Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

SOME RESULTS ON COMPLEX $(p, q)-$EXTENSION CHEBYSHEV WAVELETS

H. Mazaheri ✉, A.W. Safi, and S.M. Jesmani

Article type: Research Article
(Received: 12 February 2025, Received in revised form 02 May 2025)
(Accepted: 13 May 2025, Published Online: 14 May 2025)

Abstract. In this paper, we propose a generalized formula for well-known functions such as (p, q)-Chebyshev polynomials. Our consideration is focused on determining properties of generalized Chebyshev polynomials of the first and second kind, sparking interest in constructing a theory similar to the classical one. We define complex $(p, q)-$Chebyshev wavelets. We estimate the wavelet approximation of a continuous function $f \in L^2[0, L]$.

Keywords: Complex $(p, q)-$Chebyshev wavelets, $(p, q)-$Chebyshev polynomials, Complex $(p, q)-$Chebyshev wavelets approximation, Uniform wavelet approximation.
2020 MSC: 41A65, 41A52, 46N10.

1. Introduction

In recent years, wavelets have found their way into many different fields of science and engineering, particularly, in signal analysis, time-frequency analysis and fast algorithms. Wavelets allow an accurate representation of several functions. The wavelet approximation technique is a new tool for finding and analyzing unexpected seismic signal processing changes. There is scarcely any area of numerical analysis where Chebyshev polynomials do not drop in like surprise visitors, and indeed, there are now a number of subjects in which these polynomials take a significant position in modern developments, including orthogonal polynomials, polynomial approximation, numerical integration, and spectral methods for partial differential equations.

Chebyshev polynomials appear in many areas of mathematics. In recent years, this interest has often arisen from outside the subject of orthogonal polynomials, after their connection with the class of analytic functions.

✉ hmazaheri@yazd.ac.ir, ORCID: 0000-0001-8169-8655
https://doi.org/10.22103/jmmr.2025.24821.1764
Publisher: Shahid Bahonar University of Kerman
How to cite: H. Mazaheri, A.W. Safi and S.M. Jesmani, *Some results on complex $(p, q)-$extension Chebyshev wavelets*, J. Mahani Math. Res. 2026; 15(1): 55-69.

In the following table we have Chebyshev polynomials(see [11-12]:

$ChebyshevPolynomials$	$Formula$	$Weight\ Function$
$First\ Kind\ Chebyshev$	$T_n(x) = cos\ n\ \theta,\ x = cos\theta$	$\frac{1}{\sqrt{1-x^2}}$
$Second\ Kind\ Chebyshev$	$U_n(x) = \frac{sin\ (n+1)\ \theta}{sin\ \theta},\ x = cos\theta$	$\sqrt{1-x^2}$
$Third\ Kind\ Chebyshev$	$V_n(x) = \frac{cos(n+\frac{1}{2})\theta}{cos(\frac{\theta}{2})},\ x = cos\theta$	$\sqrt{\frac{1+x}{1-x}}$
$Fourth\ Kind\ Chebyshev$	$W_n(x) = \frac{sin(n+\frac{1}{2})\theta}{sin(\frac{\theta}{2})},\ x = cos\theta$	$\sqrt{\frac{1-x}{1+x}}$

2. $(p, q)-$extension Chebyshev polynomials

We have $q-$extension Chebyshev polynomials on $D = \{z :\ |z| < 1\}$ (see [1, 6]).

Definition 2.1. For $x = cos\theta,\ \theta \in [-\pi, \pi],\ n = 0, 1, 2, 3, \dots$ and $q \in (-1, 1]$ we have

$Kinds$	$q-extension\ Chebyshev\ polynomials$
$First-Kind\ q-extension\ Chebyshev$	$T_n(q; e^{i\theta}) = \frac{1}{2}(e^{in\theta} + q^n e^{-in\theta})$
$Second-Kind\ q-extension\ Chebyshev$	$U_n(q; e^{i\theta}) = \frac{e^{i(n+1)\theta} - q^{n+1}e^{-i(n+1)\theta}}{e^{i\theta} - qe^{-i\theta}}$
$Third-Kind\ q-extension\ Chebyshev$	$V_n(q; e^{i\theta}) = U_n(q; e^{i\theta}) - U_{n-1}(q; e^{i\theta})$
$Fourth-Kind\ q-extension\ Chebyshev$	$W_n(q; e^{i\theta}) = U_n(q; e^{i\theta}) - U_{n-1}(q; e^{i\theta}).$

Observe that $T_n(1; e^{i\theta}) = T_n(cos\theta),\ U_n(1; e^{i\theta}) = U_n(cos\theta),\ V_n(1; e^{i\theta}) = V_n(cos\theta)$ and $W_n(1; e^{i\theta}) = W_n(cos\theta)$ are ordinary Chebyshev polynomials. We have $(p, q)-$extension Chebyshev polynomials on $D = \{z :\ |z| < 1\}$,

Definition 2.2. For $x = cos\theta,\ \theta \in [-\pi, \pi],\ n = 0, 1, 2, 3, \dots$ and $p, q \in (-1, 1]$ we have

