$$

for any $E, F \in \Gamma(TB)$ and $C \in \Gamma(TB^{\perp})$.

Proof. Since the ψ is integrable then for any $E, F \in \Gamma(TB)$ we have $\overline{\mathfrak{D}}_E \psi F = \psi \overline{\mathfrak{D}}_E F$, now by using Gauss and Weingarten formulas and Equations (6) and (7) we have

$$\overline{\mathfrak{D}}_E TF + \overline{\mathfrak{D}}_E NF = \psi(\mathfrak{D}_E F + h(E,F)),$$

$$\mathfrak{D}_E TF + h(E,TF) - A_{NF}E + \mathfrak{D}^{\perp}_E NF = T\mathfrak{D}_E F + N\mathfrak{D}_E F + th(E,F) + nh(E,F),$$

by separating the tangential and normal components we have

$$(\mathfrak{D}_E T)F = A_{NF}E + th(E,TF), \tag{19}$$

$$(\overline{\mathfrak{D}}_E N)F = nh(E,F) - h(E,TF). \tag{20}$$

And for any $E \in \Gamma(B)$ and $C \in \Gamma(B^{\perp})$ we get $\overline{\mathfrak{D}}_E \psi C = \psi \overline{\mathfrak{D}}_E C$.
Similarly, we can state

$$\overline{\mathfrak{D}}_E tC + \overline{\mathfrak{D}}_E nC = \psi(-A_C E + \mathfrak{D}^{\perp}_E C),$$

$$\mathfrak{D}_E tC + h(E,tC) - A_{nC}E + \mathfrak{D}^{\perp}_E nC = -TA_C E - NA_C E + t\mathfrak{D}^{\perp}_E C + n\mathfrak{D}^{\perp}_E C.$$

So we get it by separating the tangential and normal component

$$(\mathfrak{D}_E t)C = A_{nC}E - TA_C E, \tag{21}$$

$$(\overline{\mathfrak{D}}_E n)C = -h(E,tC) - NA_C E. \tag{22}$$

And finally, by using the above equations, (8) and (9)

$$\begin{aligned} \mathcal{G}((\overline{\mathfrak{D}}_E N)F, C) &= \mathcal{G}(h(E,F), nC) - \mathcal{G}(h(E,TF), C) \\ &= \mathcal{G}(A_{nC}E - TA_C E, F) \\ &= \mathcal{G}((\mathfrak{D}_E t)C, F). \end{aligned}$$

□

Theorem 3.5. *Let B be a submanifold of an integrable poly-Norden manifold $(\overline{B}, g, \psi)$, then we have $(\overline{\mathfrak{D}}_E N)F = 0$ and $(\mathfrak{D}_E t)C = 0$, for all $E, F \in \Gamma(TB), C \in \Gamma(TB^{\perp})$ if and only if A satisfies*

$$A_{nC}E = TA_C E = A_C TE.$$

Proof. By using (20)

$$\mathcal{G}(h(E,F), nC) - \mathcal{G}(h(E,TF), C) = 0.$$

Given the relation between h and A and Equality (8) we have

$$\mathcal{G}(A_{nC}E, F) - \mathcal{G}(A_C E, TF) = 0,$$

$$\mathcal{G}(A_{nC}E, F) - \mathcal{G}(TA_C E, F) = 0.$$

So equality is achieved on the left.
On the other hand, since $(\mathfrak{D}_E t)C = 0$ then by using (21) we get

$$\mathcal{G}(A_{nC}E - TA_C E, F) = 0,$$

the right-hand equation can also be obtained by the following statement,

$$\mathcal{G}(E, A_{nC}F) - \mathcal{G}(E, A_C TF) = 0.$$

Conversely, All of the above steps are reversible. □

Theorem 3.6. *The Riemannian curvature tensor* $\overline{\mathcal{R}}$ *of an integrable almost poly-Norden manifold* $(\overline{B}, g, \psi)$ *satisfies*

