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FUZZY-BASED FUNCTIONAL CAPABILITY INDICES FOR SIMPLE LINEAR PROFILE

Z. Abbasi Ganji

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Abstract. The quality of some processes is defined by a simple linear function between a response and an explanatory variable, which is named a simple linear profile. Various capability indices have been introduced in the literature for these processes. Some of them are functional, that use all domain of the explanatory variable instead of its levels values. Therefore, these methods may fail to account capability of the process and overstate it. This paper applies fuzzy logic to deal with the explanatory variable domain and introduces two capability indices. Then, investigates and compares the performance of the proposed indices and some existing ones, using a simulation study. Results show that one of the new proposed indices has the best performance based on the mean square error and mean absolute error criteria.

Keywords: Simple linear profile, Functional capability indices, Fuzzy logic, Simulation study.
2020 MSC: 62A86.

1. Introduction

In some processes, the quality of the product is characterized by a simple linear relationship between the response variable and an explanatory/independent variable, such a simple linear regression model. This kind of model is called a simple linear profile.

In the literature, numerous studies have been conducted on process capability indices for simple linear profiles. Shahriari and Sarrafian [19] proposed a basic method to measure the capability of simple linear profiles. Hosseinifard and Abbasi [9] used the proportion of the non-conformance criterion to estimate the process capability index in a simple linear profile and investigated it for both fixed and random explanatory variable with constant and functional specification limits. In another work, Hosseinifard and Abbasi [10] considered non normal linear profiles and investigated and compared five methods to estimate the process capability index.

Ebadi and Shahriari [8] proposed two methods for measuring the process capability. The first one uses the percentage of nonconforming parts produced

✉ abbasiganji@mail.um.ac.ir, ORCID: 0000-0003-1939-0080
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at each level of the independent variable and the second one is a multivariate process capability approach with three components based on the vector of the predicted responses. Wang [21] provided an exact measure of the process yield for a simple linear profile and obtained its lower confidence bound. In another research [22], he developed two new indices for measuring the process yield for simple linear profiles with one-sided specification, and obtained the asymptotic distribution of the estimated index and also, the approximate lower confidence bound for the true process yield. Wu [23] employed two-dimensional predictions of the slope and the intercept, and applied a multivariate process capability index based on a vector of three components to assess the process capability in a simple linear profile.

Ebadi and Amiri [7] considered several correlated quality characteristics and proposed three methods for measuring process capability in multivariate simple linear profiles. Abbasi Ganji and Sadeghpour Gildeh [4] introduced a new capability index for simple linear profile.

Nemati Keshteli et al. [15] proposed a functional approach to measure the process capability of simple linear profile in all ranges of explanatory variable. This approach uses a reference profile, functional specification limits and functional natural tolerance limits to present a functional form of process capability indices. They used a non-conforming proportion method to make a comparison study and show the better performance of their proposed indices. Because using the proposed approach, all information in the entire range of the explanatory variable is utilized. Pakzad et al. [16] developed traditional loss-based indices using a functional approach and introduced two capability indices and by a simulation study, showed that the proposed indices perform better than the other one. Some recent studies in this field are [5], [12], [17], [24].

All available functional indices use the entire domain of the explanatory variable, which may fail to account the capability appropriately and overestimate it. To overcome this problem, in the present paper, fuzzy logic is applied. The motivation is that all explanatory variable domain values should not get the same attention, so the scheme is the most weigh is considered for the levels values and other weighs are concerned with other values between levels values. In capability indices formulas, fuzzy numbers are used instead of crisp numbers to set membership grade as 1 for levels values and membership grades less than 1 for other values in explanatory variable domain.

Fuzzy numbers are necessary for physical measurements. The objective of fuzzy logic control (FLC) systems is to control complex processes by means of human experience. These systems are designed for the control of technical processes. The complexity of these processes range from cameras and vacuum cleaners to cement kilns, model cars, and trains. [26]

The structure of the rest of the present paper is outlined as follows. In the following section, preliminary of capability indices for simple linear profile is presented. Fuzzy set theory and some basic definitions are reviewed in Section 3. Section 4 introduces a new functional capability index. Two functional capability indices based on fuzzy logic are recommended in Section 5. Simulation study is carried out to make comparisons among the proposed indices and some existing ones in Section 6. Section 7 provides a method for detecting the appropriate sample size. Section 8 presents an example to illustrate the applicability and effectiveness of the proposed index. Finally, conclusions are given in the last section.

2. Capability indices for simple linear profile

Let (x_i, y_{ij}); $i = 1, 2, ..., k$ be the observations for the j^{th} sample collected over time. In fact, k random samples of size n, for k fixed x points, are available. Furthermore, it is assumed that when the process is in statistical control, the relationship between the response, independent variable and error terms can be modeled by

$$Y_{ij} = A_0 + A_1 x_i + \varepsilon_{ij}; \qquad i = 1, 2, ..., k \quad , \quad j = 1, 2, ..., n, \tag{1}$$

where, the slope and the intercept of the line, A_0 and A_1, are called the parameters of the model (profile coefficients) and the random errors ε_{ij}'s are independent and $\varepsilon_{ij} \sim N(0, \sigma^2)$ [6]. The relation between the levels of independent variables is similar to a process with multiple streams. A_0 and A_1 are unknown and however, remain fixed. Then, we can estimate them by a_0 and a_1 according to the sample observations as;

$$\widehat{A}_0 = a_0 = \frac{\sum_{j=1}^{n} a_{0j}}{n}, \qquad \widehat{A}_1 = a_1 = \frac{\sum_{j=1}^{n} a_{1j}}{n}, \tag{2}$$

where,

$$a_{0j} = \bar{y}_j - a_{1j}\bar{x}, \qquad a_{1j} = \frac{S_{xy(j)}}{S_{xx}}, \tag{3}$$

and $\bar{y}_j = \sum_{i=1}^{k} y_{ij}/k$, $\bar{x} = \sum_{i=1}^{k} x_i/k$, $S_{xy(j)} = \sum_{i=1}^{k}(x_i - \bar{x})y_{ij}$, and $S_{xx} = \sum_{i=1}^{k}(x_i - \bar{x})^2$ [14].

Hence, we can write $\widehat{Y}_{ij} = a_{0j} + a_{1j}x_i$, $i = 1, 2, ..., k$, where $\widehat{Y}_{ij}$ denotes the predicted value of the response variable from j^{th} sample for a given level of x. Furthermore, $\widehat{Y}_i = a_0 + a_1 x_i$ is the predicted value of the response variable for a given level of the independent variable.

The unbiased estimator of σ^2, the mean square error (MSE), is defined by

$$\widehat{\sigma}^2 = MSE = \frac{\sum_{j=1}^{n} MSE_j}{n}, \tag{4}$$

where, $MSE_j = \sum_{i=1}^{k} e_{ij}^2/(k-2)$, and $e_{ij} = y_{ij} - \widehat{Y}_{ij}$. Here e_{ij} is the residual of the j^{th} sample. For more information [6].

2.1. **Latest capability index.** Abbasi Ganji and Sadeghpour Gildeh [1] introduced a new class of generalized capability indices as follows:

$$C_p^{'''}(u,v) = \frac{d^* - uA^*}{3\sqrt{\sigma^2 + vA^2}}; \quad u, v \geq 0, \tag{5}$$

where, $d^* = \min\{D_l,\ D_u\}$ with $D_l = T - LSL$, $D_u = USL - T$, and

$$A^* = \frac{(\mu - T)^2}{D_u} I\{\mu > T\} + \frac{(T-\mu)^2}{D_l} I\{\mu \leq T\}, \tag{6}$$

$$A^2 = \frac{d^2(\mu - T)^2}{D_u^2} I\{\mu > T\} + \frac{d^2(T-\mu)^2}{D_l^2} I\{\mu \leq T\}, \tag{7}$$

and $d = (USL - LSL)/2$, and the indicator function $I\{x\}$ is defined as

$$I\{x\} = \begin{cases} 1; & x \geq 0, \\ 0; & \text{x} < 0. \end{cases} \tag{8}$$

Abbasi Ganji and Sadeghpour Gildeh [1], [4] proposed a new capability index for a simple linear profile by setting $u = v = 1$. Similarly, in this paper, setting $u = v = 1$, some new functional capability indices are introduced. Therefore, the following index is obtained:

$$C_{pp}^{'''} = C_p^{'''}(1,1) = \frac{d^* - A^*}{3\sqrt{\sigma^2 + A^2}}. \tag{9}$$

This index can be simplified as the following;

$$C_{pp}^{'''} = \begin{cases} \frac{d^* D_u - (\mu - T)^2}{3\sqrt{\sigma^2 D_u^2 + d^2(\mu - T)^2}}; & \mu > T, \\ \\ \frac{d^* D_l - (T-\mu)^2}{3\sqrt{\sigma^2 D_l^2 + d^2(T-\mu)^2}}; & \mu \leq T. \end{cases} \tag{10}$$

Developing $C_{pp}^{'''}$, a new functional capability index is introduced in section 4.

2.2. **Some traditional functional capability indices.** Nemati Keshteli et al. [15] proposed a functional approach to measure the process capability index of simple linear profiles in all ranges of explanatory variable. The functional approach uses a reference profile, functional specification limits and functional natural tolerance limits to present a functional form of process capability indices. These functions are defined as follows:

$$\mu_Y(X) = A_0 + A_1 X, \tag{11}$$

(12)
$$LSL_Y(X) = A_{0l} + A_{1l}X, \quad USL_Y(X) = A_{0u} + A_{1u}X, \quad T_Y(X) = A_{0t} + A_{1t}X,$$

$$LNTL_Y(X) = A_0 + A_1 X - 3\sigma, \quad UNTL_Y(X) = A_0 + A_1 X + 3\sigma, \tag{13}$$

$$D_{lY}(X) = T_Y(X) - LSL_Y(X), \quad D_{uY}(X) = USL_Y(X) - T_Y(X), \tag{14}$$

$$d_Y(X) = \frac{USL_Y(X) - LSL_Y(X)}{2}. \tag{15}$$

Based on the LSL and USL of the dependent variable in each level of the explanatory variable, two regression lines are obtained, so that A_{0l} and A_{1l} are intercept and slope of regression line related to LSLs, respectively and A_{0u} and A_{1u} are intercept and slope of regression line concerned with USLs, respectively. Furthermore, A_{0t} and A_{1t} are intercept and slope of regression line concerned with the target values.

