Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505
JMMR 2012

ON AUTOCENTRAL KERNEL OF GROUPS

Sh. Bahri, A. Kaheni ✉, and M.M. Nasrabadi

Article type: Research Article
(Received: 02 January 2025, Received in revised form 06 May 2025)
(Accepted: 18 May 2025, Published Online: 21 May 2025)

Abstract. Let G be a group, where $\mathrm{Aut}(G)$ denotes the full automorphisms group of G and $L(G)$ represents the absolute center of G. An automorphism $\alpha \in \mathrm{Aut}(G)$ is called an autocentral automorphism if $g^{-1}\alpha(g) \in L(G)$, for all $g \in G$. In this paper, we introduce the notion of autocentral kernel subgroup for an arbitrary group G. We then investigate and establish several structural properties of this subgroup, providing insights into its role within the broader framework of group theory.

Keywords: Absolute central automorphisms, Absolute centre, Autocentral kernel, Autonilpotent group.
2020 MSC: 20D45, 20D25.

1. Introduction and preliminaries

Let G be a group. We denote the terms of the lower and upper central series of G by $\gamma_i(G)$ and $Z_i(G)$, respectively. The group of all automorphisms of G is denoted by $\mathrm{Aut}(G)$, while $\mathrm{Inn}(G)$ represents the group of inner automorhisms of G. For a group H and an abelian group K, we denote the group of homomorphisms from H to K by $Hom(H, K)$. We use C_m to represent the cyclic group of order m. The Frattini subgroup of G is denoted by $\Phi(G)$. Throughout this paper, p always denotes a prime number. Finally, we will use standard notation and terminology as found in [11].

For each $g \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[g, \alpha] = g^{-1}\alpha(g)$ is called the autocommutator of g and α. The autocommutator of higher weight is defined inductively as follows:

$$[g, \alpha_1, \alpha_2, ..., \alpha_n] = [[g, \alpha_1, \alpha_2, ..., \alpha_{n-1}], \alpha_n],$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G)$ and $g \in G$.

Hegarty [5] introduced the absolute center of a group G denoted by

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \ \forall \alpha \in \mathrm{Aut}(G)\}.$$

Thus, the absolute center of G consists of all elements that remain fixed under every automorphism of G, making it a characteristic subgroup of G. Since $\mathrm{Inn}(G)$ acts trivially on $Z(G)$, it follows that $L(G) \leq Z(G)$. Furthermore, if $\mathrm{Aut}(G) = \mathrm{Inn}(G)$, then $L(G) = Z(G)$. In 2015, H. Meng and X. Guo [4]

✉ azamkaheni@birjand.ac.ir, ORCID: 0000-0002-5117-1358
https://doi.org/10.22103/jmmr.2025.24620.1745
Publisher: Shahid Bahonar University of Kerman
How to cite: Sh. Bahri, A. Kaheni, M.M. Nasrabadi, *On autocentral kernel of groups*, J. Mahani Math. Res. 2025; 14(2): 343-352.

characterized the structure of finite groups G where $L(G)$ is contained in the Frattini subgroup $\Phi(G)$. Later, Orfi and Fouladi [9] proved that if G is a p-group of maximal class of order p^n and $\mathrm{Aut}(G)$ is also a p-group, then $L(G) = Z(G)$. For further details, see also [5] and [10].

Now, define $L_0(G) = \langle 1 \rangle$, $L_1(G) = L(G)$, and for $n \geq 2$, the n-th absolute center of G inductively as

$$L_n(G) = \{g \in G \mid [g, \alpha_1, ..., \alpha_n] = 1 \ , \ \forall \alpha_1, ..., \alpha_n \in \mathrm{Aut}(G)\}.$$

Clearly, $L_n(G)$ is a characteristic subgroup of G and it satisfies $L_n(G) \leq Z_n(G)$, where $Z_n(G)$ is the n-th center of G. Using this notation, we obtain the following series for G :

$$L_0(G) = \langle 1 \rangle \trianglelefteq L_1(G) = L(G) \trianglelefteq L_2(G) \trianglelefteq \cdots \trianglelefteq L_n(G) \trianglelefteq \dots .$$

A group G is called autonilpotent (See [6]) if there exists some n such that $G = L_n(G)$. Clearly, autonilpotent groups are nilpotent, but the converse does not necessarily hold. For instance, C_3 is not autonilpotent because $L(C_3) = 1$. Similarly, it is well known that the symmetric group S_3 is not nilpotent and since $L(S_3) = 1$, it follows that S_3 is not autonilpotent.