$Kinds$	$(p,\ q)-extension\ Chebyshev\ polynomials$
$First-Kind\ (p,\ q)-extension$	$T_n(p; q; e^{i\theta}) = \frac{1}{2}(p^n e^{in\theta} + q^n e^{-in\theta})$
$Second-Kind\ (p,\ q)-extension$	$U_n(p; q; e^{i\theta}) = \frac{p^{n+1}e^{i(n+1)\theta} - q^{n+1}e^{-i(n+1)\theta}}{pe^{i\theta} - qe^{-i\theta}}$
$Third-Kind\ (p,\ q)-extension$	$V_n(p; q; e^{i\theta}) = U_n(p; q; e^{i\theta}) - U_{n-1}(p; q; e^{i\theta})$
$Fourth\ Kind\ (p,\ q)-extension$	$W_n(p; q; e^{i\theta}) = U_n(p; q; e^{i\theta}) + U_{n-1}(p; q; e^{i\theta}).$

Observe that $T_n(1; 1; e^{i\theta}) = T_n(cos\theta),\ U_n(1; 1; e^{i\theta}) = U_n(cos\theta),\ V_n(1; 1; e^{i\theta}) = V_n(cos\theta)$ and $W_n(1; 1; e^{i\theta}) = W_n(cos\theta)$ are ordinary Chebyshev polynomials.

Theorem 2.3. *The trigonometric polynomials $T_n(p; q; e^{i\theta}),\ U_n(p; q; e^{i\theta}),\ V_n(p; q; e^{i\theta})$ and $W_n(p; q; e^{i\theta})$ satisfy the three-term recurrence relations for $n \geq 0$*

(i) $T_{n+2}(p; q; e^{i\theta})) = (pe^{i\theta} + qe^{-i\theta})T_{n+1}(q; e^{i\theta}) - pqT_n(q; e^{i\theta}),\ T_0(p; q; e^{i\theta}) = 1$ *and* $T_1(p; q; e^{i\theta}) = \frac{1}{2}(pe^{i\theta} + qe^{-i\theta}),$

(ii) $U_{n+2}(p; q; e^{i\theta})) = (pe^{i\theta} + qe^{-i\theta})U_{n+1}(p; q; e^{i\theta}) - pqU_n(p; q; e^{i\theta}),\ U_0(p; q; e^{i\theta}) = 1$ *and* $U_1(p; q; e^{i\theta}) = pe^{i\theta} + qe^{-i\theta},$

(iii) $V_{n+2}(p; q; e^{i\theta})) = (pe^{i\theta} + qe^{-i\theta})V_{n+1}(p; q; e^{i\theta}) - pqV_n(p; q; e^{i\theta}),\ V_0(p; q; e^{i\theta}) = 1$ *and* $V_1(p; q; e^{i\theta}) = \frac{1}{2}(pe^{i\theta} + qe^{-i\theta}) + p^{\frac{1}{2}}q^{\frac{1}{2}},$

(iv) $W_{n+2}(p;q;e^{i\theta})) = (pe^{i\theta}+qe^{-i\theta})W_{n+1}(p;q;e^{i\theta})-pqW_n(p;q;e^{i\theta})$, $W_0(p;q;e^{i\theta}) = 1$ *and* $W_1(p;q;e^{i\theta}) = pe^{i\theta}+qe^{-i\theta}-p^{\frac{1}{2}}q^{\frac{1}{2}}$.

Proof. Parts (i) and (ii) are proved in [6]. Here is the proof of (iii):

$$\begin{aligned} V_{n+2}(p;q;e^{i\theta})) &= U_{n+2}(p;q;e^{i\theta}) - U_{n+1}(p;q;e^{i\theta}) \\ &= (pe^{i\theta}+qe^{-i\theta})U_{n+1}(p;q;e^{i\theta}) - pqUn(p;q;e^{i\theta}) \\ &- (pe^{i\theta}+qe^{-i\theta})U_n(p;q;e^{i\theta}) + pqUn-1(p;q;e^{i\theta}) \\ &= (pe^{i\theta}+qe^{-i\theta})(U_{n+1}(p;q;e^{i\theta}) - U_n(p;q;e^{i\theta}) \\ &- pq(U_n(p;q;e^{i\theta}) - U_{n-1}(p;q;e^{i\theta})) \\ &= (pe^{i\theta}+qe^{-i\theta})V_{n+1}(p;q;e^{i\theta}) - pqV_n(p;q;e^{i\theta}). \end{aligned}$$

The part (iv) is proved below:

$$\begin{aligned} W_{n+2}(p;q;e^{i\theta})) &= U_{n+2}(p;q;e^{i\theta}) + U_{n+1}(p;q;e^{i\theta}) \\ &= (pe^{i\theta}+qe^{-i\theta})U_{n+1}(p;q;e^{i\theta}) - pqUn(p;q;e^{i\theta}) \\ &+ (pe^{i\theta}+qe^{-i\theta})U_n(p;q;e^{i\theta}) + pqUn-1(p;q;e^{i\theta}) \\ &= (pe^{i\theta}+qe^{-i\theta})(U_{n+1}(p;q;e^{i\theta}) + U_n(p;q;e^{i\theta}) \\ &- pq(U_n(p;q;e^{i\theta}) + U_{n-1}(p;q;e^{i\theta})) \\ &= (pe^{i\theta}+qe^{-i\theta})V_{n+1}(p;q;e^{i\theta}) - pqV_n(p;q;e^{i\theta}). \end{aligned}$$

□

Note that $T_n(1;1;e^{i\theta}) = T_n(cos\theta)$, $U_n(1;1;e^{i\theta}) = T_n(cos\theta)$, $V_n(1;1;e^{i\theta}) = T_n(cos\theta)$ and $W_n(1;1;e^{i\theta}) = W_n(cos\theta)$ are ordinary Chebyshev polynomials.