Nemati Keshteli et al. [15] introduced two functional indices $C_p(profile)$ and $C_{pk}(profile)$ as what follows;

$$C_p(profile) = \frac{\int_{x_1}^{x_k}[USL_Y(X) - LSL_Y(X)]dX}{\int_{x_1}^{x_k}[UNTL_Y(X) - LNTL_Y(X)]dX}, \tag{16}$$

and

$$C_{pk}(profile) = \min\left\{C_{pkl}(profile), C_{pku}(profile)\right\}, \tag{17}$$

where

$$C_{pkl}(profile) = \frac{\int_{x_1}^{x_k}[\mu_Y(X) - LSL_Y(X)]dX}{\int_{x_1}^{x_k}[\mu_Y(X) - LNTL_Y(X)]dX},$$

$$C_{pku}(profile) = \frac{\int_{x_1}^{x_k}[USL_Y(X) - \mu_Y(X)]dX}{\int_{x_1}^{x_k}[UNTL_Y(X) - \mu_Y(X)]dX}.$$

Based on the sample gathered from the in-control process, all parameters of the model are estimated and so, the above indices are estimated.

Pakzad et al. [16] suggested two functional indices $C_{pm}(profile)$ and $C_{pmk}(profile)$, based on this scheme, as:

$$C_{pm}(profile) = \begin{cases} \dfrac{\int_{x_1}^{x_k}[USL_Y(X) - LSL_Y(X)]dX}{\int_{x_1}^{x_k}[6\sqrt{\sigma^2 + \left(\mu_Y(X) - T_Y(X)\right)^2}]dX}; & T_Y(X) = M_Y(X), \\ \dfrac{\int_{x_1}^{x_k}[d_Y^*(X)]dX}{\int_{x_1}^{x_k}[3\sqrt{\sigma^2 + \left(\mu_Y(X) - T_Y(X)\right)^2}]dX}; & T_Y(X) \neq M_Y(X), \end{cases} \tag{18}$$

and

$$C_{pmk}(profile) = \min\left\{C_{pmkl}(profile), C_{pmku}(profile)\right\}, \tag{19}$$

in which,

$$C_{pmkl}(profile) = \frac{\int_{x_1}^{x_k}[\mu_Y(X) - LSL_Y(X)]dX}{\int_{x_1}^{x_k}[3\sqrt{\sigma^2 + \left(\mu_Y(X) - T_Y(X)\right)^2}]dX}, \tag{20}$$

$$C_{pmku}(profile) = \frac{\int_{x_1}^{x_k} [USL_Y(X) - \mu_Y(X)]dX}{\int_{x_1}^{x_k} [3\sqrt{\sigma^2 + \big(\mu_Y(X) - T_Y(X)\big)^2}]dX}, \tag{21}$$

$M_Y(X) = \big(LSL_Y(X) + USL_Y(X)\big)/2$, and

$$d_Y^*(X) = \min\big\{\big(T_Y(X) - LSL_Y(X)\big),\ \big(USL_Y(X) - T_Y(X)\big)\big\}.$$

These indices are estimated from the in-control process. Simulation study concluded that the two indices $C_{pm}(profile)$ and $C_{pmk}(profile)$ perform better than the indices $C_p(profile)$ and $C_{pk}(profile)$, in terms of mean square error (MSE) and mean absolute error (MAE) [16].

Interpretation of these four functional indices are such as the traditional ones. The process is considered as "capable" if the value of the functional index is at least 1.

3. Preliminary concepts on fuzzy numbers

Fuzzy sets were introduced by Zadeh [25] to deal with the data and information possess nonstatistical uncertainties. Fuzzy logic is a form of a many-valued logic which deals with an approximating reasoning rather than a fixed and exact one. Compared to traditional binary sets (where variables may take on true or false values), fuzzy logic variables may have a true value that ranges from 0 to 1. Here, some basic definitions are presented that are mentioned in [2] and [3], too.

Definition 3.1. (Fuzzy set) Suppose X is a nonempty set. A fuzzy set (subset) $\widetilde{A}$ of X is described by its membership function $\mu_{\widetilde{A}} : X \to [0,1]$ and $\mu_{\widetilde{A}}(x)$ is expressed as the degree of membership of element x in fuzzy set $\widetilde{A}$ for each $x \in X$. This function is shown by $\widetilde{A}$, that is $\widetilde{A} : X \to [0,1]$. In the present paper, the second notation is used.

Definition 3.2. (Support) Let $\widetilde{A}$ be a fuzzy set of X. The support of $\widetilde{A}$, denoted by $supp(\widetilde{A})$, is defined as follows:

$$supp(\widetilde{A}) = \big\{x \in X | \widetilde{A}(x) > 0\big\}.$$

Definition 3.3. (Normal fuzzy set) A fuzzy set $\widetilde{A}$ of X is called normal if there exists an $x \in X$ that $\widetilde{A}(x) = 1$. Otherwise, $\widetilde{A}$ is subnormal.

Definition 3.4. (α-cut) An α-level set of a fuzzy set $\widetilde{A}$ of X is a crisp set, written by $\widetilde{A}_\alpha$, and is defined by

$$\widetilde{A}_\alpha = \begin{cases} \{t \in X | \widetilde{A}(t) \geq \alpha\}; & \alpha > 0, \\ cl(supp\widetilde{A}); & \alpha = 0, \end{cases}$$

where, $cl(supp\widetilde{A})$ is the closure of the support of $\widetilde{A}$.

Definition 3.5. (Convex fuzzy set) A fuzzy set $\widetilde{A}$ of X is convex if $\widetilde{A}_\alpha$ is a convex subset of X, $\forall\alpha \in [0,1]$.

Definition 3.6. (Fuzzy number) A fuzzy number $\widetilde{A}$ is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by **F**.

Definition 3.7. (LR fuzzy quantity) Any fuzzy quantity $\widetilde{A}$ is described as;

$$\widetilde{A}(t) = \begin{cases} L(\frac{a-t}{\alpha}); & t \in [a-\alpha, a], \\ 1; & t \in [a,b], \\ R(\frac{t-b}{\beta}); & t \in [b, b+\beta], \\ 0; & otherwise, \end{cases}$$

where, $L, R : [0,1] \to [0,1]$ are continuous and non-increasing shape functions which $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. This function is called LR-type fuzzy quantity and denoted by $\widetilde{A} = (a, b, \alpha, \beta)_{LR}$.

Definition 3.8. (Triangular fuzzy number) A fuzzy set $\widetilde{A}$ is called a triangular fuzzy number, denoted by $T(a,b,c)$, if its membership function is as follows:

$$\widetilde{A}(t) = \begin{cases} (t-a)/(b-a); & a \leq t < b, \\ (c-t)/(c-b); & b \leq t < c, \\ 0; & otherwise. \end{cases}$$

Definition 3.9. (Trapezoidal fuzzy quantity) A fuzzy set $\widetilde{A}$ is called a trapezoidal fuzzy quantity, denoted by $T_r(a,b,c,d)$, if its membership function is as the follows:

$$\widetilde{A}(t) = \begin{cases} (t-a)/(b-a); & a \leq t < b, \\ 1; & b \leq t < c, \\ (d-t)/(d-c); & c \leq t < d, \\ 0; & otherwise. \end{cases}$$

It is noted that a fuzzy quantity has all properties of a fuzzy number with at least one normal element.

4. New functional capability indix

In this section, a new functional capability index is recommended, which is the extension of the index $C_{pp}^{'''}$, mentioned in section 2.1. To introduce the new capability index $C_{pp}^{'''}(profile)$, four situations are considered. The two cases are that the mean line is parallel to the target line in the interval $[x_1,\ x_k]$ and the two other ones are that the mean line has an intersection with the target line, somewhere between two points x_1 and x_k.

For the first two ones, if the mean line is on the left side of the target line in the interval $[x_1,\ x_k]$, then it is concluded that $\mu_Y(X) > T_Y(X)$, otherwise, $\mu_Y(X) < T_Y(X)$. Hence, the index $C^{'''}_{pp.g}(profile)$ is defined as:

$$C^{'''}_{pp}(profile) = \begin{cases} \dfrac{\int_{x_1}^{x_k}\left[d^*_Y(X)D_{uY}(X) - \left(\mu_Y(X)-T_Y(X)\right)^2\right]dX}{\int_{x_1}^{x_k}\left[3\sqrt{\sigma^2 D^2_{uY}(X)+d^2_Y(X)\left(\mu_Y(X)-T_Y(X)\right)^2}\right]dX}; & \mu_Y(X) > T_Y(X), \\ \dfrac{\int_{x_1}^{x_k}\left[d^*_Y(X)D_{lY}(X) - \left(T_Y(X)-\mu_Y(X)\right)^2\right]dX}{\int_{x_1}^{x_k}\left[3\sqrt{\sigma^2 D^2_{lY}(X)+d^2_Y(X)\left(T_Y(X)-\mu_Y(X)\right)^2}\right]dX}; & \mu_Y(X) < T_Y(X). \end{cases} \tag{22}$$

If the mean and target lines intersect somewhere between two points x_1 and x_k, in other words $\mu_Y(x_m) = T_Y(x_m)$, where $x_j \leq x_m < x_{j+1};\quad j = \{1, 2, ..., k-1\}$, there are two cases that are discussed as follows:

• Case 1. For $x_1 \leq x \leq x_m$, the process mean line is located on the left side of the target line, that is, $\mu_Y(X) > T_Y(X)$, and for $x_m < x \leq x_k$, the mean line is on the right side of the target line, i.e., $\mu_Y(X) \leq T_Y(X)$. Therefor, the new index is obtained as the following;

$$C^{'''}_{pp}(profile) = \frac{C^{'''}_{pp}(profile)[num1]}{C^{'''}_{pp}(profile)[denum1]}, \tag{23}$$

where,

$$\begin{aligned} C^{'''}_{pp}(profile)[num1] = & \int_{x_1}^{x_m}[d^*_Y(X)D_{uY}(X) - \left(\mu_Y(X) - T_Y(X)\right)^2]dX \\ + & \int_{x_m}^{x_k}[d^*_Y(X)D_{lY}(X) - \left(T_Y(X) - \mu_Y(X)\right)^2]dX, \end{aligned} \tag{24}$$

and

$$\begin{aligned} C^{'''}_{pp}(profile)[denum1] = & \int_{x_1}^{x_m}[3\sqrt{\sigma^2 D^2_{uY}(X) + d^2_Y(X)\left(\mu_Y(X) - T_Y(X)\right)^2}]dX \\ + & \int_{x_m}^{x_k}[3\sqrt{\sigma^2 D^2_{lY}(X) + d^2_Y(X)\left(T_Y(X) - \mu_Y(X)\right)^2}]dX. \end{aligned} \tag{25}$$

• Case 2. For $x_1 \leq x \leq x_m$, $\mu_Y(X) \leq T_Y(X)$, and for $x_m < x \leq x_k$, $\mu_Y(X) > T_Y(X)$. Hence, the index is gain as:

$$C^{'''}_{pp}(profile) = \frac{C^{'''}_{pp}(profile)[num2]}{C^{'''}_{pp}(profile)[denum2]}, \tag{26}$$

that

$$C_{pp}^{'''}(profile)[num2] = \int_{x_1}^{x_m}[d_Y^*(X)D_{lY}(X) - \big(T_Y(X) - \mu_Y(X)\big)^2]dX$$

$$+ \int_{x_m}^{x_k}[d_Y^*(X)D_{uY}(X) - \big(\mu_Y(X) - T_Y(X)\big)^2]dX, \quad (27)$$

and

$$C_{pp}^{'''}(profile)[denum2] = \int_{x_1}^{x_m}[3\sqrt{\sigma^2 D_{lY}^2(X) + d_Y^2(X)\big(T_Y(X) - \mu_Y(X)\big)^2}]dX$$

$$+ \int_{x_m}^{x_k}[3\sqrt{\sigma^2 D_{uY}^2(X) + d_Y^2(X)\big(\mu_Y(X) - T_Y(X)\big)^2}]dX. \quad (28)$$

First of all, all model parameters should be estimated from the in-control process and then, the index $C_{pp}^{'''}(profile)$ is estimated. The process is supposed to be "capable" if $\widehat{C}_{pp}^{'''}(profile) \geq 1$.