i) $\overline{\mathcal{R}}(E, F)\psi = \psi\overline{\mathcal{R}}(E, F)$.
ii) $\overline{\mathcal{R}}(\psi E, F) = \overline{\mathcal{R}}(E, \psi F)$.
iii) $\overline{\mathcal{R}}(\psi E, \psi F) = m\overline{\mathcal{R}}(\psi E, F) - \overline{\mathcal{R}}(E, F)$.
iv) $\mathcal{G}(\overline{\mathcal{R}}(E, F)\psi W, \psi W) = m\mathcal{G}(\overline{\mathcal{R}}(E, F)W, \psi W) - \mathcal{G}(\overline{\mathcal{R}}(E, F)W, W)$.
v) $\mathcal{G}(\overline{\mathcal{R}}(E, F)\psi W, W) = \mathcal{G}(\overline{\mathcal{R}}(E, F)W, \psi W)$.

Proof. By using (2), (3), equation $\overline{\mathcal{R}}(E, F)W = [\overline{\mathfrak{D}}_E, \overline{\mathfrak{D}}_F]W - \overline{\mathfrak{D}}_{[E,F]}W$ and the integrability property ψ the result is achieved. □

Theorem 3.7. *Let* $\overline{\mathcal{S}}$ *be the curvature Ricci tensor of* $\overline{B}$ *and almost poly-Norden structure* ψ *on almost poly-Norden* $(\overline{B}, g, \psi)$ *be integrable, then*

i) $\overline{\mathcal{S}}(\psi^2 E, F) = m\overline{\mathcal{S}}(\psi E, F) - \overline{\mathcal{S}}(E, F)$.
ii) $\overline{\mathcal{S}}(E, \psi^2 F) = m\overline{\mathcal{S}}(E, \psi F) - \overline{\mathcal{S}}(E, F)$.
iii) $\overline{\mathcal{S}}(\psi E, F) = \overline{\mathcal{S}}(E, \psi F)$.
iv) $\overline{\mathcal{S}}(\psi E, \psi F) = m\overline{\mathcal{S}}(\psi E, F) - \overline{\mathcal{S}}(E, F)$.

Proof. Let $\{e_i\}$, $i = 1, ..., n$ be an orthonormal basic on $T_p\overline{B}$ then by using (1) we obtain

$$\begin{aligned} i)\ \overline{\mathcal{S}}(\psi^2 E, F) &= \Sigma_{i=1}^n \overline{\mathcal{R}}(e_i, \psi^2 E, F, e_i) \\ &= m\Sigma_{i=1}^n \overline{\mathcal{R}}(e_i, \psi E, F, e_i) - \Sigma_{i=1}^n \overline{\mathcal{R}}(e_i, E, F, e_i) \\ &= m\overline{\mathcal{S}}(\psi E, F) - \overline{\mathcal{S}}(E, F). \end{aligned}$$

ii) Similar to part (*i*).

$$\begin{aligned} iii)\ \overline{\mathcal{S}}(\psi E, F) &= \Sigma_{i=1}^n \overline{\mathcal{R}}(e_i, \psi E, F, e_i) \\ &= \Sigma_{i=1}^n \overline{\mathcal{R}}(\psi e_i, E, F, e_i) \\ &= \Sigma_{i=1}^n \psi\overline{\mathcal{R}}(F, e_i, e_i, E) \\ &= \Sigma_{i=1}^n \overline{\mathcal{R}}(e_i, E, \psi F, e_i) \\ &= \overline{\mathcal{S}}(E, \psi F). \end{aligned}$$

iv) By using items (*i*) and (*iii*) we have

$$\overline{\mathcal{S}}(\psi E, \psi F) = \overline{\mathcal{S}}(\psi^2 E, F) = m\overline{\mathcal{S}}(\psi E, F) - \overline{\mathcal{S}}(E, F).$$

□

Theorem 3.8. *On the integrable almost poly-Norden manifold* $(\overline{B}, g, \psi)$*, the following relations hold:*

i) $(\overline{\mathfrak{D}}_W\overline{\mathcal{R}})(E, F)\psi W = \psi(\overline{\mathfrak{D}}_W\overline{\mathcal{R}})(E, F)W$.

ii) $(\overline{\mathfrak{D}}_W\overline{\mathcal{S}})(\psi E,F)=(\overline{\mathfrak{D}}_W\overline{\mathcal{S}})(E,\psi F)$.