Theorem 2.4. *The trigonometric polynomials satisfy the following orthogonality relations:*

(i)

$$\int_{-\pi}^{\pi} T_n(p;q;e^{i\theta})\overline{T_m}(p;q,e^{i\theta})d\theta = \begin{cases} 0 & if\ m \neq n \\ \frac{\pi}{2}(p^{2n}+q^{2n}) & if\ m = n \neq 0 \\ 2\pi & if\ m = n = 0, \end{cases} .$$

(ii)

$$\int_{-\pi}^{\pi} U_n(p;q;e^{i\theta})\overline{U_m}(p;q,e^{i\theta})|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta = \begin{cases} 0 & if\ m \neq n \\ \frac{\pi}{2}(p^{2n}+q^{2n}) & if\ m = n \neq 0 \\ 2\pi & if\ m = n = 0, \end{cases} .$$

(iii)

$$\int_{-\pi}^{\pi} V_n(p;q;e^{i\theta})\overline{V_m}(p;q,e^{-i\theta})|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta = \begin{cases} 0 & if\ m \neq n \\ \pi(p^{2n}+q^{2n}) & if\ m = n \neq 0 \\ 4\pi & if\ m = n = 0, \end{cases} .$$

(iv)

$$\int_{-\pi}^{\pi} W_n(p;q;e^{i\theta})\overline{W_m}(p;q,e^{-i\theta})|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta = \begin{cases} 0 & if\ m\neq n \\ \pi(p^{2n}+q^{2n}) & if\ m=n\neq 0 \\ 4\pi & if\ m=n=0, \end{cases}.$$

Proof. The parts (i) and (ii) are proved in [6] and [12].
Part (iii) if $m=n$ or $m=n-1$ is proved below:

$$\begin{aligned} \textstyle\int_{-\pi}^{\pi} & V_n(p;q;e^{i\theta})\overline{V_m}(p;q,e^{-i\theta})|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta \\ = & \int_{-\pi}^{\pi} (U_n(p;q;e^{i\theta}) - U_{n-1}(p;q;e^{i\theta}))(\overline{U_m}(p;q;e^{-i\theta}) - \overline{U_{m-1}}(p;q;e^{-i\theta})|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta \\ = & \int_{-\pi}^{\pi} [U_n(p;q;e^{i\theta})\overline{U_m}(p;q;e^{-i\theta}) - U_n(p;q;e^{i\theta})\overline{U_{m-1}}(p;q;e^{-i\theta}) \\ - & U_{n-1}(p;q;e^{i\theta}))\overline{U_m}(p;q;e^{-i\theta}) + U_{n-1}(p;q;e^{i\theta})\overline{U_{m-1}}(p;q;e^{-i\theta})]|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta \\ = & \pi(p^{2n}+q^{2n}), \end{aligned}$$

If $m\neq n, n-1$

$$\textstyle\int_{-\pi}^{\pi} \quad V_n(p;q;e^{i\theta})\overline{V_m}(p;q,e^{i\theta})|\frac{pe^{i\theta}+qe^{i\theta}}{2i}|^2 d\theta = 0,$$

and

$$\textstyle\int_{-\pi}^{\pi} \quad V_0(p;q;e^{i\theta})\overline{V_0}(p;q,e^{i\theta})|\frac{pe^{i\theta}+qe^{-i\theta}}{2i}|^2 d\theta = 4\pi.$$

part (iv) if $m=n$ or $m=n-1$ is proved below:

$$\begin{aligned} \textstyle\int_{-\pi}^{\pi} & W_n(p;q;e^{i\theta})\overline{W_m}(p;q,e^{i\theta})|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta \\ = & \int_{-\pi}^{\pi} (U_n(p;q;e^{i\theta}) + U_{n-1}(p;q;e^{i\theta}))(\overline{U_m}(p;q;e^{i\theta}) + \overline{U_{m-1}}(p;q;e^{i\theta})|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta \\ = & \int_{-\pi}^{\pi} [U_n(p;q;e^{i\theta})\overline{U_m}(p;q;e^{i\theta}) + U_n(p;q;e^{i\theta})\overline{U_{m-1}}(p;q;e^{i\theta}) \\ + & U_{n-1}(p;q;e^{i\theta}))\overline{U_m}(p;q;e^{i\theta}) + U_{n-1}(p;q;e^{i\theta})\overline{U_{m-1}}(p;q;e^{i\theta})]|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta \\ = & \pi(p^{2n}+q^{2n}), \end{aligned}$$

if $m\neq n, n-1$

$$\textstyle\int_{-\pi}^{\pi} \quad W_n(p;q;e^{i\theta})\overline{W_m}(p;q,e^{i\theta})|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta = 0,$$

and

$$\textstyle\int_{-\pi}^{\pi} \quad W_0(p;q;e^{i\theta})\overline{W_0}(p;q,e^{i\theta}|\frac{pe^{i\theta}-qe^{-i\theta}}{2i}|^2 d\theta = 4\pi.$$