5. New fuzzy-based functional capability indices

In the simple linear profile processes the explanatory variable is assumed to have k levels with fixed values x_1, x_2, ..., x_k. Nevertheless, the traditional and new functional indices use the entire domain of the explanatory variable. In other word, for calculating these indices, integration domain is $[x_1, \ \ x_k]$, that uses all values between these levels.

Further explanation is that the level i has the crisp value equal to x_i, for $i = \{1, 2, ..., k\}$, while the functional indices use all values in ranges $[x_i, \ \ x_{i+1}]$ for $i = \{1, 2, ..., k-1\}$ instead of x_i. More illustrated that although more focus should be on values x_1, x_2, ..., x_k and less focus for the other numbers in range $[x_1, \ \ x_k]$, the same focus is given to all numbers. It means the weigh of all values in $[x_1 \ \ x_k]$ is equal to 1. To overcome the drawback of this issue, the use of fuzzy numbers/quantities $\tilde{x}_i$; $i = \{1, 2, ..., k\}$ instead of x_i is suggested. In this way, the maximum weigh, equal to 1, is given to the values x_1, x_2, ..., x_k, and other numbers have less weigh.

The scheme is that the explanatory variable levels is set to $\tilde{x}_1$, $\tilde{x}_2$, ..., $\tilde{x}_k$, which is equivalent to *approximately* x_1, *approximately* x_2, ..., *approximately* x_k. It should be mentioned that these fuzzy numbers are used only for capability indices calculations, not in general. Because of the wide field of applications of triangular fuzzy numbers, this research is based on these numbers. To detect support of fuzzy numbers, first the following values should be obtained first.

$$a_i = \frac{x_i - x_{i-1}}{2}, \quad b_i = \frac{x_{i+1} - x_i}{2}; \quad i = 2, 3, ..., k-1$$

and $a_k = b_{k-1}$, $b_1 = a_2$. Then, fuzzy numbers $\tilde{x}_1$, and $\tilde{x}_k$ are defined as:

$$\tilde{x}_1 = \begin{cases} \frac{x_1 + b_1 - x}{b_1}; & x_1 \leq x < x_1 + b_1, \\ 0; & \text{otherwise}, \end{cases} \quad (29)$$

and

$$\tilde{x}_k = \begin{cases} \frac{x-x_k+a_k}{a_k}; & x_k - a_k < x \le x_k, \\ 0; & \text{otherwise.} \end{cases} \tag{30}$$

Other levels fuzzy numbers, for $i = \{2, ..., k-1\}$, are obtained as what follows;

$$\tilde{x}_i = \begin{cases} \frac{x-x_i+a_i}{a_i}; & x_i - a_i \le x < x_i, \\ \frac{x_i+b_i-x}{b_i}; & x_i \le x < x_i + b_i, \\ 0; & \text{otherwise.} \end{cases} \tag{31}$$

It should be noted that due to the preference of the traditional index C_{pmk} index over the indices C_p, C_{pk}, and C_{pm}, here a functional index based on fuzzy logic is introduced according to this index. Furthermore, As shown in the paper [1], the class of capability indices $C_p^{'''}(u,v)$ performs better than the other traditional indices, in expressing process capability. Therefore, the index $C_{pp}^{'''}$ has the better performance than the other ones. For this reason, functional and fuzzy-based functional indices according to this index are proposed, too. In addition, the performance of two introduced fuzzy-based functional indices is studied comparatively with the performance of functional indices.

5.1. **Fuzzy-based functional index $\mathbf{C_{pmk.g}}$(profile).** Now, based on the functional capability index $C_{pmk}(profile)$, presented in section 2.2, the fuzzy-based functional capability index $C_{pmk.g}(profile)$ is introduced as the following;

$$C_{pmk.g}(profile) = \min\left\{C_{pmkl.g}(profile), C_{pmku.g}(profile)\right\}, \tag{32}$$

that

$$C_{pmkl.g}(profile) = \frac{\sum_{i=1}^{k} \int_{x_1}^{x_k} \tilde{x}_i[\mu_Y(X) - LSL_Y(X)]dX}{\sum_{i=1}^{k} \int_{x_1}^{x_k} [3\tilde{x}_i \sqrt{\sigma^2 + \left(\mu_Y(X) - T_Y(X)\right)^2}]dX}, \tag{33}$$

$$C_{pmku.g}(profile) = \frac{\sum_{i=1}^{k} \int_{x_1}^{x_k} \tilde{x}_i[USL_Y(X) - \mu_Y(X)]dX}{\sum_{i=1}^{k} \int_{x_1}^{x_k} [3\tilde{x}_i \sqrt{\sigma^2 + \left(\mu_Y(X) - T_Y(X)\right)^2}]dX}. \tag{34}$$

5.2. **Fuzzy-based functional index $\mathbf{C_{pp.g}^{'''}}$(profile).** To introduce the fuzzy functional capability index $C_{pp.g}^{'''}(profile)$, four situations should be considered, as mentioned in section 4. For the first two ones that there is no intersection between the mean and target lines in the interval $[x_1,\ x_k]$, the index $C_{pp.g}^{'''}(profile)$ is defined as:

(35)

$$C'''_{pp.g}(profile) = \begin{cases} \dfrac{\sum_{i=1}^{k}\int_{x_1}^{x_k} \tilde{x}_i[d^*_Y(X)D_{uY}(X)-\big(\mu_Y(X)-T_Y(X)\big)^2]dX}{\sum_{i=1}^{k}\int_{x_1}^{x_k}[3\tilde{x}_i\sqrt{\sigma^2 D^2_{uY}(X)+d^2_Y(X)\big(\mu_Y(X)-T_Y(X)\big)^2}]dX}; & \mu_Y(X) > T_Y(X), \\ \dfrac{\sum_{i=1}^{k}\int_{x_1}^{x_k} \tilde{x}_i[d^*_Y(X)D_{lY}(X)-\big(T_Y(X)-\mu_Y(X)\big)^2]dX}{\sum_{i=1}^{k}\int_{x_1}^{x_k}[3\tilde{x}_i\sqrt{\sigma^2 D^2_{lY}(X)+d^2_Y(X)\big(T_Y(X)-\mu_Y(X)\big)^2}]dX}; & \mu_Y(X) < T_Y(X). \end{cases}$$

Now, suppose the mean and target lines intersect somewhere between two points x_1 and x_k, that is $\mu_Y(x_m) = T_Y(x_m)$, where $x_j \le x_m < x_{j+1}$; $j = \{1, 2, ..., k-1\}$. Hence two cases are discussed here.

• Case 1. For $x_1 \le x \le x_m$, $\mu_Y(X) > T_Y(X)$, and for $x_m < x \le x_k$, $\mu_Y(X) \le T_Y(X)$. Therefor, the index is calculated from the following equations.

$$C'''_{pp.g}(profile) = \frac{C'''_{pp.g}(profile)[num1]}{C'''_{pp.g}(profile)[denum1]}, \tag{36}$$

where,

$$\begin{aligned} C'''_{pp.g}(profile)[num1] = & \sum_{i=1}^{j-1}\int_{x_1}^{x_{j-1}} \tilde{x}_i[d^*_Y(X)D_{uY}(X) - \big(\mu_Y(X) - T_Y(X)\big)^2]dX \\ + & \int_{\tilde{x}_j^-[0]}^{x_m} \tilde{x}_j[d^*_Y(X)D_{uY}(X) - \big(\mu_Y(X) - T_Y(X)\big)^2]dX \\ + & \int_{x_m}^{\tilde{x}_j^+[0]} \tilde{x}_j[d^*_Y(X)D_{lY}(X) - \big(T_Y(X) - \mu_Y(X)\big)^2]dX \\ + & \sum_{i=j+1}^{k}\int_{x_{j+1}}^{x_k} \tilde{x}_i[d^*_Y(X)D_{lY}(X) - \big(T_Y(X) - \mu_Y(X)\big)^2]dX, \end{aligned} \tag{37}$$

and

$$\begin{aligned} C'''_{pp.g}(profile)[denum1] = & \sum_{i=1}^{j-1}\int_{x_1}^{x_{j-1}} [3\tilde{x}_i\sqrt{\sigma^2 D^2_{uY}(X) + d^2_Y(X)\big(\mu_Y(X) - T_Y(X)\big)^2}]dX \\ + & \int_{\tilde{x}_j^-[0]}^{x_m} [3\tilde{x}_j\sqrt{\sigma^2 D^2_{uY}(X) + d^2_Y(X)\big(\mu_Y(X) - T_Y(X)\big)^2}]dX \\ + & \int_{x_m}^{\tilde{x}_j^+[0]} [3\tilde{x}_i\sqrt{\sigma^2 D^2_{lY}(X) + d^2_Y(X)\big(T_Y(X) - \mu_Y(X)\big)^2}]dX \\ + & \sum_{i=j+1}^{k}\int_{x_{j+1}}^{x_k} [3\tilde{x}_i\sqrt{\sigma^2 D^2_{lY}(X) + d^2_Y(X)\big(T_Y(X) - \mu_Y(X)\big)^2}]dX, \end{aligned} \tag{38}$$

and $\tilde{x}_j^-[0]$ and $\tilde{x}_j^+[0]$ are the minimum and maximum values in the 0-cut interval of $\tilde{x}_j$, respectively. In other word, $\tilde{x}_j[0] = \big[\tilde{x}_j^-[0] \quad \tilde{x}_j^+[0]\big]$.