Proof. i) By using Theorem 3.6 part (*i*) and the integrability property ψ we get:

$$\begin{aligned}(\overline{\mathfrak{D}}_W\overline{\mathcal{R}})(E,F)\psi W &= \overline{\mathfrak{D}}_W\overline{\mathcal{R}}(E,F)\psi W-\overline{\mathcal{R}}(\overline{\mathfrak{D}}_W E,F)\psi W-\overline{\mathcal{R}}(E,\overline{\mathfrak{D}}_W F)\psi W\\ &\quad -\overline{\mathcal{R}}(E,F)\overline{\mathfrak{D}}_W\psi W\\ &=\psi(\overline{\mathfrak{D}}_W\overline{\mathcal{R}})(E,F)W.\end{aligned}$$

ii) We use the Theorem 3.7 to prove this part

$$\begin{aligned}(\overline{\mathfrak{D}}_W\overline{\mathcal{S}})(\psi E,F) &= \overline{\mathfrak{D}}_W\overline{\mathcal{S}}(\psi E,F)-\overline{\mathcal{S}}(\overline{\mathfrak{D}}_W\psi E,F)-\overline{\mathcal{S}}(\psi E,\overline{\mathfrak{D}}_W F)\\ &=\overline{\mathfrak{D}}_W\overline{\mathcal{S}}(E,\psi F)-\overline{\mathcal{S}}(\overline{\mathfrak{D}}_W E,\psi F)-\overline{\mathcal{S}}(E,\psi\mathfrak{D}_W F)\\ &=(\overline{\mathfrak{D}}_W\overline{\mathcal{S}})(E,\psi F).\end{aligned}$$

□

4. Slant submanifolds of integrable poly-Norden manifolds

Definition 4.1. A submanifold $(B,\mathcal{G})$ of an almost poly-Norden manifold $(\overline{B},g,\psi)$ is said to be a slant submanifold if $\forall E\in\mathcal{T}_pB$, the $\alpha(E)$ angle between $\mathcal{T}_pB$ and ψE, does not depend on the choice of $p\in B$ and $E\in\mathcal{T}_pB$. Also, $\alpha:=\alpha(E)$ is called the slant angle.

It should be noted that, if

i) $\alpha=0$, B is ψ-invariant,
ii) $\alpha=\dfrac{\pi}{2}$, B is ψ-anti-invariant,
iii) $0<\alpha<\dfrac{\pi}{2}$, B is a proper slant.

According to the above definition we have:

(23)
$$cos\alpha=\frac{\mathcal{G}(\psi E_p,TE_p)}{\|TE_p\|\|\psi E_p\|}=\frac{\mathcal{G}(TE_p,TE_p)+\mathcal{G}(NE_p,TE_p)}{\|TE_p\|\|\psi E_p\|}=\frac{\|TE_p\|^2}{\|TE_p\|\|\psi E_p\|}=\frac{\|TE_p\|}{\|\psi E_p\|}.$$

Example 4.2. *([11]) Consider Euclidean space $\mathbb{R}^4$ with map ψ, such that*

$$\begin{aligned}\psi:\mathbb{R}^4&\longrightarrow\mathbb{R}^4\\ (x_1,x_2,x_3,x_4)&\mapsto(B_mx_1,\overline{B}_mx_2,B_mx_3,\overline{B}_mx_4)\end{aligned}$$

where $B_m = \dfrac{m+\sqrt{m^2-4}}{2}$ *and* $g = \Sigma_{i=1}^{4} dx_i \otimes dx_i$.
$(\mathbb{R}^4, \psi, \mathcal{G})$ *is an almost poly-Norden manifold because*

$$\begin{aligned}\psi^2(x_1, x_2, x_3, x_4) &= (B_m^2 x_1, \overline{B}_m^2 x_2, B_m^2 x_3, \overline{B}_m^2 x_4)\\ &= m(B_m x_1, \overline{B}_m x_2, B_m x_3, \overline{B}_m x_4) - (x_1, x_2, x_3, x_4)\\ &= m\psi(x_1, x_2, x_3, x_4) - (x_1, x_2, x_3, x_4).\end{aligned}$$