Hegarty [5] introduced the notion of absolute central automorphisms. An automorphism α of G is called an absolute central automorphism if $[g, \alpha] \in L(G)$, for all $g \in G$. The set of all absolute central automorphisms, denoted by $\mathrm{Aut}_L(G)$, forms a normal subgroup of $\mathrm{Aut}(G)$. In 2016, Nasrabadi and Farimani [8, Main Theorem] proved that if G is a finite autonilpotent p-group of class 2, then $\mathrm{Aut}_L(G) = \mathrm{Inn}(G)$ if and only if $L(G) = Z(G)$ and $Z(G)$ is cyclic.

Moghaddam and Safa [7] further examined the properties of $\mathrm{Aut}_L(G)$ and demonstrated that it fixes certain elements of G. They introduced the subgroup $E(G)$, defined as follows:

$$E(G) = \langle [g, \alpha] \mid g \in G \ , \alpha \in C_{\mathrm{Aut}(G)}(\mathrm{Aut}_L(G)) \rangle,$$

in which

$$C_{\mathrm{Aut}(G)}(\mathrm{Aut}_L(G)) = \{\alpha \in \mathrm{Aut}(G) \mid \alpha\beta = \beta\alpha \ , \forall \beta \in \mathrm{Aut}_L(G)\}.$$

The subgroup $E(G)$ is characteristic in G and contains the derived subgroup of G.

Lemma 1.1. *([7, Lemma 1]) If G is an arbitrary group, then $Aut_L(G)$ acts trivially on the subgroup $E(G)$ of G.*

Example 1.2. *Let G be a group with the following presentation*

$$G = \langle x, y \mid x^{2^n - 1} = 1 \ , \ x^y = x^2 \rangle \ , \ \textit{where } n \geq 2.$$

Then $Aut_L(G) = 1$, and $E(G) = \langle x \rangle$. (See Example 1 of [7]).

An automorphism α of G is called a central automorphism if $[g, \alpha] \in Z(G)$, for all $g \in G$. The set of all central automorphisms of G is denoted by $\mathrm{Aut}_c(G)$, and this set forms a normal subgroup of the automorphism group $\mathrm{Aut}(G)$. Moreover, $\mathrm{Aut}_c(G)$ contains the set $\mathrm{Aut}_L(G)$.

In 1955, Haimo [3] introduced the concept of the central kernel of a given group. The central kernel of G consists of all elements of G that are fixed by every central automorphism. Davoudirad *et al.* [2] denoted this subgroup by $L_c(G)$, and proved that $L_c(G)$ is a charachteristic subgroup of G, which contains both $L(G)$ and $G^{'}$.

The following result is well known:

Lemma 1.3. *Let U, V and W be finite abelian groups. Then*

1. $Hom(U \times V, W) = Hom(U, W) \times Hom(V, W)$.
2. $Hom(U, V \times W) = Hom(U, V) \times Hom(U, W)$.
3. *If* $(|U|, |V|) \neq 1$, *then* $Hom(U, V) \neq 1$.
4. $Hom(C_m, C_n) = C_s$, *where s is the greatest common divisor of m and n.*

In this paper, we introduce the notion of autocentral kernel for any group, which is a set of elements containing the derived subgroup, the subgroup $E(G)$, and the central kernel $L_c(G)$. We then establish some of its fundamental properties.

2. Autocentral kernel of a group

The previous section established that every autocentral automorphism acts trivially on the derived subgroup and certain subgroups detailed in [7]. However, autocentral automorphisms may also fix elements outside the derived subgroup. Therefore, we define the set of all elements fixed by every autocentral automorphism. This set provides a more comprehensive understanding of the action of these automorphisms beyond their known behavior on the subgroups mentioned in [7].

Definition 2.1. Let G be a group. The set of all elements that remain fixed under every absolute central automorphism of G is called the autocentral kernel of G, denoted by $L_L(G)$.

$$L_L(G) = \{g \in G \mid [g, \alpha] = 1 \ , \ \forall \alpha \in \mathrm{Aut}_L(G)\}.$$

It is straightforward to verify that the autocentral kernel $L_L(G)$ is a characteristic subgroup of G and contains both $L_c(G)$ and $E(G)$.