• Case 2. For $x_1 \leq x \leq x_m$, $\mu_Y(X) \leq T_Y(X)$, and for $x_m < x \leq x_k$, $\mu_Y(X) > T_Y(X)$. Hence, the index is obtained as what follows;

$$C^{'''}_{pp.g}(profile) = \frac{C^{'''}_{pp.g}(profile)[num2]}{C^{'''}_{pp.g}(profile)[denum2]}, \tag{39}$$

where,

$$\begin{aligned} C^{'''}_{pp.g}(profile)[num2] = & \sum_{i=1}^{j-1} \int_{x_1}^{x_{j-1}} \tilde{x}_i [d^*_Y(X) D_{lY}(X) - \big(T_Y(X) - \mu_Y(X)\big)^2] dX \\ + & \int_{\tilde{x}_j^-[0]}^{x_m} \tilde{x}_j [d^*_Y(X) D_{lY}(X) - \big(T_Y(X) - \mu_Y(X)\big)^2] dX \\ + & \int_{x_m}^{\tilde{x}_j^+[0]} \tilde{x}_j [d^*_Y(X) D_{uY}(X) - \big(\mu_Y(X) - T_Y(X)\big)^2] dX \\ + & \sum_{i=j+1}^{k} \int_{x_{j+1}}^{x_k} \tilde{x}_i [d^*_Y(X) D_{uY}(X) - \big(\mu_Y(X) - T_Y(X)\big)^2] dX, \end{aligned} \tag{40}$$

and

$$\begin{aligned} C^{'''}_{pp.g}(profile)[denum2] = & \sum_{i=1}^{j-1} \int_{x_1}^{x_{j-1}} [3\tilde{x}_i \sqrt{\sigma^2 D^2_{lY}(X) + d^2_Y(X)\big(T_Y(X) - \mu_Y(X)\big)^2}] dX \\ + & \int_{\tilde{x}_j^-[0]}^{x_m} [3\tilde{x}_i \sqrt{\sigma^2 D^2_{lY}(X) + d^2_Y(X)\big(T_Y(X) - \mu_Y(X)\big)^2}] dX \\ + & \int_{x_m}^{\tilde{x}_j^+[0]} [3\tilde{x}_j \sqrt{\sigma^2 D^2_{uY}(X) + d^2_Y(X)\big(\mu_Y(X) - T_Y(X)\big)^2}] dX \\ + & \sum_{i=j+1}^{k} \int_{x_{j+1}}^{x_k} [3\tilde{x}_i \sqrt{\sigma^2 D^2_{uY}(X) + d^2_Y(X)\big(\mu_Y(X) - T_Y(X)\big)^2}] dX. \end{aligned} \tag{41}$$

Based on the sample data gathered from the in-control process, the mean line $\mu_Y(X)$ is estimated by the refernce line $Y(X)$, and the variance σ^2 is estimated by MSE and then, the index is estimated. The process is considered to be "capable" if the estimated value of the index $C^{'''}_{pp.g}(profile)$ is at least one.

6. **Simulation Study**

In this section, simulation study with 10000 iterations are engaged to investigate and compare the performance of the mentioned traditional and new capability indices. Kang and Albin [11] used the linear profile $Y_{ij} = 3+2X_i+\varepsilon_{ij}$; $\varepsilon \sim N(0, 1)$. They considered four levels for the independent variable and the fix values was $x = 2,\ 4,\ 6,$ and 8. The specification limits and target values of the dependent variable in each level of the explanatory variable are presented in Table 1. Here, simulation scheme is used to generate the necessary data from this model by $\varepsilon_{ij} \sim N(0, \sigma^2)$, for various values of $\sigma \in \{0.5, 0.8, 1.0, 1.2\}$ and sample size $n \in \{25, 50, 100, 200\}$.

Capability indices are compared in terms of the mean square error (MSE) and mean absolute error (MAE). To mind these criteria, suppose statistic $\widehat{\theta}$ is used for estimating parameter θ. These criteria are as:

$$MSE = E(\widehat{\theta} - \theta)^2, \qquad MAE = E|\widehat{\theta} - \theta|.$$

The less the values of these criteria, the better the index in measuring process capability.

Table 1. Specification limits and target value for each level of the independent variable.

i	X_i	LSL	USL	T
1	2	2.5	10	6.25
2	4	6.85	14.35	10.6
3	6	11.25	18.75	15
4	8	16.25	23.75	20

Fitting regression line for LSL values in Table 1, functional lower specification limit is gain. Similarly, regression line for USL values gives functional upper specification limit. These lines are obtained as the following:

$$LSL_Y(X) = -2.2 + 2.2825X, \quad USL_Y(X) = 5.3 + 2.2825X$$

Furthermore, it is obviously seen that for each level, the target value is in the middle of the LSL and USL values, that means the tolerance is symmetric. According to this, for symmetric tolerance, the target line is obtained as $T_Y(X) = 1.55 + 2.2825X$. To further evaluate the performance of the proposed index in comparison with the other ones, two other models $Y = 3.5 + 2X + \varepsilon$, and $Y = 3.4 + 1.8X + \varepsilon$ are investigated, too. Figure 1 shows the graph of these lines.

For asymmetric case, consider the target line $T_Y(X) = 3.425 + 2.2825X$, and three profile lines $Y = 3.4 + 2.4X + \varepsilon$, $Y = 3.6 + 2.2X + \varepsilon$, and $Y = 3.3 + 2.3X + \varepsilon$. Graph of these lines are presented in Figure 1.

Fuzzy values of *approximately* 2, *approximately* 4, *approximately* 6, and *approximately* 8 are considered for the explanatory variable, as what follows;

$$\tilde{x}_1 = \begin{cases} 3 - x; & 2 \le x < 3, \\ 0; & \text{otherwise.} \end{cases} \qquad \tilde{x}_4 = \begin{cases} x - 7; & 7 < x \le 8, \\ 0; & \text{otherwise.} \end{cases}$$

$$\tilde{x}_2 = \begin{cases} x - 3; & 3 \le x < 4, \\ 5 - x; & 4 \le x < 5, \\ 0; & \text{otherwise.} \end{cases} \qquad \tilde{x}_3 = \begin{cases} x - 5; & 5 \le x < 6, \\ 7 - x; & 6 \le x < 7, \\ 0; & \text{otherwise.} \end{cases}$$

Figure 2 shows graph of these fuzzy numbers as well as the traditional case that gives to the explanatory levels the same weigh equal to 1.

For each profile model and fix values of σ and sample size n, the simulation procedure is done by six steps as follows:

• Step 1. Calculate the values of capability indices $C_{pmk}(profile)$, $C'''_{pp}(profile)$, $C_{pmk.g}(profile)$, $C'''_{pp.g}(profile)$, and set them as true values.

• Step 2. Generate four samples of size n from standard normal distribution by variance σ^2 and set them as values of ε for each level of the explanatory variable.

• Step 3. Obtain the values of Y in samples, based on the linear equation $Y = a_0 + a_1 X + \varepsilon$, and substituting the values of X levels by $2, 4, 6$, and 8 and ε values gotten in Step 2.

• Step 4. Calculate estimated values of capability indices as $\widehat{C}_{pmk}(profile)$, $\widehat{C}'''_{pp}(profile)$, $\widehat{C}_{pmk.g}(profile)$, $\widehat{C}'''_{pp.g}(profile)$, based on the sample values obtained in Step 3.

• Step 5. Obtain AE and SE of each index based on the formula $AE_C = |\widehat{C} - C_{truevalue}|$ and $SE_C = (\widehat{C} - C_{truevalue})^2$.

• Step 6. Replicate steps 2 to 5 for 10000 times and obtain mean of the estimated indices, AEs and SEs, and set them as estimated indices, MAEs and MSEs of indices, respectively.

Results of simulations are tabulated in tables 2-13.

Simulation study demonstrates that for various sigma values and various sample sizes and various profile models and also, in both case symmetric and asymmetric specification lines, the index $C'''_{pp.g}(profile)$ performs the best for assessment of process capability, because of having the lowest values of MSE and MAE. Moreover, this index has accurate and flexible information of the explanatory variable, because it uses fuzzy numbers for its levels value. On the other hand, the sample size affects the MSE and MAE. In fact, the larger the sample size, the smaller the values MSE and MAE.

For a better representation of the results, figures 3 and 4 show MSE plots of the discussed estimators in simulation study for one model in symmetric tolerance and asymmetric case, respectively. These figures show that the mentioned values for the index $C'''_{pp.g}(profile)$ are putted in bellow of others indices values.

TABLE 2. Comparison among the previous PCIs and the new indices for symmetric functional tolerance and $y_{ij} = 3 + 2x_i + \varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 0.5^2)$ true value		1.808677	1.79566	1.79067	1.776072
		$\widehat{C}_{pmk}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp}(profile)$ $(MAE\&MSE)$	$\widehat{C}_{pmk.g}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp.g}(profile)$ $(MAE\&MSE)$
sample size	25	1.76346 (0.08652&0.01153)	1.75662 (0.08392&0.01086)	1.74654 (0.08538&0.01123)	1.73810 (0.08295&0.01062)
	50	1.78659 (0.05870&0.00536)	1.77609 (0.05784&0.00518)	1.76912 (0.05800&0.00524)	1.75702 (0.05726&0.00508)
	100	1.79846 (0.04023&0.00255)	1.78634 (0.039637&0.00247)	1.78073 (0.03978&0.00249)	1.76702 (0.03928&0.00243)
	200	1.80393 (0.02836&0.00126)	1.79113 (0.02788&0.00122)	1.78605 (0.02806&0.00123)	1.77167 (0.02764&0.00120)
$\varepsilon \sim N(0, 0.8^2)$ true value		1.32817	1.31861	1.32002	1.30926
sample size	25	1.27822 (0.07973&0.00962)	1.27960 (0.07421&0.00839)	1.27077 (0.07905&0.00946)	1.27091 (0.07370&0.00829)
	50	1.30277 (0.05284&0.00430)	1.29904 (0.05052&0.00393)	1.29497 (0.05246&0.00424)	1.29001 (0.05027&0.00389)
	100	1.31592 (0.03568&0.00199)	1.30910 (0.03440&0.00186)	1.30796 (0.03545&0.00197)	1.299928 (0.03426&0.00184)
	200	1.32266 (0.02495&0.00098)	1.31399 (0.02413&0.00091)	1.31460 (0.02479&0.00096)	1.30472 (0.02404&0.00090)

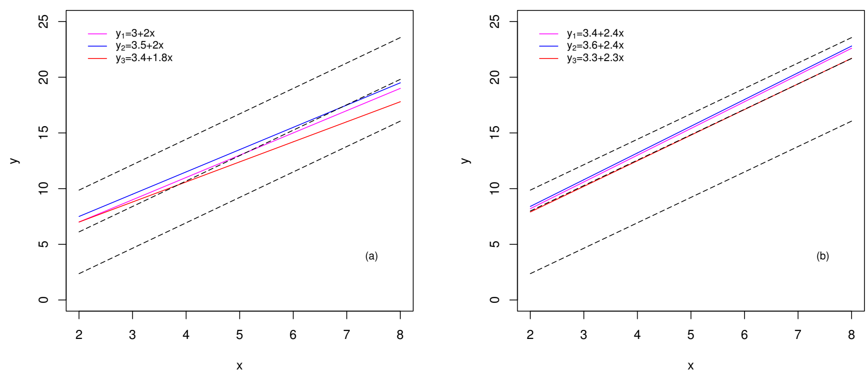

FIGURE 1. Specification limits lines, target line and functional profiles for symmetric (a) and asymmetric (b) functional tolerances.