Suppose $B = \{(u, v) \mid u, v \in \mathbb{R}\}$ *is a submanifold of* $\mathbb{R}^4$ *and* $f : B \longrightarrow \mathbb{R}^4$*, that* $f(u, v) = (B_m u, \overline{B}_m u, B_m v, \overline{B}_m v)$.
In this case, the tangent space of the TB *is generated by the following vectors*

$$e_1 = (B_m, \overline{B}_m, 0, 0), \qquad e_2 = (0, 0, B_m, \overline{B}_m),$$

On the other hand we have:

$$\begin{aligned}\|e_1\| &= \|e_2\| = \sqrt{m^2-2},\\ \|\psi e_1\| &= \|\psi e_2\| = \sqrt{m^4-4m^2-2},\\ \cos\gamma &= \frac{\mathcal{G}(\psi e_1, e_1)}{\|\psi e_1\|\,\|e_1\|} = \frac{m(m^2-3)}{\sqrt{(m^2-4)(m^4-4m-2)}},\\ \cos\beta &= \frac{\mathcal{G}(\psi e_2, e_2)}{\|\psi e_2\|\,\|e_2\|} = \frac{m(m^2-3)}{\sqrt{(m^2-4)(m^4-4m-2)}},\end{aligned}$$

so $\gamma = \beta = \arccos(\dfrac{m(m^2-3)}{\sqrt{(m^2-4)(m^4-4m-2)}})$.
Therefore B *is a proper slant submanifold of* $\mathbb{R}^4$.

Lemma 4.3. *On a slant submanifold* $(B, \mathcal{G})$ *of an almost poly-Norden manifold* $(\overline{B}, g, \psi)$*, we get*

i) $\mathcal{G}(TE, TF) = cos^2\alpha[m\mathcal{G}(E, TF) - \mathcal{G}(E, F)]$,
ii) $\mathcal{G}(NE, NF) = sin^2\alpha[m\mathcal{G}(E, TF) - \mathcal{G}(E, F)]$.

Proof. i) From Equation (23) we have

$$\mathcal{G}(TE, TE) = cos^2\alpha\mathcal{G}(\psi E, \psi E), \tag{24}$$

in the above equation, put $E + F$ instead of E,

$$\mathcal{G}(TE + TF, TE + TF) = cos^2\alpha\mathcal{G}(\psi E + \psi F, \psi E + \psi F),$$

now by using the linearity feature of the metric g we have:

$$\mathcal{G}(TE, TE) + \mathcal{G}(TE, TF) + \mathcal{G}(TF, TE) + \mathcal{G}(TF, TF) = cos^2\alpha[\mathcal{G}(\psi E, \psi E) + \mathcal{G}(\psi E, \psi F) + \mathcal{G}(\psi F, \psi E) + \mathcal{G}(\psi F, \psi F)],$$

given the Equation (24)

$$\mathcal{G}(TE, TF) = cos^2\alpha\mathcal{G}(\psi E, \psi F),$$

and the finally by using (3), (1) and (6) we have:

$$\mathcal{G}(TE,TF) = cos^2\alpha[m\mathcal{G}(E,TF) - \mathcal{G}(E,F)].$$

ii) To prove this part, we use part (*i*) and (6), the similarly to the previous case we have:

$$\begin{aligned}\mathcal{G}(NE,NF) &= \mathcal{G}(\psi E - TE, \psi F - TF)\\ &= \mathcal{G}(\psi E, \psi F) - \mathcal{G}(TE,TF) - \mathcal{G}(TE,TF) + \mathcal{G}(TE,TF)\\ &= \mathcal{G}(E, \psi^2 F) - m\, cos^2\alpha \mathcal{G}(E,TF) + cos^2\alpha\mathcal{G}(E,F)\\ &= m\mathcal{G}(E, TF + NF) - \mathcal{G}(E,F) - m\, cos^2\alpha\mathcal{G}(E,TF) + cos^2\alpha\mathcal{G}(E,F)\\ &= (1 - cos^2\alpha)m\mathcal{G}(E,TF) - (1-cos^2\alpha)\mathcal{G}(E,F)\\ &= sin^2\alpha[m\mathcal{G}(E,TE) - \mathcal{G}(E,F)].\end{aligned}$$

□

Theorem 4.4. *Let* $(B,\mathcal{G})$ *be a submanifold of an almost poly-Norden manifold* $(\overline{B}, g, \psi)$. *Then* B *is a slant submanifold of* $\overline{B}$ *if and only if we have* λ *belong to interval* $[0,1]$ *such that* $T^2 = \lambda(mT - I)$.

Proof. Let B be a slant submanifold of an almost poly-Norden manifold $\overline{B}$ with the constant slant angle α, so put $\lambda = cos^2\alpha \in [0,1]$.
By using the previous lemma

$$\begin{aligned}\mathcal{G}(T^2E,F) &= \mathcal{G}(TE,TF) = cos^2\alpha[m\mathcal{G}(TE,F) - \mathcal{G}(E,F)]\\ &= cos^2\alpha\mathcal{G}(mTE - E, F) = cos^2\alpha\mathcal{G}((mT-I)E,F),\end{aligned} \tag{25}$$

for any $E, F \in \Gamma(TB)$, therefore we have $T^2 = \lambda(mT - I)$.
Conversely, suppose there exist a real number λ in the interval $[0,1]$ such that $T^2 = \lambda(mT - I)$. Let α be the angele between ψ and the tangent space of B. Thus from Equation (23), for any $E \in \Gamma(TB)$, we have

$$cos\alpha = \frac{\|TE\|}{\|\psi E\|}. \tag{26}$$

On the other hand, by using (3), we obtain

$$cos\alpha = \frac{\mathcal{G}(\psi E, TE)}{\|TE\|\|\psi E\|} = \frac{\mathcal{G}(E, \psi TE)}{\|TE\|\|\psi E\|} = \frac{\mathcal{G}(E, T^2E)}{\|TE\|\|\psi E\|}. \tag{27}$$

So, (26) and (27) imply

$$cos^2\alpha = \frac{\mathcal{G}(E,T^2E)}{\|\psi E\|^2} = \frac{\mathcal{G}(E,T^2E)}{\mathcal{G}(\psi E,\psi E)}. \tag{28}$$

In account of the Equation (2), by putting $T^2 = \lambda(mT - I)$ in (28), we get

$$cos^2\alpha = \frac{\lambda\mathcal{G}(E, mTE - E)}{\mathcal{G}(mTE - E, E)}. \tag{29}$$

This means $\lambda = cos^2\alpha$, hence α is constant and independent of the choice E. □

Theorem 4.5. *Let $(B, \mathcal{G})$ be a submanifold of an almost poly-Norden manifold $(\overline{B}, g, \psi)$. If B is slant submanifold with slant angle α, then*

$$(\mathfrak{D}_E T^2)F = m\, cos^2\alpha(\mathfrak{D}_E T)F,$$

for any $E, F \in \Gamma(TB)$.

Proof. By using Lemma 4.3 we have:

$$\begin{aligned}(\mathfrak{D}_E T^2)F &= \mathfrak{D}_E T^2 F - T^2 \mathfrak{D}_E F \\ &= m\, cos^2\alpha \mathfrak{D}_E TF - cos^2\alpha \mathfrak{D}_E F - m\, cos^2\alpha\, T\mathfrak{D}_E F + cos^2\alpha \mathfrak{D}_E F \\ &= m\, cos^2\alpha(\mathfrak{D}_E T)F.\end{aligned}$$

□

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