□

In the following corollary, we found L^2- norm of T_n, T_0, U_n, U_0, V_n, V_0 and W_n, W_0

Corollary 2.5. *If* $n \neq 0$, *it is proved below:*

$$\begin{aligned} \| & T_n(p;q;e^{i\theta})\|_2^2 \\ = & \|U_n(p;q;e^{i\theta})\|_2^2 \\ = & \frac{\pi}{2}(p^{2n}+q^{2n}), \end{aligned}$$

$$\begin{aligned} \| & V_n(p;q;e^{i\theta})\|_2^2 \\ = & \|W_n(p;q;e^{i\theta})\|_2^2 \\ = & \pi(p^{2n}+q^{2n}), \end{aligned}$$

and

$$\begin{aligned} \| & T_0(p;q;e^{i\theta})\|_2^2 \\ = & \|U_0(p;q;e^{i\theta})\|_2^2 \end{aligned}$$

$$\begin{aligned} = & \|V_0(p;q;e^{i\theta})\|_2^2 \\ = & \|W_0(p;q;e^{i\theta})\|_2^2 \\ = & 2\pi \end{aligned}$$

3. Complex $(p, q)-$Extension Chebyshev Wavelets

In this section, we consider multiresolution analysis **(MRA)**. The multiresolution analysis of wavelets is an important property in the multilevel approximation of engineering problems. Multiresolution analysis (MRA) can be viewed as a sequence of approximations of a given function $f(t)$ at different resolutions. Multiresolution analysis is an important tool to construct an orthonormal basis of $L^2(\mathbb{R})$. It used this tool to construct a class of compactly supported wavelets with arbitrary regularity and Multiresolution analysis is a general method for constructing wavelet bases.

Definition 3.1. Multiresolution Analysis: An **MRA** with scaling function ϕ is a collection of closed subspaces $\{V_j\}_{j\in Z}$ of $L^2((\mathbb{R})$, such that
(i) $V_j \subset V_{j+1}$;
(ii) $f(x) \in V_j \iff f(2x) \in V_{j+1}$;
(iii) $\overline{\cup V_j} = L^2((\mathbb{R})$,
(iv) $\cap V_j = 0$;
(v) There exists a function $\phi \in V_0$ such that the collection $\{\phi(x–k) :\ k \in \mathbb{Z}\}$ is a Riesz basis of V_0

The sequence of wavelet subspaces W_j of $L^2(\mathbb{R})$, are such that $V_j \perp W_j$, for all j and $V_{j+1} = V_j \bigoplus W_j$. Closure of $\bigoplus W_j$ is dense in $L^2(\mathbb{R})$ for L^2- norm.

Now we state Mallat's theorem, which guarantees that in the presence of an orthogonal **MRA**, an orthonormal basis for $L^2(\mathbb{R})$ exists. These basic functions are fundamental in the wavelet theory, which helps us to develop advanced computational techniques.

Lemma 3.2. [Mallat's Theorem, 9] *Given an orthogonal MRA with scaling function ϕ, there is a wavelet $\psi \in L^2(\mathbb{R})$ such that for each $j \in \mathbb{Z}$, the family $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j. Hence, the family $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

Definition 3.3. If the family $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

(i) Let $P_n(f)$ be the orthogonal projection of $L^2([-1,1])$ onto V_n. Then

$$P_n(f) = \sum_{-\infty}^{\infty} < f, \phi_{n,k} > \phi_{n,k}, \ n = 1, 2, 3, \ldots$$

(ii) The wavelet approximation of these polynomials a defined by

$$E_n(f) = \|f - P_n(f)\|_2^2 = \int_{-1}^{1} |f(t) - P_n(f)(t)|^2 dt = o(\phi(n)),$$

and

$$lim_{n\to\infty} \phi(n) = 0.$$

(iii) Suppose P_n is a set of all polynomials of degree n and smaller on $[a,b]$ in $L^2(a,b]$, if for $f \in L^2[a,b]$, there exists $q* \in P_n$ such that q^* is wavelet approximation and $lim_{n\to\infty} E_n(f) = 0$. Then q^* is called uniform wavelet approximation to f on $[a,b]$.