Table 3. Comparison among the previous PCIs and the new indices for symmetric functional tolerance and $y_{ij} = 3 + 2x_i + \varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 1.0^2)$ true value		1.11504	1.10701	1.10992	1.10087
		$\widehat{C}_{pmk}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp}(profile)$ $(MAE\&MSE)$	$\widehat{C}_{pmk.g}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp.g}(profile)$ $(MAE\&MSE)$
sample size	25	1.06573 (0.073170&0.00802)	1.07066 (0.06616&0.00663)	1.06111 (0.07268&0.00793)	1.06498 (0.06585&0.00658)
	50	1.08930 (0.04799&0.00353)	1.08875 (0.044738&0.00307)	1.08444 (0.04773&0.00350)	1.08284 (0.04461&0.00306)
	100	1.10222 (0.03216&0.00161)	1.09808 (0.03038&0.00145)	1.09724 (0.03200&0.00160)	1.09207 (0.03032&0.00144)
	200	1.10917 (0.02231&0.00078)	1.10267 0.02126&0.00071)	1.10412 (0.02222&0.00077)	1.09659 (0.02123&0.00070)
$\varepsilon \sim N(0, 1.2^2)$ true value		0.95645	0.94956	0.95309	0.94532
sample size	25	0.90866 (0.06725&0.00672)	0.91611 (0.05900&0.00525)	0.90564 (0.06690&0.00666)	0.91217 (0.05882&0.00523)
	50	0.93091 (0.04374&0.00293)	0.93274 (0.03968&0.00241)	0.92773 (0.04357&0.00290)	0.92865 (0.03963&0.00241)
	100	0.94333 (0.02912&0.00132)	0.94130 (0.02689&0.00113)	0.94008 (0.02901&0.00131)	0.4 (0.02688&0.00113)
	200	0.95026 (0.02006&0.00063)	0.94554 (0.01879&0.00055)	0.94695 (0.01999&0.00063)	0.94134 (0.01879&0.09006)

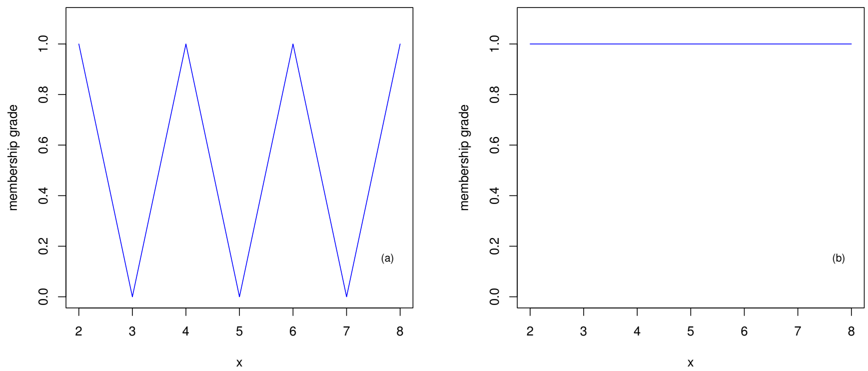

Figure 2. The membership functions plots of fuzzy values $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$ (a) and integration interval plot of crisp values in $[x_1, x_4]$ (b)

.

TABLE 4. Comparison among the previous PCIs and the new indices for symmetric functional tolerance and $y_{ij} = 3.5 + 2x_i + \varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 0.5^2)$	true value	1.29462	1.45445	1.28536	1.44262
		$\widehat{C}_{pmk}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp}(profile)$ $(MAE\&MSE)$	$\widehat{C}_{pmk.g}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp.g}(profile)$ $(MAE\&MSE)$
sample size	25	1.27375 (0.06242&0.00607)	1.43029 (0.06208&0.00600)	1.26496 (0.06164&0.00592)	1.41903 (0.06139&0.00587)
	50	1.28438 (0.04368&0.00300)	1.44251 (0.04311&0.00291)	1.27534 (0.04314&0.00292)	1.43096 (0.04265&0.00285)
	100	1.28931 (0.03064&0.00147)	1.44841 (0.03006&0.00142)	1.28018 (0.03027&0.00144)	1.43674 (0.02975&0.00139)
	200	1.29196 (0.02151&0.00073)	1.45144 (0.02110&0.00070)	1.28276 (0.02125&0.00071)	1.43970 (0.02089&0.00069)
$\varepsilon \sim N(0, 0.8^2)$	true value	1.01799	1.14367	1.01330	1.13728
sample size	25	0.99434 (0.06302&0.00616)	1.11587 (0.06178&0.00591)	0.98992 (0.06252&0.00607)	1.10982 (0.06141&0.00584)
	50	1.00635 (0.04408&0.00304)	1.12987 (0.04261&0.00283)	1.00178 (0.04373&0.00299)	1.12364 (0.04238&0.00280)
	100	1.01194 (0.03090&0.00149)	1.13669 (0.02959&0.00137)	1.00733 (0.03066&0.00147)	1.13040 (0.02942&0.00136)
	200	1.01498 (0.02167&0.00074)	1.14022 (0.02076&0.00068)	1.01033 (0.02151&0.00073)	1.13388 (0.02066&0.00067)

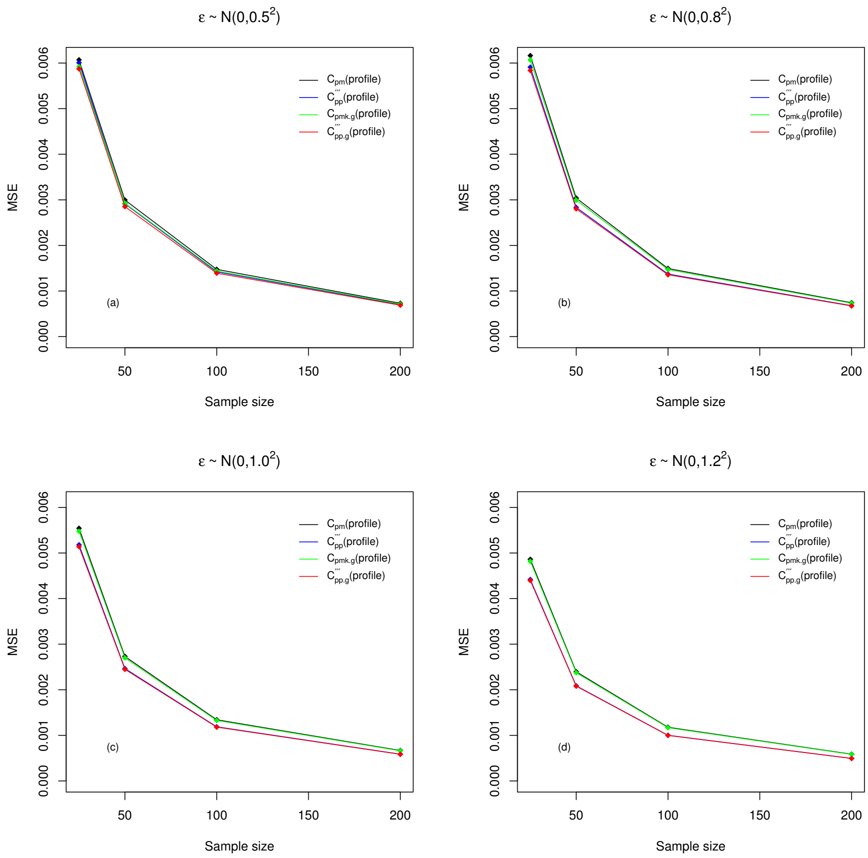

FIGURE 3. Graph comparison based on the MSE criterion in simulation study, for various values of variances and sample sizes, for profile model $y_{ij} = 3.5 + 2x_i + \varepsilon_{ij}$.

Table 5. Comparison among the previous PCIs and the new indices for symmetric functional tolerance and $y_{ij} = 3.5 + 2x_i + \varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 1.0^2)$ true value		0.88003	0.98868	0.87688	0.98417
		$\widehat{C}_{pmk}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp}(profile)$ $(MAE\&MSE)$	$\widehat{C}_{pmk.g}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp.g}(profile)$ $(MAE\&MSE)$
sample size	25	0.85671 (0.05979&0.00554)	0.96095 (0.05793&0.00518)	0.85374 (0.05944&0.00548)	0.95668 (0.05770&0.00514)
	50	0.86853 (0.04180&0.00273)	0.97488 (0.03976&0.00246)	0.86546 (0.04156&0.00270)	0.97048 (0.03963&0.00245)
	100	0.87405 (0.02930&0.00134)	0.98170 (0.02756&0.00119)	0.87095 (0.02913&0.00133)	0.97726 (0.02746&0.00118)
	200	0.87706 (0.02055&0.00067)	0.98524 (0.01932&0.00059)	0.87393 (0.02044&0.00066)	0.98076 (0.01926&0.00058)
$\varepsilon \sim N(0, 1.2^2)$ true value		0.77064	0.86578	0.76844	0.86246
sample size	25	0.74833 (0.05600&0.00486)	0.83892 (0.05358&0.00442)	0.74626 (0.05575&0.00482)	0.83577 (0.05344&0.00439)
	50	0.75962 (0.03916&0.00240)	0.85238 (0.03660&0.00209)	0.75749 (0.03898&0.00237)	0.84914 (0.03653&0.00208)
	100	0.76489 (0.02746&0.00119)	0.85900 (0.02532&0.00100)	0.76273 (0.02734&0.00117)	0.85574 (0.02527&0.00099)
	200	0.76779 (0.01926&0.00059)	0.86245 (0.01773&0.00049)	0.76561 (0.01918&0.00058)	0.85915 (0.01770&0.00049)

