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FROM STRUCTURES TO WEAK HYPERSTRUCTURES

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Abstract. The hyperstructures, born in 1934, are a generalization of the operation by the hyper-operation or multivalued operation. In 1970, the β^* relation, called now fundamental, connects structures with corresponding hyperstructures. In 1990, a generalization on the properties or axioms, which the hyper-operations fulfill, called weak properties, appeared. So the H_v-structures were born. The study of hyperstructures, especially of the weak ones, gives new topics in the field which have numerous of applications in mathematics and other sciences. Here, we present an overview of applications of H_v-structures.

Keywords: Weak hyper-operation, Weak axiom, H_v-structure, H_v-field.
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1. Hyperstructures and fundamental relations

F. Marty, in his pioneer paper in 1934, introduced the notion of Hypergroup. Then, several generalizations appeared and were studied in depth. The largest class of hyperstructures is the H_v-structures, named after T. Vougiouklis, introduced in 1990, 4th AHA Congress [20], [22]. They satisfy the weak axioms where the equality is replaced by the non-empty intersection. For definitions and applications, see books and papers such as [1], [2], [3], [4], [5], [6], [10], [11], [17], [19], [20], [21], [22], [23], [24], [37].

Definitions 1.1 A hyper-operation in a set H is called any map $(\cdot) : H \times H \rightarrow \mathcal{P}(H) - \{\emptyset\}$. A hyperstructure is called any set equipped with at least one hyper-operation.

In a set with a hyper-operation $(H, \cdot)$, weak associativity means

$$(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset, \quad \forall x, y, z \in H,$$

and weak commutativity means

$$x \cdot y \cap y \cdot x \neq \emptyset, \quad \forall x, y \in H.$$

A hyperstructure $(H, \cdot)$ is called an H_v-semigroup if it is weakly associative, and it is an H_v-group if, moreover, it is reproductive: $x \cdot H = H \cdot x = H, \forall x \in H$.

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The powers are defined in H_v-semigroups using the circle hyper-operation, which is the union of all hyper-products, with all patterns of parentheses on them.

Several general new definitions are given, first, for H_v-semigroups or H_v-groups which are extended analogously to stricter H_v-structures [22], [30].

Definitions 1.2 Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups. We call $(\cdot)$ smaller than $(*)$, and $(*)$ greater than $(\cdot)$, if there exists an automorphism f of $(H,*)$ for which

$$x\cdot y\subseteq f(x*y),\quad \forall x,y\in H.$$

We say that $(H,*)$ contains $(H,\cdot)$. A minimal H_v-group is one that does not contain any H_v-group.

Little Theorem. Hyper-operations containing weakly associative or commutative hyper-operations are weakly associative or weakly commutative, respectively.

Large classes of H_v-structures, which are near to the corresponding structure, are defined as follows [19].

Definition 1.3 An H_v-structure is very-thin if all hyper-operations are operations except one, with all results singletons except one, which is a set with more than one element.

The enumeration and classification of H_v-groups, as well as of all H_v-structures, defined in a set is complicated. We can see this even in sets with three elements. The number of H_v-groups with three elements, up to isomorphism, is 1.026.462. There are 7.926 abelian; the 1.013.598 are cyclic [10].

Hyperstructures are connected with the corresponding classical structures by the fundamental relations. In 1970, M. Koskas introduced the relation β^* in hypergroups [22]. In 1990, T. Vougiouklis introduced the relations γ^* and ε^*, for hyper-rings and hyper-vector spaces, respectively, by giving a new sort of proof of the main theorem, and named all of the fundamentals. First, we are referred to hypergroups [20], [22].

Definition 1.4 The fundamental relation β^* in H_v-groups is the smallest equivalence such that the quotient is a group.

The main theorem for H_v-groups is the following:

Theorem 1.5 Let $(H,\cdot)$ be an H_v-group and U be the set of finite products with elements of H. Define the relation β in H by

$$x\beta y \quad \text{if and only if} \quad \{x,y\}\subset u, \quad \text{where} \quad u\in U,$$

then, β^* is the transitive closure of β.

Proof. See [18], [20].

An element is called single if its fundamental class is a singleton.

If $(G,\cdot)$ is a group and R any partition in G, then $(G/R,\cdot)$ is an H_v-group and $(G/R,\cdot)/\beta^*$ is the fundamental group.

A related method is the following:

Definition 1.6 Uniting elements is a method to obtain strong structures, introduced by Corsini & Vougiouklis, using fundamental relations, as follows: Let

G be a structure where a property d is not valid. Take a partition in G and put in the same class all pairs for which d is not valid. G/d is an H_v-structure and $(G/d)/\beta^*$ is a structure for which d is valid [3], [22].
Motivation: In classical algebra, we know that the quotient of a group by an invariant subgroup is a group. In classical hyperstructures, the quotient of a group by a subgroup is a hypergroup. Finally, the quotient of a group by any partition is an H_v-group. It is clear that the number of 'partitions' is very great compared to the numbers of invariant subgroups and subgroups; this is the main reason for weak hyperstructures to admit many applications, even as organized devices.