Since autocentral automorphisms fix every element of $G^{'}$, we conclude that $G^{'} \leq L_L(G)$. Consequently, the quotient group $G/L_L(G)$ is abelian.

Example 2.2. *It can be easily verified that:* $L_L(C_6) = C_6$, $L_L(C_{12}) \cong C_6$, $L_L(C_{16}) \cong C_8$. *Additionally, for the dihedral group* $D_8 = \langle a, b | a^4 = b^2 = 1, bab = a^{-1} \rangle$, *we have* $L_L(D_8) = \{1, a^2\}$.

The reason of calling $L_L(G)$ as the autocentral kernel becomes evident in the following result, which follows directly from Definition 2.1.

Lemma 2.3. *Let G be a group. Then* $L_L(G) = \cap_{\alpha \in Aut_L(G)} ker\alpha$.

In the following, we will identify the conditions under which the autocenteral kernel of a group coincides with the group itself, i.e., when $L_L(G) = G$.

Lemma 2.4. *Let G be a group in which $L(G)$ or $Aut_L(G)$ is trivial. Then $L_L(G) = G$.*

Proof. By Definition 2.1, it follows directly that if either $L(G)$ or $\mathrm{Aut}_L(G)$ is trivial, then the autocentral kernel $L_L(G)$ must equal the entire group G. This result can be easily verified by applying the definition of the autocentral kernel. □

The next results provide the exact structure of autocentral kernel for each cyclic p-group.

Theorem 2.5. *Let G be a cyclic group of order p^n, where p is an odd prime number. Then $L_L(G) = G$.*

Proof. Let $G = \langle a : a^{p^n} = 1\rangle$. Consider the automorphism $\phi : a \longmapsto a^2$. Assume there exists some $i < p^n$ such that $\phi(a^i) = a^i$. Then applying ϕ to a^i, we get $a^{2i} = a^i$. This implies $a^i = 1$, which is a contradiction unless $i = 0$. Thus, there is no non-trivial element in G that is fixed by ϕ. Therefore, $L(G) = 1$ and by Lemma 2.4, we conclude that $L_L(G) = G$. □

Theorem 2.6. *Let G be a finite cyclic group of order 2^n, where $n \geq 2$. Then $L_L(G)$ is a cyclic group of order 2^{n-1}.*

Proof. Let $G = \langle x : x^{2^n} = 1\rangle$, where $n \geq 2$. Each automorphism of G can be expressed as $\alpha : x \longmapsto x^r$, where r is an odd number, and $1 \leq r \leq 2^n - 1$. We aim to show that $L_L(G) =< x^2 >$. Assume that $r = 2t + 1$, for some integer t. Consider the following calculation:

$$r2^{n-1} = (2t + 1)(2^{n-1}) = 2^n t + 2^{n-1} \equiv 2^{n-1} \ (mod\ 2^n).$$

We conclude that $2^n | 2^{n-1}(r - 1)$. This implies $\alpha(x^{2^{n-1}}) = (x^{2^{n-1}})^r = x^{2^{n-1}}$, showing that $x^{2^{n-1}} \in L(G)$. Now, let $x^s \in L(G)$ for some $1 \leq s < 2^n$. Then, $\alpha(x^s) = x^s$ or $x^{rs} = x^s$. Then $2^n | s(r - 1)$, and $s = 2^{n-1}$. This gives us $L(G) = \{1, x^{2^{n-1}}\}$. Now, using [7, Proposition 1], we find the automorphism group $\mathrm{Aut}_L(G)$ is isomorphic to

$$\mathrm{Aut}_L(G) \cong Hom(\frac{C_{2^n}}{C_2}, C_2) \cong Hom(C_{2^{n-1}}, C_2) \cong C_2.$$

Thus, $\mathrm{Aut}_L(G) = \{1, \sigma\}$, where $\sigma : x \longmapsto x^{2^{n-1}+1}$. Hence, $L_L(G) = \langle x^2\rangle \cong C_{2^{n-1}}$. □

It is known from Proposition 1.6 in [10] that $L(G) \neq 1$ when G is a non trivial autonilpotent group. The next result provides a similar statement for the autocentral kernel of G.