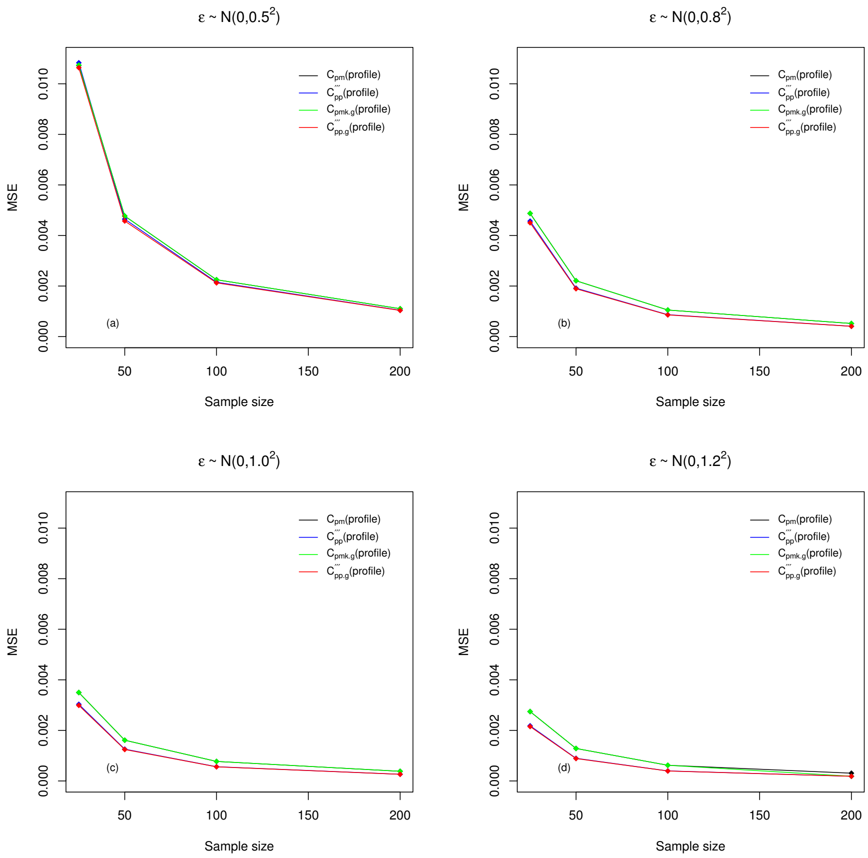

Figure 4. Graph comparison based on the MSE criterion in simulation study, for various values of variances and sample sizes, for profile model $y_{ij} = 3.3 + 2.3x_i + \varepsilon_{ij}$ in asymmetric tolerance case.

TABLE 6. Comparison among the previous PCIs and the new indices for symmetric functional tolerance and $y_{ij} = 3.4 + 1.8x_i + \varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 0.5^2)$	true value	1.03888	1.13401	1.02751	1.11827
		$\widehat{C}_{pmk}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp}(profile)$ $(MAE\&MSE)$	$\widehat{C}_{pmk.g}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp.g}(profile)$ $(MAE\&MSE)$
sample size	25	1.02760 (0.03866&0.00232)	1.12116 (0.03860&0.00233)	1.01644 (0.03818&0.00226)	1.10568 (0.03821&0.00228)
	50	1.03317 (0.02746&0.00117)	1.12749 (0.02737&0.00116)	1.02190 (0.02713&0.0011)	1.11187 (0.02710&0.00114)
	100	1.03644 (0.019074&0.00057)	1.13116 (0.01893&0.00056)	1.02512 (0.018848&0.00056)	1.11548 (0.01875&0.00055)
	200	1.03775 (0.01339&0.00028)	1.13266 (0.01331&0.00028)	1.02640 (0.01323&0.00027)	1.11694 (0.01318&0.00027)
$\varepsilon \sim N(0, 0.8^2)$	true value	0.86986	0.94952	0.86202	0.93815
sample size	25	0.85466 (0.04517&0.00316)	0.93188 (0.04452&0.00308)	0.84716 (0.04462&0.00308)	0.92097 (0.04412&0.00303)
	50	0.86221 (0.03210&0.00160)	0.94062 (0.03148&0.00153)	0.85454 (0.03172&0.00156)	0.92948 (0.03123&0.00150)
	100	0.86647 (0.02228&0.00078)	0.94549 (0.02167&0.00074)	0.85871 (0.02202&0.00076)	0.93424 (0.02151&0.00073)
	200	0.86828 (0.01565&0.00038)	0.94760 (0.01521&0.00036)	0.86047 (0.01547&0.00037)	0.93629 (0.01510&0.00035)

7. Determining sample size

Although larger sample sizes provide better capability estimators, they may not be appropriate in some processes due to sampling cost or some other factors. Determination a good sample size is an important issue of estimating the process capability. Absolute percentage error (APE) can be applied to derive the optimal sample size. As mentioned in [13], APE is popular numerical tool, due to its advantages of scale-independency and interpretability. For instance, Sedighi Maman, et al. [18] and Pakzad, et al. [16] engaged this metric for detecting the sample size. In this section, first this metric for the index $C'''_{pp.g}(profile)$ is defined and then, the minimum necessary sample size is offered.

APE for $C'''_{pp.g}(profile)$ is defined as follows:

$$APE_{C'''_{pp.g}(profile)} = \left| \frac{C'''_{pp.g}(profile) - \widehat{C}'''_{pp.g}(profile)}{C'''_{pp.g}(profile)} \right|. \tag{42}$$

Cumulative distribution function (cdf) of $APE_{C'''_{pp.g}(profile)}$ is calculted as:

$$F_{APE_{C'''_{pp.g}(profile)}}(APE_0) = p(APE_{C'''_{pp.g}(profile)} \leq APE_0). \tag{43}$$

TABLE 7. Comparison among the previous PCIs and the new indices for symmetric functional tolerance and $y_{ij} = 3.4 + 1.8x_i + \varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 1.0^2)$	true value	0.77479	0.84574	0.768905	0.83682
		$\widehat{C}_{pmk}(profile)$	$\widehat{C}'''_{pp}(profile)$	$\widehat{C}_{pmk.g}(profile)$	$\widehat{C}'''_{pp.g}(profile)$
		$(MAE\&MSE)$	$(MAE\&MSE)$	$(MAE\&MSE)$	$(MAE\&MSE)$
sample size	25	0.75859 (0.04590&0.00326)	0.82670 (0.04477&0.0031)	0.75304 (0.04541&0.00319)	0.81822 (0.04447&0.00306)
	50	0.76665 (0.03263&0.00165)	0.83614 (0.03156&0.00153)	0.76093 (0.03230&0.00162)	0.827433 (0.03139&0.00152)
	100	0.77114 (0.02266&0.00081)	0.84134 (0.02168&0.00074)	0.76534 (0.02243&0.00079)	0.83253 (0.02158&0.00073)
	200	0.77308 (0.01593&0.00040)	0.84364 (0.01522&0.00036)	0.76724 (0.01577&0.00039)	0.83477 (0.01515&0.00036)
$\varepsilon \sim N(0, 1.2^2)$	true value	0.69407	0.75763	0.68965	0.75056
sample size	25	0.67766 (0.04527&0.00317)	0.73805 (0.04367&0.00294)	0.67353 (0.04487&0.00311)	0.73136 (0.04346&0.00292)
	50	0.68583 (0.03220&0.00161)	0.74776 (0.03068&0.00145)	0.68155 (0.03193&0.00158)	0.74087 (0.03058&0.00144)
	100	0.69035 (0.02238&0.00079)	0.75306 (0.02105&0.00070)	0.68600 (0.02219&0.00078)	0.74609 (0.02099&0.00069)
	200	0.69233 (0.01574&0.00039)	0.75544 (0.01477&0.00034)	0.68794 (0.01561&0.00038)	0.74842 (0.01474&0.00034)

Figure 5 depicts the empirical cdf plots of $APE_{C'''_{pp.g}(profile)}$ of two profile models $Y = 3 + 2X + \varepsilon$ and $Y = 3.4 + 2.4X + \varepsilon$, discussed in section 6, for symmetric and asymmetric functional tolerances, respectively.

TABLE 8. Comparison among the previous PCIs and the new indices for asymmetric functional tolerance and $y_{ij} = 3.4+2.4x_i+\varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 0.5^2)$ true value		0.57156	0.45120	0.57097	0.45050
		$\widehat{C}_{pmk}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp}(profile)$ $(MAE\&MSE)$	$\widehat{C}_{pmk.g}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp.g}(profile)$ $(MAE\&MSE)$
sample size	25	0.56273 (0.04094&0.00260)	0.44902 (0.03201&0.00161)	0.56210 (0.04084&0.00259)	0.44823 (0.03190&0.00160)
	50	0.56743 (0.02918&0.00133)	0.45034 (0.02294&0.00083)	0.56683 (0.02910&0.00132)	0.44960 (0.02285&0.00082)
	100	0.56916 (0.02057&0.00066)	0.45049 (0.01620&0.00041)	0.56856 (0.02052&0.00066)	0.44978 (0.01614&0.00041)
	200	0.57035 (0.01452&0.00033)	0.45081 (0.01142&0.00020)	0.56975 (0.01448&0.00033)	0.45010 (0.01138&0.00020)
$\varepsilon \sim N(0, 0.8^2)$ true value		0.44090	0.40028	0.44054	0.39958
sample size	25	0.43058 (0.04024&0.00251)	0.39519 (0.03760&0.00220)	0.43020 (0.04018&0.00250)	0.39436 (0.03750&0.00218)
	50	0.43599 (0.02871&0.00129)	0.39806 (0.02708&0.00115)	0.43562 (0.02866&0.00128)	0.39730 (0.02700&0.00114)
	100	0.43812 (0.02027&0.00064)	0.39883 (0.01920&0.00058)	0.43775 (0.02023&0.00064)	0.39809 (0.01914&0.00057)
	200	0.43950 (0.01431&0.00032)	0.39953 (0.01357&0.00029)	0.43914 (0.01429&0.00032)	0.39880 (0.01353&0.00029)

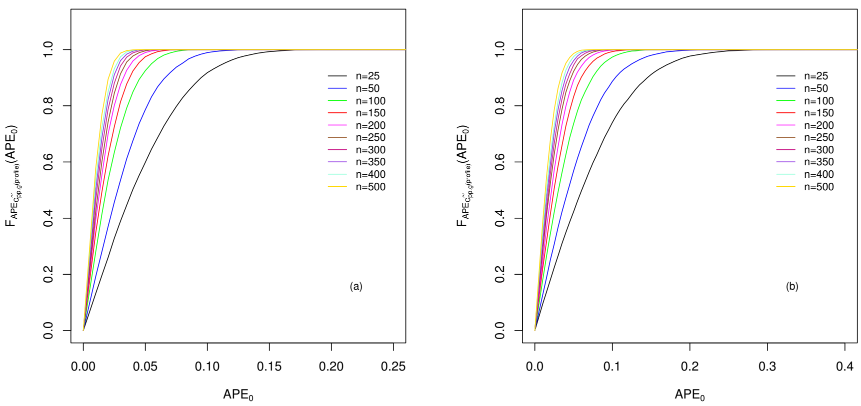

FIGURE 5. Empirical cdf plots of $APE_{C'''_{pp.g}}(profile)$ for various values of sample sizes for symmetric (a) and asymmetric (b) tolerance cases.