Definition 3.4 ((a.k.a. vanishing) moments)**.** For a wavelet ψ, the scalar $k \in \mathbb{N}$ is degree of multiresolution, if

$$\int_{\mathbb{R}} x^p \psi(x) dx = 0 \ for \ p = 0, 1, 2, ..., k$$

Definition 3.5. Suppose $k \in \mathbb{N}$ (degree of multiresolution), $m \geq 0$, $n = 1, 2, ..., 2^{k-1}$. We define

i) first kind complex $(p,q)-$extension Chebyshev wavelets on $[0, L]$

$$T_{n,m}(p, q, e^{it}) = \sqrt{\frac{2^{k+1}}{n}} T_m(p; q; e^{i(\frac{2^k}{L}t - 2n+1)}) \boldsymbol{\chi}_{[\frac{(n-1)L}{2^{k-1}}, \frac{nL}{2^{k-1}})}(t).$$

ii) second kind complex $(p,q)-$extension Chebyshev wavelets on $[0, L]$

$$U_{n,m}(p, q, e^{it}) = \sqrt{\frac{2^{k+1}}{n}} U_m(p; q; e^{i(\frac{2^k}{L}t - 2n+1)}) \boldsymbol{\chi}_{[\frac{(n-1)L}{2^{k-1}}, \frac{nL}{2^{k-1}})}(t).$$

iii) third kind complex $(p,q)-$extension Chebyshev wavelets on $[0, L]$

$$V_{n,m}(p, q, e^{it}) = \sqrt{\frac{2^{k+1}}{n}} V_m(p; q; e^{i(\frac{2^k}{L}t - 2n+1))} \boldsymbol{\chi}_{[\frac{(n-1)LL}{2^{k-1}}, \frac{n}{2^{k-1}})}(t).$$

iv) fourth kind complex $(p, q)-$extension Chebyshev wavelets on $[0, L]$

$$W_{n,m}(p, q, e^{it}) = \sqrt{\frac{2^{k+1}}{n}} W_m(p; q; e^{i(\frac{2^k}{L}t-2n+1)}) \boldsymbol{\chi}_{[\frac{(n-1)L}{2^{k-1}}, \frac{nL}{2^{k-1}})}(t).$$

Theorem 3.6. *Let $f \in L^2[0, L]$ be a continuous function and there is $Q > 0$ such that $|f'(t)| \leq Q$.*

(i) If $f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} t_{n,m} T_{n,m}(p, q, e^{it})$, then complex $(p, q)-$extesion Chebyshev wavelet approximation f, for every l, is the partial sum $s_{2^k,l-1}(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{l-1} t_{n,m} T_{n,m}(p, q, e^{it})$, and

$$E_{2^k,l-1}(f) = o(\sum_{n=1}^{2^k} \sum_{m=l}^{\infty} \frac{1}{m^2}),$$

also, $s_{2^k,l-1}$ is a uniform wavelet approximation,

(ii) If $f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} t_{n,m} U_{n,m}(p, q, e^{it})$, then complex $(p, q)-$Chebyshev wavelet approximation f, for every l is the partial sum $s_{2^k,l-1}(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{l-1} t_{n,m} U_{n,m}(p, q, e^{it})$, and

$$E_{2^k,l-1}(f) = o(\sum_{n=1}^{2^k} \sum_{m=l}^{\infty} (\frac{1}{m} + \frac{1}{m+1})^2,$$

also, $s_{2^k,l-1}$, is a uniform wavelet approximation.

Proof. Since f is continuous, there is $P > 0$ such that $|f(t)| \leq P$.
(i) We have

$$\begin{aligned}
|c_{n,m}| &= |< f(t), T_{n,m} >| \\
&= |\int_0^L -f(t)T_{n,m}(p,q,e^{it})dt| \\
&= |\sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} f(t)T_{n,m}(p,q,e^{it})dt| \\
&= |\sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} f(t)T_m(p,q,e^{i(\frac{2^k}{L}t-2n+1)})dt| \\
put\ u &= \frac{2^k}{L}t - 2n + 1 \\
&= 2^{\frac{k}{2}}|\int_0^1 f(\frac{u(p;q:L+2nL-L}{2^k})T_m(p,q,e^{iu})du| \\
&= 2^{\frac{k}{2}-1}|\int_0^1 |f(\frac{uL+2nL-L}{2^k}||(p^m e^{imu} + q^m e^{-imu})|du| \\
&\leq 2^{\frac{k}{2}}|[(f(\frac{cos\theta+2n+1}{2^k})\frac{1}{im}(-p^m e^{imcu} + q^m e^{-imu})_0^1| \\
&+ \frac{1}{2^{\frac{k}{2}}}|\int_0^1 if'(\frac{cos\theta+2n+1}{2^k})\frac{1}{im}(-p^m e^{imu} + q^m e^{-imu})du| \\
&\leq 2^{\frac{k}{2}+1}(\frac{1}{m}(|p|^m + |q|^m) + \frac{1}{2^{\frac{k}{2}}}\frac{2}{m} \\
&= \frac{k_1}{m},
\end{aligned}$$