Lemma 2.7. *If G is an autonilpotent group of class 2, then $L_L(G) = L(G)$.*

Proof. Let α be an arbitrary element of $\mathrm{Aut}(G)$. Since G is an autonilpotent group of class 2, we know that $L_2(G) = G$. This implies that for every $\beta \in \mathrm{Aut}(G)$, we have the commutator identity $[g, \alpha, \beta] = 1$. In particular, this means that $[g, \alpha] \in L(G)$, where g is any element of G. Therefore, $\alpha \in \mathrm{Aut}_L(G)$, and since α was chosen arbitrarily, we conclude that $\mathrm{Aut}(G) = \mathrm{Aut}_L(G)$, as required. □

Lemma 2.8. *Let G be a group such that $G/L(G)$ is an abelian group. Then $L_L(G) \leq Z(G)$.*

Proof. Let $G/L(G)$ be an abelian group. Therefore, $G^{'} \leq L(G)$, which implies that $\mathrm{Inn}(G) \leq \mathrm{Aut}_L(G)$. Now, let g be an arbitrary element of $L_L(G)$. Since $\mathrm{Inn}(G)$ acts trivially on g, we conclude that $L_L(G) \leq Z(G)$. □

An immediate consequence of Lemma 2.8 is the following result:

Corollary 2.9. *If G is a finite group in which $G^{'} \leq L(G) = Z(G)$, then $L_L(G) = Z(G)$.*

Theorem 2.10. *Let G be a finite p-group such that $L(G) = \Phi(G)$. Then $L_L(G) \leq Z(G)$.*

Proof. Since G is a p-group, it follows that $\Phi(G) = G^{'}G^p$. Therefore, we have $L(G) = G^{'}G^p$, which implies that $G^{'} \leq L(G)$. Then, the result is followed by applying Lemma 2.8. □

Lemma 2.11. *([1, Corollary 2.2]) If G is a finite group in which $G^{'} \leq L(G)$ and also $Z(G) = L(G)$ is cyclic, then $Aut_L(G) = Inn(G)$.*

Recall that, a p-group G is called extra special if $G^{'} = Z(G) = \Phi(G)$ and $Z(G)$ has order p. For a finite extra special p-group, the following result can be derived by using Corollary 3.5 from [12].

Corollary 2.12. *Let G be a finite extra special p-group. Then*

(i) *If $p > 2$, then $L_L(G) = G$.*
(ii) *If $p = 2$, then $L_L(G) = Z(G)$.*

Example 2.13. *Let $G = \langle x_1, x_2, x \mid [x_1, x_2] = x^s \ (s \neq 1)\ , [x_i, x] = 1,\ i = 1, 2\rangle$. Then $L_L(G) = Z(G) = \langle x\rangle$.*

Using Lemma 2.8 and Theorem 3.2 from [8] one obtains the next result.

Corollary 2.14. *Let G be a non-abelian finite p-group. If G is autonilpotent of class 2 and $L(G) = Z(G)$ is cyclic, then $L_L(G) = Z(G)$.*

3. Some results for absolute central automorphism

In this section, we delve into the properties of absolute central automorphisms, leveraging the concept of the autocentral kernel of a group. We begin by establishing a property of autocentral automorphisms analogous to [7, Proposition 1], originally demonstrated by Moghaddam and Safa. It is important to note that while the quotient group $G/L(G)$ is not necessarily abelian, the quotient group $G/L_L(G)$ is always abelian. This distinction arises from the fact that $L_L(G)$ is defined as the autocentral kernel, which enforces a stronger commutativity condition. Furthermore, when G is finite, the order of $G/L_L(G)$ is strictly less than that of $G/L(G)$, reflecting the refined structure captured by the autocentral kernel. This inequality highlights that $L_L(G)$ is a larger subgroup of G than $L(G)$, thus providing a more precise measure of centrality within the group. The fact that $G/L_L(G)$ is abelian, even when $G/L(G)$ is not, provides an important tool for analyzing the structure of non-abelian groups. We will explore the implications of these properties in the subsequent results.

Theorem 3.1. *Let G be a group. Then $Aut_L(G) \cong Hom(\dfrac{G}{L_L(G)}, L(G))$.*