TABLE 9. Comparison among the previous PCIs and the new indices for asymmetric functional tolerance and $y_{ij} = 3.4+2.4x_i+\varepsilon_{ij}$

		$C_{pmk}(profile)$	$C''''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C''''_{pp.g}(profile)$
$\varepsilon \sim N(0, 1.0^2)$	true value	0.37680	0.36670	0.37655	0.36606
		$\widehat{C}_{pmk}(profile)$ $(MAE\&MSE)$	$\widehat{C}''''_{pp}(profile)$ $(MAE\&MSE)$	$\widehat{C}_{pmk.g}(profile)$ $(MAE\&MSE)$	$\widehat{C}''''_{pp.g}(profile)$ $(MAE\&MSE)$
sample size	25	0.36670 (0.03818&0.00226)	0.35969 (0.03796&0.00223)	0.36644 (0.03814&0.00225)	0.35892 (0.03791&0.00222)
	50	0.37197 (0.02728&0.00116)	0.36354 (0.02741&0.00118)	0.37171 (0.02724&0.00116)	0.36283 (0.02735&0.00117)
	100	0.37407 (0.01927&0.00058)	0.36477 (0.01948&0.00059)	0.37382 (0.01924&0.00058)	0.36408 (0.01944&0.00059)
	200	0.37543 (0.01362&0.00029)	0.36571 (0.01379&0.00030)	0.37518 (0.01360&0.00029)	0.36505 (0.01376&0.00030)
$\varepsilon \sim N(0, 1.2^2)$	true value	0.32696	0.33563	0.32678	0.33506
sample size	25	0.31741 (0.03616&0.00203)	0.32696 (0.03704&0.00212)	0.31722 (0.03613&0.00202)	0.32626 (0.03702&0.00211)
	50	0.32237 (0.02587&0.00104)	0.33161 (0.02678&0.00112)	0.32219 (0.02585&0.00104)	0.33098 (0.02675&0.00112)
	100	0.32436 (0.01829&0.00052)	0.33327 (0.01907&0.00057)	0.32419 (0.01827&0.00052)	0.33266 (0.01904&0.00057)
	200	0.32566 (0.01293&0.00026)	0.33444 (0.01351&0.00029)	0.32548 (0.01292&0.00026)	0.33385 (0.01349&0.00028)

The sample size is obtained approximataly based on the empirical cdf of $APE_{C''''_{pp.g}}(profile)$, so by simulation scheme, the minimum sample size required to achive the pre-determined APE, i.e. APE_0, with the least probability $1-\alpha$, is considered as the appropriate sample size, equivalent to solve the following equation.

$$p(APE_{C''''_{pp.g}(profile)} \leq APE_0) \geq 1-\alpha. \tag{44}$$

Here, for two profile models $Y = 3 + 2X + \varepsilon$ and $Y = 3.4 + 2.4X + \varepsilon$, discussed in section 6, respectively with symmetric and asymmetric functional tolerances, and for $\sigma = 0.5$, simulated samples by 10000 iteration was generated by diffenernt sample sizes values from 3 to 4000 for symmetric tolerance and from 3 to 9000 for asymmetric case. For values of the sample size that apply to equation (44), the same number is assumed to be appropriate sample size. Otherwise, this value is obtained by linear interpolation between the two closest numbers smaller and larger amounts of the desired value, with the probability value between the two related probability values. The results are tabulated in tables 14 and 15.

TABLE 10. Comparison among the previous PCIs and the new indices for asymmetric functional tolerance and $y_{ij} = 3.6 + 2.4x_i + \varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 0.5^2)$ true value		0.40350	0.31934	0.40330	0.31900
		$\widehat{C}_{pmk}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp}(profile)$ $(MAE\&MSE)$	$\widehat{C}_{pmk.g}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp.g}(profile)$ $(MAE\&MSE)$
sample size	25	0.39955 (0.02931&0.00134)	0.31870 (0.02160&0.00074)	0.39933 (0.02927&0.00134)	0.31834 (0.02157&0.00073)
	50	0.40172 (0.02095&0.00069)	0.31914 (0.01542&0.00038)	0.40151 (0.02092&0.00069)	0.31880 (0.01540&0.00038)
	100	0.40238 (0.01478&0.00034)	0.31905 (0.01087&0.00019)	0.40217 (0.01476&0.00034)	0.31872 (0.01086&0.00018)
	200	0.40291 (0.01043&0.00017)	0.31916 (0.00765&0.00009)	0.40272 (0.01042&0.00017)	0.31883 (0.00764&0.00009)
$\varepsilon \sim N(0, 0.8^2)$ true value		0.33253	0.29670	0.33236	0.29634
sample size	25	0.32683 (0.03281&0.00167)	0.29487 (0.02888&0.00131)	0.32664 (0.03277&0.00167)	0.29445 (0.02884&0.00130)
	50	0.32989 (0.02352&0.00087)	0.29600 (0.02072&0.00068)	0.32971 (0.02349&0.00086)	0.29561 (0.02069&0.00068)
	100	0.33094 (0.01662&0.00043)	0.29610 (0.01464&0.00034)	0.33077 (0.01660&0.00043)	0.29572 (0.01462&0.00034)
	200	0.33172 (0.01174&0.00022)	0.29637 (0.01033&0.00017)	0.33154 (0.01173&0.00022)	0.29600 (0.01031&0.00017)

8. **Illustrative Example**

In this section, the new proposed index is engaged for a real data set of industrial springs collected by Shi et al. [20]. The quality of the industrial spirings can be characterized by a functional relationship between spirings elasticity and length. According to Hooke's law, when the spring has reached a state of equilibrium, its elasticity is a simple linear profile of the amount by which the free end of the spring is displaced (when it is not stretched), that is length.

Independent variable has 6 levels by fixed values as 11, 12.5, 13.5, 15, 16, and 17 in centimeter. The lower and upper specification limits of the elasticity at each level of the length are provided in Table 16. To have asymmtric tolerance related to this servey, it is supposed to target value of each level is as $T_i = 2LSL_i + USL_i/3; \quad i = 1, 2, ..., 6$, as presented in Table 16. Based on these values, the functional specification limits and target lines are obtained as $LSL_Y(X) = 5.5377 - 0.3223X$, $USL_Y(X) = 4.8190 - 0.2464X$, and $T_Y(X) = 5.2980 - 0.2970X$.

Values of the elasticity of 9 springs in newton at different lengths are measured from the in-control process and shown in Table 17.

Table 11. Comparison among the previous PCIs and the new indices for asymmetric functional tolerance and $y_{ij} = 3.6 + 2.4x_i + \varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 1.0^2)$ true value		0.29244	0.27979	0.29231	0.27943
		$\widehat{C}_{pmk}(profile)$	$\widehat{C}'''_{pp}(profile)$	$\widehat{C}_{pmk.g}(profile)$	$\widehat{C}'''_{pp.g}(profile)$
		$(MAE\&MSE)$	$(MAE\&MSE)$	$(MAE\&MSE)$	$(MAE\&MSE)$
sample size	25	0.28633 (0.03285&0.00168)	0.27685 (0.03144&0.00154)	0.28618 (0.03282&0.00167)	0.27641 (0.03140&0.00154)
	50	0.28959 (0.02358&0.00087)	0.27858 (0.02264&0.00081)	0.28944 (0.02356&0.00087)	0.27818 (0.02260&0.00080)
	100	0.29075 (0.01668&0.00044)	0.27890 (0.01603&0.00040)	0.29061 (0.01667&0.00044)	0.27852 (0.01601&0.00040)
	200	0.29158 (0.01179&0.00022)	0.27931 (0.01132&0.00020)	0.29144 (0.01178&0.00022)	0.27894 (0.01131&0.00020)
$\varepsilon \sim N(0, 1.2^2)$ true value		0.25893	0.26272	0.25882	0.26237
sample size	25	0.25277 (0.03227&0.00161)	0.25856 (0.03269&0.00166)	0.25265 (0.03224&0.00162)	0.25812 (0.03266&0.00166)
	50	0.25603 (0.02320&0.00084)	0.26091 (0.02361&0.00088)	0.25592 (0.02319&0.00084)	0.26053 (0.02358&0.00087)
	100	0.25722 (0.01643&0.00042)	0.26152 (0.01675&0.00044)	0.25711 (0.01642&0.00042)	0.26115 (0.01674&0.00044)
	200	0.25806 (0.01162&0.00021)	0.26209 (0.01184&0.00022)	0.25795 (0.01161&0.00021)	0.26173 (0.01183&0.00021)

To check the normality of the springs lengths of collected sample data at each level, Anderson-Darling test is applied and the p-values are obtained as 0.7361, 0.5956, 0.2047, 0.9443, 0.2879, and 0.3516, respectively. Since all the p-values are greater than 0.05, it can be concluded that the data in each level are likely to follow normal distribution, at 95% significant level.

The reference line based on the least squaere error estimation is obtained as $Y(X) = 5.2340 - 0.2952X$, and MSE is measured to be 0.00456. The specification limits and target lines as well as the reference line are drawn in Figure 6.