therefore

$$|c_{n,m}|^2 \leq \frac{k_1^2}{m^2}.$$

and

$$\begin{aligned}
\|T_{n,m}\|_2^2 &= 2^{-\frac{k}{2}}\int_0^L |T_m(\frac{2^k}{L}t-2n+1)|^2dt \\
&= 2^{-\frac{k}{2}}\sum_{n=1}^{2^k}\int_{\frac{(n-1)L}{2^k}}^{\frac{nL}{2^k}} |T_m(\frac{2^k}{L}t-2n-1)|^2dt \\
put\ u &= \frac{2^k}{L}t-2n-1 \\
&= 2^{\frac{k}{2}}\sum_{n=1}^{2^k}\int_0^1 T_m(u)^2du \\
&\leq \frac{\pi}{2}2^{\frac{k}{2}}\sum_{n=1}^{2^k}\|T_m(p;q,e^{it})|_2^2 \\
&= \pi 2^{\frac{k}{2}-1}\sum_{n=1}^{2^k}(|p|^{2n}+|q|^{2n})^2.
\end{aligned}$$

$$\begin{aligned}
E_{2^k,l-1}^2 &= \|f-s_{2^k,l-1}\|_2^2 \\
&= \|\sum_{n=1}^{2^k}\sum_{m=0}^{\infty}c_{n,m}T_{n,m}-\sum_{n=1}^{2^k}\sum_{m=0}^{l-1}c_{n,m}T_{n,m}\|_2^2 \\
&= \|\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}T_{n,m}\|_2^2 \\
&\leq \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}|c_{n,m}|^2\|T_{n,m}\|_2^2 \\
&for\ some\ k\in\mathbb{R} \\
&= \sum_{m=l}^{\infty}\frac{k}{m^2}.
\end{aligned}$$

It follows that

$$E_{2^k-1,l} = o(\sum_{n=1}^{\infty}(\frac{k}{m^2}),$$

and

$$lim_{l\to\infty}\sum_{m=l}^{\infty}\frac{k}{m^2} = 0.$$

(ii) We have

$$\begin{aligned}
|c_{n,m}| &= |< f(t), U_{n,m} >| \\
&= 2^{\frac{k}{2}}|\int_0^L f(t)U_{n,m}(p;q;e^{it})|\frac{pe^{it}-qe^{-it}}{2i}|^2dt| \\
&= |\sum_{n=1}^{2^{k-1}}\int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} f(t)U_{n,m}(p,q,e^{it})|\frac{pe^{it}-qe^{-it}}{2i}|^2dt| \\
&= |\sum_{n=1}^{2^{k-1}}\int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} f(t)U_m(p,q,e^{i(\frac{2^k}{L}t-2n+1)})|\frac{pe^{i(\frac{2^k}{L}t-2n+1}-qe^{i(\frac{2^k}{L}t-2n+1}}{2i}|^2dt| \\
put\ u &= \frac{2^k}{L}t-2n+1 \\
&= 2^{\frac{k}{2}}|\int_0^1 f(\frac{uL+2nL-L}{2^k})U_m(p,q,e^{iu})|\frac{pe^{iu}-qe^{-iu}}{2i}|^2du| \\
&= 2^{\frac{k}{2}}|\int_0^L f(\frac{uL+2nL-L}{2^k})\frac{p^{m+1}e^{i(m+1)u}-q^{m+1}e^{-i(m+1)u}}{pe^{iu}-qe^{-iu}}\frac{(pe^{iu}-qe^{-iu})(pe^{iu}-qe^{iu})}{4}d\theta| \\
&= 2^{\frac{k}{2}-2}|\int_0^L f(\frac{uL+2nL-L}{2^k})(p^{m+1}e^{i(m+1)u}-q^{m+1}e^{-i(m+1)u})(pe^{-iu}-qe^{iu})du| \\
&= 2^{\frac{k}{2}-2}|\int_0^L f(\frac{uL+2nL-L}{2^k})(p^{m+2}e^{imu}-qp^{m+1}e^{i(m+2)u}+q^{m+2}e^{-imu}-p^{m+1}qe^{i(m+2)u})du| \\
&\leq 2^{\frac{k}{2}-1}P(\frac{p^{m+2}}{m}+\frac{qp^{m+1}}{m+2}+\frac{q^{m+2}}{m}+\frac{qp^{m+1}}{m+2}) \\
&- \frac{L^2Q}{2^k}(\frac{p^{m+2}}{m}+\frac{qp^{m+1}}{m+2}+\frac{q^{m+2}}{m}+\frac{qp^{m+1}}{m+2}) \\
&= (2^{\frac{k}{2}-1}P+L^2Q2^{\frac{k}{2}})(\frac{p^{m+2}}{m}+\frac{qp^{m+1}}{m+2}+\frac{q^{m+2}}{m}+\frac{qp^{m+1}}{m+2}) \\
&\leq k(\frac{1}{m}+\frac{1}{m+1}).
\end{aligned}$$