Proof. Let $g \in G$ and $\alpha \in \mathrm{Aut}_L(G)$. By Definition 2.1, we have $[g, \alpha] \in L(G)$ so the map $f_\alpha : \dfrac{G}{L_L(G)} \longrightarrow L(G)$ given by $f_\alpha(gL_L(G)) = [g, \alpha]$ is a well-defined homomorphism. To see this, consider $g_1, g_2 \in G$. If $g_1L_L(G) = g_2L_L(G)$, then $g_2^{-1}g_1 \in L_L(G)$. Therefore, there exists $l \in L_L(G)$ such that $g_2^{-1}g_1 = l$, and so using Definition 2.1

$$\begin{aligned} f_\alpha(g_1L_L(G)) = [g_1, \alpha] = [g_2l, \alpha] = (g_2l)^{-1}\alpha(g_2l) &= l^{-1}g_2^{-1}\alpha(g_2)\alpha(l) \\ &= g_2^{-1}\alpha(g_2)l^{-1}\alpha(l) \\ &= g_2^{-1}\alpha(g_2) \\ &= [g_2, \alpha] = f_\alpha(g_2L_L(G)) \end{aligned}$$

Now let g_1, $g_2 \in G$, then

$$\begin{aligned} f_\alpha(g_1g_2L_L(G)) = [g_1g_2, \alpha] &= (g_1g_2)^{-1}\alpha(g_1g_2) \\ &= g_2^{-1}g_1^{-1}\alpha(g_1)\alpha(g_2) \\ &= g_1^{-1}\alpha(g_1)g_2^{-1}\alpha(g_2) \\ &= f_\alpha(g_1L_L(G))f_\alpha(g_2L_L(G)). \end{aligned}$$

Now, consider the map

$$\begin{aligned} \psi : \mathrm{Aut}_L(G) &\longrightarrow Hom(\frac{G}{L_L(G)}, L(G)) \\ \alpha &\longmapsto f_\alpha \end{aligned}$$

such that $f_\alpha : \dfrac{G}{L_L(G)} \longrightarrow L(G)$ given by $f_\alpha(gL_L(G)) = [g, \alpha]$, for all $g \in G$. We show that ψ is an isomorphism. Clearly, the map ψ is a well-defined monomorphism. Therefore, it remains to verify that ψ is onto. For an arbitrary element h of $Hom(\frac{G}{L_L(G)}, L(G))$, consider the map $\beta : G \longrightarrow G$, defined by

$\beta(g) = gh(gL_L(G))$, for all $g \in G$. We will show that $\beta \in \mathrm{Aut}_L(G)$, and $\psi(\beta) = h$. Clearly, β is well-defined homomorphism. To see that β is injective, let $k \in ker\beta$, then $\beta(k) = 1$, which implies that $kh(kL_L(G)) = 1$. Since $h(kL_L(G)) \in L(G)$,we conclude that, $k \in L(G) \leq L_L(G)$. Therefore, we have $1 = \beta(k) = kh(kL_L(G)) = k$. Thus, $k = 1$, as required. To prove that β is onto, first note that, $Imh \leq Im\beta$. Now, for each $g \in G$,

$$g^{-1}\beta(g) = g^{-1}gh(gL_L(G)) = h(gL_L(G)) \in Imh \leq Im\beta.$$

Therefore, $G = Im\beta$. Clearly, β is an autocentral automorphism of G and $\psi(\beta) = h$, as desirable. □

Example 3.2. *Let $G \cong C_{12}$. It is straightforward to verify that $L(G) \cong C_2$, $Aut_L(G) \cong C_2$, and $L_L(G) \cong C_6$. Therefore, we conclude that $Aut_L(G) \cong Hom(\dfrac{G}{L_L(G)}, L(G))$.*

It is evident that $\mathrm{Inn}(G) \subseteq \mathrm{Aut}_L(G)$, when $G/L(G)$ is abelian. In the following, using Theorem 3.1 we will demonstrate that $\mathrm{Inn}(D_8) \cong \mathrm{Aut}_L(D_8)$.

Example 3.3. *Let $G \cong D_8$. It is easy to verify that $L_L(G) = L(G) = Z(G) \cong C_2$. Therefore $G/Z(G) \cong C_2 \times C_2$. Since $G/Z(G) \cong Inn(G)$ is abelian, we conclude that $Inn(G) \subseteq Aut_L(G)$. Using Theorem 3.1, it follows that $|Aut_L(G)| = 4$ and hence $Inn(G) \cong Aut_L(G)$.*

Some conditions under which the autocentral automorphism group is trivial are stated below:

Theorem 3.4. *Assume that G is a group in which $L(G)$ is torsion-free. Then $Aut_L(G)$ is trivial if $G/L_L(G)$ is torsion.*