To calculate the capability index $C'''_{pp.g}(profile)$, set the springs lengths as approximately 11, approximately 12.5, approximately 13.5, approximately 15, approximately 16, and approximately 17, with the following membership functions as shown in Figure 6.

$$\tilde{x}_1 = \begin{cases} \frac{11.75-x}{0.75}; & 11 \leq x < 11.75, \\ 0; & \text{otherwise.} \end{cases} \qquad \tilde{x}_6 = \begin{cases} \frac{x-16.5}{0.5}; & 16.5 < x \leq 17, \\ 0; & \text{otherwise.} \end{cases}$$

Table 12. Comparison among the previous PCIs and the new indices for asymmetric functional tolerance and $y_{ij} = 3.3 + 2.3x_i + \varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 0.5^2)$	true value	1.26913	1.22678	1.26900	1.22629
		$\widehat{C}_{pmk}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp}(profile)$ $(MAE\&MSE)$	$\widehat{C}_{pmk.g}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp.g}(profile)$ $(MAE\&MSE)$
sample size	25	1.21893 (0.08459&0.01072)	1.17154 (0.08519&0.01083)	1.21865 (0.08465&0.01074)	1.17253 (0.08436&0.01064)
	50	1.24406 (0.05561&0.00477)	1.20116 (0.05494&0.00465)	1.24384 (0.05563&0.00477)	1.20181 (0.05447&0.00457)
	100	1.25627 (0.03800&0.00225)	1.21634 (0.03699&0.00215)	1.25609 (0.03800&0.00225)	1.21678 (0.03675&0.00212)
	200	1.26288 (0.02655&0.00110)	1.22436 (0.02578&0.00104)	1.26273 (0.02655&0.00110)	1.22468 (0.02569&0.00104)
$\varepsilon \sim N(0, 0.8^2)$	true value	0.79543	0.77514	0.79540	0.77501
sample size	25	0.76368 (0.05642&0.00487)	0.73638 (0.05569&0.00457)	0.76354 (0.05646&0.00488)	0.73695 (0.05526&0.00450)
	50	0.77958 (0.03767&0.00221)	0.75597 (0.03539&0.00192)	0.77949 (0.03769&0.00221)	0.75632 (0.03517&0.00189)
	100	0.78722 (0.02591&0.00105)	0.76607 (0.02353&0.00087)	0.78716 (0.02592&0.00105)	0.76627 (0.02341&0.00086)
	200	0.79142 (0.01813&0.00052)	0.77142 (0.01619&0.00041)	0.79138 (0.01814&0.00052)	0.77155 (0.01613&0.00041)

$$\tilde{x}_2 = \begin{cases} \frac{x-11.75}{0.75}; & 11.75 \leq x < 12.5, \\ \frac{13-x}{0.5}; & 12.5 \leq x < 13, \\ 0; & \text{otherwise.} \end{cases} \qquad \tilde{x}_3 = \begin{cases} \frac{x-13}{0.5}; & 13 \leq x < 13.5, \\ \frac{14.25-x}{0.75}; & 13.5 \leq x < 14.25, \\ 0; & \text{otherwise.} \end{cases}$$

$$\tilde{x}_4 = \begin{cases} \frac{x-14.25}{0.75}; & 14.25 \leq x < 15, \\ \frac{15.5-x}{0.5}; & 15 \leq x < 15.5, \\ 0; & \text{otherwise.} \end{cases} \qquad \tilde{x}_5 = \begin{cases} \frac{x-15.5}{0.5}; & 15.5 \leq x < 16, \\ \frac{16.5-x}{0.5}; & 16 \leq x < 16.5, \\ 0; & \text{otherwise.} \end{cases}$$

TABLE 13. Comparison among the previous PCIs and the new indices for asymmetric functional tolerance and $y_{ij} = 3.3 + 2.3x_i + \varepsilon_{ij}$

		$C_{pmk}(profile)$	$C'''_{pp}(profile)$	$C_{pmk.g}(profile)$	$C'''_{pp.g}(profile)$
$\varepsilon \sim N(0, 1.0^2)$ true value		0.63676	0.62171	0.63674	0.62164
		$\widehat{C}_{pmk}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp}(profile)$ $(MAE\&MSE)$	$\widehat{C}_{pmk.g}(profile)$ $(MAE\&MSE)$	$\widehat{C}'''_{pp.g}(profile)$ $(MAE\&MSE)$
sample size	25	0.61126 (0.04749&0.00350)	0.58933 (0.04543&0.00303)	0.61115 (0.04752&0.00350)	0.58978 (0.04510&0.00299)
	50	0.62404 (0.03211&0.00161)	0.60548 (0.02871&0.00126)	0.62398 (0.03212&0.00161)	0.60574 (0.02854&0.00124)
	100	0.63012 (0.02219&0.00077)	0.61381 (0.01899&0.00056)	0.63008 (0.02219&0.00077)	0.61396 (0.01890&0.00056)
	200	0.63351 (0.01553&0.00038)	0.61823 (0.01301&0.00026)	0.63348 (0.01553&0.00038)	0.61832 (0.01297&0.00026)
$\varepsilon \sim N(0, 1.2^2)$ true value		0.53082	0.51882	0.53081	0.51878
sample size	25	0.50950 (0.041890&0.00274)	0.49080 (0.03857&0.00218)	0.50942 (0.04191&0.00274)	0.49117 (0.03830&0.00215)
	50	0.52020 (0.02864&0.00128)	0.50466 (0.02424&0.00090)	0.52015 (0.02864&0.00128)	0.50488 (0.02410&0.00088)
	100	0.52524 (0.01987&0.00062)	0.51184 (0.01596&0.00040)	0.52521 (0.01988&0.00062)	0.51196 (0.01589&0.00039)
	200	0.52808 (0.01392&0.00031)	0.51563 (0.01089&0.00018)	0.52806 (0.01392&0.00031)	0.51570 (0.01086&0.00018)

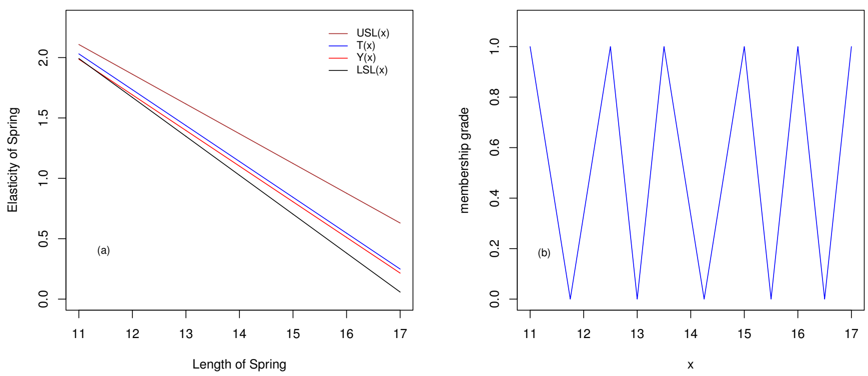

FIGURE 6. Lines of spring production data (a) and membership functions plots of fuzzy values of the springs lengths (b).

Table 14. Minimum sample sizes required based on the APE criterion for symmetric functional tolerance and $y_{ij} = 3+2x_i+\varepsilon_{ij}$, and differnet levels of confidence.

$1-\alpha$	APE_0									
	0.010	0.015	0.020	0.025	0.030	0.035	0.040	0.045	0.050	0.055
0.80	1196	565	318	200	142	104	86	71	55	46
0.85	1584	706	400	258	179	132	101	87	73	58
0.90	2017	914	516	332	231	172	131	105	91	79
0.95	2914	1215	721	465	322	241	185	146	119	101
	0.060	0.065	0.070	0.075	0.080	0.085	0.090	0.095	0.100	0.110
0.80	41	36	31	26	23	20	19	18	16	14
0.85	48	44	39	35	30	25	23	21	19	17
0.90	66	53	48	44	40	36	31	25	24	20
0.95	91	82	73	62	50	47	44	40	37	29
	0.120	0.130	0.140	0.150	0.160	0.170	0.180	0.190	0.200	
0.80	12	11	10	9	9	8	8	8	7	
0.85	14	13	12	10	10	9	9	9	8	
0.90	18	15	14	13	11	10	10	10	9	
0.95	23	20	18	15	14	13	12	10	10	

The $\widehat{C}'''_{pp.g}(profile)$ is obtained as 1.76651. Since this value is greater than 1, it is concluded that the process is capable.

9. Conclusion

In this article a new capability index $C'''_{pp}(profile)$ was introduced based on the functional scheme to assess the capability of simple linear profile and then, by setting levels of explanatory variable as fuzzy numbers and applying functional scheme, two more new indices $C_{pmk.g}(profile)$ and $C'''_{pp.g}(profile)$ were proposed.

Simulation scheme was engaged to investigate the performance of the proposed indices and compare them with the latest existing one, in terms of mean absolute error (MAE) and mean square error (MSE). Results showed that the index $C'''_{pp.g}(profile)$ performs the best in estimating the process capability.

Since determination sample size is an important issue in estimating the process capability, a method based on the absolute percentage error (APE) was presented to derive the optimal sample size.

In the present paper, the new indices were derived under the assumption that the values of response and explanatory variables and specification limits are crisp values. In some processes, data and information possess non-statistical uncertainties, so fuzzy set theory should be employed to deal with them. This subject will be investigated in the future inquiries.

Table 15. Minimum sample sizes required based on the APE criterion for asymmetric functional tolerance and $y_{ij} = 3.4 + 2.4x_i + \varepsilon_{ij}$, and differnet levels of confidence.

$1-\alpha$	APE_0									
	0.010	0.015	0.020	0.025	0.030	0.035	0.040	0.045	0.050	0.055
0.80	3394	1465	836	533	372	268	202	162	133	110
0.85	4200	1837	955	677	466	345	254	201	166	138
0.90	5532	2501	1383	886	610	442	344	272	217	178
0.95	8018	3572	1918	1252	887	636	493	385	317	258
	0.060	0.065	0.070	0.075	0.080	0.085	0.090	0.095	0.100	0.110
0.80	94	84	74	64	53	47	43	40	36	28
0.85	116	100	91	81	72	62	52	48	44	37
0.90	149	131	114	99	91	83	75	68	58	47
0.95	211	186	162	142	125	109	98	93	87	74
	0.120	0.130	0.140	0.150	0.160	0.170	0.180	0.190	0.200	
0.80	23	20	17	15	13	11	10	9	8	
0.85	31	25	21	19	16	14	13	11	10	
0.90	42	36	30	24	21	19	16	14	14	
0.95	60	49	44	40	34	28	24	21	19	

Table 16. Specification limits and target value of springs elasticity for each level of springs length

	X					
	11	12.5	13.5	15	16	17
LSL_i	1.9923	1.5089	1.1866	0.7031	0.3808	0.0585
USL_i	2.1086	1.7390	1.4926	1.1230	0.8766	0.6302
T_i	2.0311	1.5856	1.2886	0.8431	0.5461	0.2491

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Table 17. Springs elasticity at differnet levels of springs length

		X					
		11	12.5	13.5	15	16	17
Sample	1	2.1767	1.6667	1.3633	0.8600	0.5700	0.2467
	2	2.0533	1.5767	1.2467	0.7767	0.4900	0.2100
	3	1.9567	1.5300	1.2333	0.8100	0.5367	0.2600
	4	2.0400	1.5633	1.2600	0.7900	0.4800	0.1900
	5	1.8300	1.4467	1.2000	0.8000	0.5567	0.2800
	6	1.9767	1.5300	1.2267	0.7633	0.4667	0.1833
	7	2.0967	1.6333	1.3233	0.8267	0.5367	0.2467
	8	1.9900	1.5300	1.2200	0.7433	0.4067	0.1433
	9	1.7967	1.4100	1.1833	0.7833	0.5367	0.2600

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Zainab Abbasi Ganji
Orcid number: 0000-0003-1939-0080
Khorasan Razavi Agricultural and
Natural Resources Research and
Education Center, AREEO
Mashhad, Iran
Email address: z.ganji@areeo.ac.ir, abbasiganji@mail.um.ac.ir