therefore

$$|c_{n,m}|^2 \leq k^2(\frac{1}{m}+\frac{1}{m+1})^2,$$

and

$$\begin{aligned}
\|U_{n,m}\|_2^2 &= 2^{\frac{k}{2}}\int_0^L |U_{n,m}(p;q;e^{it})|^2|\frac{pe^{it}-qe^{-it}}{2i}|^2 dt| \\
&= \sum_{n=1}^{2^{k-1}}\int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} |U_{n,m}(p,q,e^{it})|^2|\frac{pe^{it}-qe^{-it}}{2i}|^2 dt| \\
&= \sum_{n=1}^{2^{k-1}}\int_{\frac{(n-1)L}{2^{k-1}}}^{\frac{nL}{2^{k-1}}} |U_m(p,q,e^{i(\frac{2^k}{L}t-2n+1)})|^2|\frac{pe^{i(\frac{2^k}{L}t-2n+1}-qe^{i(\frac{2^k}{L}t-2n+1}}{2i}|^2 dt \\
put\ u &= \frac{2^k}{L}t-2n+1 \\
&= 2^{\frac{k}{2}}\sum_{n=1}^{2^k}\int_0^1 U_m(u)^2|\frac{pe^{iu}-qe^{-iu}}{2i}|^2 du \\
&\leq \frac{\pi}{2}2^{\frac{k}{2}}\sum_{n=1}^{2^k}\|U_m(p;q,e^{it})\|_2^2 \\
&= \pi 2^{\frac{k}{2}-1}\sum_{n=1}^{2^k}(p^{2n}+q^{2n})^2.
\end{aligned}$$

$$\begin{aligned}
E^2_{2^k,l-1} &= \|f-s_{2^k,l-1}\|_2^2 \\
&= \|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}U_{n,m}-\sum_{n=1}^{2^k}\sum_{m=0}^{l-1}c_{n,m}U_{n,m}\|_2^2 \\
&= \|\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}U_{n,m}\|_2^2 \\
&\leq k(\frac{1}{m}+\frac{1}{m+1})^2.
\end{aligned}$$

It follows that

$$E_{2^k-1,l} = o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(\frac{1}{m}+\frac{1}{m+1})^2,$$

and

$$lim_{l\to\infty}\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(\frac{1}{m}+\frac{1}{m+1})^2 = 0.$$

□

Corollary 3.7. *Let $f \in L^2[0, L]$ be a continuous function and there is $Q > 0$ such that $|f'(t)| \leq Q$.*

(i) If $f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}t_{n,m}V_{n,m}(p,q,e^{it})$*., then complex* $(p,q)-$*extension Chebyshev wavelet approximation* f*, for every* l *is the partial sums* $s_{2^k,l-1}(t)=\sum_{n=1}^{2^k}\sum_{m=0}^{l-1}t_{n,m}V_{n,m}(p,q,e^{it})$*. and*

$$E_{2^k-1,l}=o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m}+\frac{1}{m+1})^2+k_2(\frac{1}{m-1}+\frac{1}{m})^2)),$$

also $s_{2^k,l-1}$ *is uniform wavelet approximation.*

(ii) If $f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}t_{n,m}W_{n,m}(p,q,e^{it})$*, then complex* $(p,q)-$*extension Chebyshev wavelet approximation* f*, for every* l *is the partial sums* $s_{2^k,l-1}(t)=\sum_{n=1}^{2^k}\sum_{m=0}^{l-1}t_{n,m}W_{n,m}(p,q,e^{it})$*, and*

$$E_{2^k-1,l}=o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m}+\frac{1}{m+1})^2+k_2(\frac{1}{m-1}+\frac{1}{m})^2)),$$

also, $s_{2^k,l-1}$ *is uniform wavelet approximation.*

Proof. (i) We have

$$\begin{aligned}
E_{2^k,l-1} &= \|f-s_{2^k,l-1}\|_2\\
&= \|\sum_{n=1}^{2^k}\sum_{m=0}^{\infty}c_{n,m}V_{n,m}-\sum_{n=1}^{2^k}\sum_{m=0}^{l-1}c_{n,m}V_{n,m}\|_2\\
&= \|\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}V_{n,m}\|_2\\
&\le \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}\|V_{n,m}\|_2\\
&\le \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}\|U_{n,m}-U_{n,m-1}\|_2\\
&\le \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}c_{n,m}\|U_{n,m}\|_2+U_{n,m-1}\|_2\\
&\le \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m}+\frac{1}{m+1})^2+k_1(\frac{1}{m-1}+\frac{1}{m})^2).
\end{aligned}$$

It follows that

$$E_{2^k-1,l}=o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m}+\frac{1}{m+1})^2+k_2(\frac{1}{m-1}+\frac{1}{m})^2)).$$

(ii) We have

$$\begin{aligned}
E_{2^k,l-1} &= \|f - s_{2^k,l-1}\|_2 \\
&= \|\sum_{n=1}^{2^k}\sum_{m=0}^{\infty} c_{n,m}W_{n,m} - \sum_{n=1}^{2^k}\sum_{m=0}^{l-1} c_{n,m}W_{n,m}\|_2 \\
&= \|\sum_{n=1}^{2^k}\sum_{m=l}^{\infty} c_{n,m}W_{n,m}\|_2 \\
&\leq \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} c_{n,m}\|W_{n,m}\|_2 \\
&\leq \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} c_{n,m}\|U_{n,m} + U_{n,m-1}\|_2 \\
&\leq \sum_{n=1}^{2^k}\sum_{m=l}^{\infty} c_{n,m}\|U_{n,m}\|_2 + U_{n,m-1}\|_2 \\
&\leq \sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m} + \frac{1}{m+1})^2 + k_1(\frac{1}{m-1} + \frac{1}{m})^2).
\end{aligned}$$