Proof. Let $g \in G$. It is sufficient to show that $\alpha(g) = g$, for each $\alpha \in \mathrm{Aut}_L(G)$. Since $G/L_L(G)$ is torsion, there exists some $n \in \mathbb{N}$ such that $g^n \in L_L(G)$. Now, consider the action of α on g :

$$\alpha(g)^n = \alpha(g^n) = g^n$$

Note that $g^{-1}\alpha(g) \in L(G) \subseteq Z(G)$. Therefore $g^{-1}\alpha(g)$ commutes with $\alpha(g)$, which implies that $(g^{-1}\alpha(g))^n = 1$. On the other hand, since $g^{-1}\alpha(g) \in L(G)$ and $L(G)$ is torsion-free, we must have $g^{-1}\alpha(g) = 1$. This completes the proof. □

Theorem 3.5. *Let G be a finite group. If $(|G|, |\dfrac{G}{L_L(G)}|) = 1$, then $Aut_L(G) = 1$.*

Proof. By the way of contradiction, assume that $\mathrm{Aut}_L(G) \neq 1$. This implies that there exists a homomorphism $f \in Hom(\frac{G}{L_L(G)}, L(G))$ such that $Imf \neq 1$, using Theorem 3.1. Thus, $(|\frac{G}{L_L(G)}|, |L(G)|) = (|kerf||Imf|, r|Imf|) \neq 1$, which is a contradiction as required. □

Example 3.6. *Let $G \cong C_{10}$. It is not difficult to verify that $L(G) \cong C_2$, $Aut_L(G) \cong 1$, and $L_L(G) = G$. Thus C_{10} satisfies the conditions in Theorem 3.5.*

Theorem 3.7. *Let G be a finite p-group such that $exp(\dfrac{G}{L_L(G)})|exp(L(G))$. Then*

$$Hom(\frac{G}{L_L(G)}, L(G)) \cong \frac{G}{L_L(G)}$$

if and only if $L(G)$ is cyclic.

Proof. Let G be a finite p-group such that $exp(\frac{G}{L_L(G)})$ divides $exp(L(G))$. If $L(G)$ is cyclic, then $exp(L(G)) = |L(G)|$. Since $G/L_L(G)$ and $L(G)$ are both abelian groups, the result follows from parts (1) and (4) of Lemma 1.3. Conversely, let $Hom(\frac{G}{L_L(G)}, L(G)) \cong \frac{G}{L_L(G)}$. We proceed by contradiction. Suppose that $L(G)$ is not cyclic and let us assume that $L(G) \cong C_{p^i} \times B$, where $exp(L(G)) = p^i$ and B is a non- trivial abelian group. By Lemma 1.3 (2), we can decompose the homomorphism group as follows:

$$Hom(\tfrac{G}{L_L(G)}, L(G)) \cong Hom(\tfrac{G}{L_L(G)}, C_{p^i}) \times Hom(\tfrac{G}{L_L(G)}, B).$$

By our assumption, we have $\frac{G}{L_L(G)} \cong \frac{G}{L_L(G)} \times Hom(\frac{G}{L_L(G)}, B)$. It follows that the homomorphism group $Hom(\frac{G}{L_L(G)}, B)$ is trivial. This leads to a contradiction because B is assumed to be non-trivial. Therefore, $L(G)$ is cyclic and the proof is complete. □

Corollary 3.8. *Let G be a finite p-group such that $exp(\dfrac{G}{L(G)})|exp(L(G))$. Suppose that $L(G)$ is cyclic and $G/L(G)$ is an abelian group. Then $L_L(G) = L(G)$.*

Proof. Let G be a finite p-group that $G/L(G)$ is abelian and $L(G)$ is cyclic. Suppose that, $exp(\frac{G}{L(G)})|exp(L(G))$. By invoking parts (1) and (4) of Lemma 1.3, we have $Hom(\frac{G}{L(G)}, L(G)) \cong \frac{G}{L(G)}$. Since $L(G) \leq L_L(G)$, it follows that $exp(\frac{G}{L_L(G)})|exp(L(G))$. Using Theorem 3.7, we then obtain $Hom(\frac{G}{L_L(G)}, L(G)) \cong \frac{G}{L_L(G)}$. Thus, $\frac{G}{L(G)} \cong \frac{G}{L_L(G)}$ and by Theorem 3.1 and Proposition 1 from [7] we conclude that $L(G) = L_L(G)$. □

Let $C_{\mathrm{Aut}_L(G)}(Z(G))$ denote the set of all absolute central automorphisms of G that fix $Z(G)$, elementwise.

In the following, we provide a structural property of $C_{\mathrm{Aut}_L(G)}(Z(G))$.

Theorem 3.9. *Let G be a group. Then*

$$C_{Aut_L(G)}(Z(G)) \cong Hom(\frac{G}{L_L(G)Z(G)}, L(G)).$$

Proof. For each $\alpha \in C_{\mathrm{Aut}_L(G)}(Z(G))$, consider the map f_α :

$$f_\alpha : \quad \frac{G}{L_L(G)Z(G)} \longrightarrow L(G)$$
$$gL_L(G)Z(G) \longmapsto [g, \alpha]$$

Using the method from the proof of Theorem 3.1, it is not difficult to show that f_α is a homomorphism. Furthermore, the map $\Psi : C_{\mathrm{Aut}_L(G)}(Z(G)) \longrightarrow Hom(\frac{G}{L_L(G)Z(G)}, L(G))$ defined by $\alpha \longmapsto f_\alpha$, is an isomorphism as required.

□

By applying Lemma 2.8 and Theorem 3.9, one can derive the following result.

Corollary 3.10. *Let G be a group such that $G/L(G)$ is an abelian group. Then*

$$C_{Aut_L(G)}(Z(G)) \cong Hom(\frac{G}{Z(G)}, L(G)).$$

References

[1] Arora, H. (2019). Various bounds of group of Autocentral Automorphisms. MathLAB Journal, 3, 1-12.
http://purkh.com/index.php/mathlab

[2] Davoudirad, S., Moghaddam, M. R. R., & Rostamyari, M. A. (2016). Some properties of central kernel and central autocommutator subgroups. Journal of Algebra and Applications, 15 (7), 1-7.
https://doi.org/10.1142/S0219498816501280

[3] Haimo, F. (1955). Normal automorphisms and their fixed points. Trans. Amer. Math. Soc. 78 (1), 150-167.
https://doi.org/10.1090/s0002-9947-1955-0067894-0

[4] Hangyang, M., & Guo, X. (2015). The absolute center of finite groups. Journal of Group Theory 18 (8), 887–904.
https://doi.org/10.1515/jgth-2015-0015

[5] Hegarty, P. (1994). The absolute centre of a group. J. Algebra 169, 929 – 935.
https://doi.org/10.1006/jabr.1994.1318

[6] Moghaddam, M.R.R. (2009). Some properties of autocommutator groups. First Two Days Group Theory Seminar, University of Isfahan, Iran, 12-13, March .
https://profdoc.um.ac.ir/articles/a/1010256.pdf

[7] Moghaddam, M. R. R., & Safa, H. (2010). Some properties of autocentral automorphisms of a group. Ric. Mat, 59, 257–264.
https://doi.org/10.1007/s11587-010-0085-6

[8] Nasrabadi, M.M., & Kaboutari Farimani, Z. (2015). Absolute central automorphisms that are inner. Indagationes Mathematicae, 26, 137–141.
https://doi.org/10.1016/j.indag.2014.08.003

[9] Orfi, R., & Fouladi, S. (2020). The Absolute Center of p-Groups of Maximal Class. Filomat 34(13), 4483–4487.
https://doi.org/10.2298/FIL2013483O

[10] Parvaneh, F. (2015). Some properties of autonilpotent groups. Italian Jornal of Pure and Applied Mathematics, 35 ,1-8.

[11] Robinson, D. J. S. (1996). A Course in the Theory of Groups. 2nd ed. Springer-verlag;New York, Heidelberg, Berlin.
https://link.springer.com/book/10.1007/978-1-4419-8594-1

[12] Soleimani, R. (2022). A Note on Absolute Central Automorphisms of Finite p-Groups. Iranian Journal of Mathematical Sciences and Informatics,17(2), 97-108. https://doi.org/10.52547/ijmsi.17.2.97

Shafigh Bahri
Orcid number: 0009-0009-7775-8892
Department of Mathematics
University of birjand
Birjand, Iran
Email address: shafighbahri@birjand.ac.ir

Azam Kaheni
Orcid number: 0000-0002-5117-1358
Department of Mathematics
University of birjand
Birjand, Iran
Email address: azamkaheni@birjand.ac.ir

Mohammad Mehdi Nasrabadi
Orcid number: 0000-0002-0808-1904
Department of Mathematics
University of birjand
Birjand, Iran
Email address: mnasrabadi@birjand.ac.ir