It follows that

$$E_{2^k-1,l} = o(\sum_{n=1}^{2^k}\sum_{m=l}^{\infty}(k_1(\frac{1}{m} + \frac{1}{m+1})^2 + k_2(\frac{1}{m-1} + \frac{1}{m})^2)).$$

□

4. Conclusion

In this paper, we will explore the sequences of polynomials of the generalized Chebyshev polynomial of the second kind $U_n(p, q; e^{i\theta})$ and of the first kind $T_n(p, q; e^{i\theta})$. Each of these sequences is useful in applications for a particular reason. The Chebyshev polynomials of the second kind are defined by the fact its connections with the generalized typically real functions; similarly as was in the classical case.
Also, we define wavelets on complex $(p, q)-$extension, which is useful in Wavelet theory.

In further research we can
1. define complex $(p, q)-$extension $\alpha-$Chebyshev,
2. Solve differential equations for $|x| \geq 1$.

Declarations

Not applicable

Funding: No funding

Availability of data and materials: Data sharing is not applicable to this article.

References

[1] Altınkaya, & S., Yalçın, *The (p, q)-Chebyshev polynomial bounds of a general bi-univalent function class.* Bol. Soc. Mat. Mex. (3) 26 (2020), no. 2, 341–348. DOI:10.1007/s40590-019-00246-2.

[2] Brandi, & Ricci, P. E., *Some properties of the pseudo-Chebyshev polynomials of half-integer degree* Tbilisi Math. J. 12 (2019), no. 4, 111-121. DOI:10.32513/tbilisi/1578020571.

[3] Çakmak, & M. Uslu, K. A., *generalization of Chebyshev polynomials with well-known kinds and transition relations.* Acta Univ. Apulensis Math. Inform. No. 57 (2019), 19-30. Doi: 10.17114/j.aua.2019.57.02.

[4] Dehghan, & M. Eslahchi, M. R., *Best uniform polynomial approximation of some rational functions.* Comput. Math. Appl. 59 (2010), no. 1, 382–390. DOI:10.1016/j.camwa.2009.07.016.

[5] Eslahchi, M. R. & Dehghan, M., *The best uniform polynomial approximation to the class of the form* $\frac{1}{(a^2 \pm x^2)}$. Nonlinear Anal. 71 (2009), no. 3-4, 740-750. DOI:10.1016/j.camwa.2009.07.016.

[6] Kiepiela, K., & Klimek, D., *An extension of the Chebyshev polynomials.* J. Comput. Appl. Math. 178 (2005), no. 1-2, 305-312. https://doi.org/10.1016/j.cam.2004.06.021.

[7] Kumar, L. S., Mishra, S., &, Awasthi, S. K., *Error bounds of a function related to generalized Lipschitz class via the pseudo-Chebyshev wavelet and its applications in the approximation of functions.* Carpathian Math. Publ. 14 (2022), no. 1, 29-48. https://doi.org/10.15330/cmp.14.1.29-48.

[8] Nigam, H. K. Mohapatra, R. N. & Murari, K., *Wavelet approximation of a function using, Chebyshev wavelets.* Thai J. Math. (2020), 197-208. https://doi.org/10.1186/s13660-020-02453-2.

[9] Mallet, Y., de Vel, O. , & Coomans D., *Fundamentals of wavelet transform in Wavelets in Chemistry (Data Handling in Science and Technology).* vol. 22, B.Walczak, Ed. Amsterdam, The Netherlands: Elsevier, 2000, pp. 57-84. https://doi.org/10.1186/s13660-020-02453-2.

[10] Jesmani, S. M., Mazaheri, H., & Shojaeian, S.,*Wavelet approximation with Chebyshev.* Iranian Journal of Numerical Analysis and Optimization. 14 (2024), no. 28, 315-329. doi: 10.22067/IJNAO.2023.84431.1315.

[11] Mason, J. C. & Handscomb, D. C. *Chebyshev polynomials.* Chapman and Hall/CRC, Boca Raton, FL, 2003.

[12] Rivlin, T. J., *An introduction to the approximation of functions.* Corrected reprint of the original. Dover Books on Advanced Mathematics. Dover Publications, Inc., New York, 1981. https://doi.org/10.1201/9781420036114.

[13] Tatarczak, A., *An extension of the Chebyshev polynomials.* Complex Anal. Oper. Theory 10 (2016), no. 7, 1519-1533. https://doi.org/10.1007/s11785-015-0503-6.

H. Mazaheri
Orcid number: 0000-0001-8169-8655
Department of Mathematics Sciences
Yazd University
Yazd, Iran
Email address: hmazaheri@yazd.ac.ir

A.W. Safi
Orcid number: 0009-0009-2254-3667
Department of Mathematics Sciences
Yazd University
Yazd, Iran
Email address: safiabdulwali@gmail.com

S.M. Jesmani
Orcid number: 0000-0002-1491-4159
Department of Materials and Metallurgical Engineering
National University of Skills(NUS)
Tehran, Iran
Email address: smjesmani@nus.ac.ir