2. Weak hyperstructures

The main problem of classical hyperstructures was to give general definitions for stricter hyperstructures such as hyper-rings, hyper-fields, hyper-vector spaces, and hyper-Lie algebras. Therefore, only special classes of hyperstructures appeared in research, such as Krasner hyper-rings, canonical groups, and polygroups. We overcome this problem in weak hyperstructures; thus, generally stricter weak hyperstructures are defined and new topics appeared, as well [10], [22].
Definitions 2.1 The H_v-structure $(R, +, \cdot)$ is an H_v-ring if both $(+)$ and $(\cdot)$ are weakly associative, $(+)$ is reproductive, and $(\cdot)$ is weakly distributive to $(+)$:

$$x \cdot (y + z) \cap (x \cdot y + x \cdot z) \neq \emptyset, \quad (x + y) \cdot z \cap (x \cdot z + y \cdot z) \neq \emptyset, \quad \forall x, y, z \in R.$$

A weak commutative H_v-group $(M, +)$ is called an H_v-module over an H_v-ring $(R, +, \cdot)$ if there is an external hyper-operation

$$\cdot : R \times M \to \mathcal{P}(M) - \{\emptyset\} : (a, x) \mapsto a \cdot x,$$

such that, $\forall a, b \in R, \forall x, y \in M$, we have

$$a\cdot(x+y)\cap(a\cdot x+a\cdot y) \neq \emptyset, \quad (a+b)\cdot x\cap(a\cdot x+b\cdot x) \neq \emptyset, \quad (a\cdot b)\cdot x\cap a\cdot(b\cdot x) \neq \emptyset.$$

In order to define the H_v-vector space, we need an H_v-field instead of an H_v-ring, which is defined later.

Recall that in 1990, T. Vougiouklis introduced the relations γ^* and ε^*, giving a new proof of the main theorem, and named all of them fundamentals [10], [22].
Definition 2.2 The fundamental relations γ^* and ε^* in an H_v-ring and H_v-vector space, respectively, are the smallest equivalences such that the quotient is a ring and vector space, respectively.

The main theorems on the topic are analogous to the following:
Theorem 2.3 Let $(R, +, \cdot)$ be an H_v-ring and U the set of all finite polynomials of R. Define the relation γ in R by

$$x\gamma y \quad \text{if and only if} \quad \{x, y\} \subset u, \quad \text{where} \quad u \in U,$$

then, γ^* is the transitive closure of γ.

Proof. A sort proof is the following: Let $\underline{\gamma}$ be the transitive closure of γ and denote $\underline{\gamma}(a)$ the class of a. In $R/\underline{\gamma}$ the hyper-operations $(\oplus)$, $(\otimes)$ are defined by

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \{\underline{\gamma}(c) : c \in \underline{\gamma}(a) + \underline{\gamma}(b)\},$$
$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \{\underline{\gamma}(d) : d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)\}, \quad \forall a, b \in R.$$

Take elements $a' \in \underline{\gamma}(a)$ and $b' \in \underline{\gamma}(b)$. Then, we obtain that: $a'\underline{\gamma}a$ means that there are elements $x_1, \dots, x_{m+1}$ such that

$$x_1 = a', \quad x_{m+1} = a, \quad u_1, \dots, u_m \in U, \quad \{x_i, x_{i+1}\} \subset u_i, \quad i = 1, \dots, m.$$

Similarly, $b'\underline{\gamma}b$ means that: there are elements $y_1, \dots, y_{n+1}$ such that

$$y_1 = b', \quad y_{n+1} = b, \quad v_1, \dots, v_n \in U, \quad \{y_j, y_{j+1}\} \subset v_j, \quad j = 1, \dots, n.$$

Thus, we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1, \quad i = 1, \dots, m-1, \quad x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j, \quad j = 1, \dots, n.$$

So, the $u_i + v_1 = t_i, i = 1, \dots, m-1, u_m + v_j = t_{m+j-1}, j = 1, \dots, n$, are polynomials of U.

Then, select elements $z_1, \dots, z_{m+n}$ such that $z_i \in x_i + y_1, i = 1, \dots, n, z_{m+j} \in x_{m+1} + y_{j+1}, j = 1, \dots, n$, from which we have $\{z_k, z_{k+1}\} \subset t_k, k = 1, \dots, m+n-1$. Consequently, any $z_1 \in x_1 + y_1 = a' + b'$ is γ equivalent to any $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$.

This proves that $\underline{\gamma}(a) \oplus \underline{\gamma}(b)$ is a singleton and we write $\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \underline{\gamma}(c), \forall c \in \underline{\gamma}(a) + \underline{\gamma}(b)$.

Similarly, $\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \underline{\gamma}(d), \forall d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)$.

From the weak associativity and distributivity on R, we obtain the associativity and distributivity in R/γ^*, so R/γ^* is a ring.

Let σ be an equivalence in R for which R/σ is a ring. Then, $\forall a, b \in R$, we have

$$\sigma(a) \oplus \sigma(b) = \sigma(c), \quad \forall c \in \sigma(a) + \sigma(b) \quad \text{and} \quad \sigma(a) \otimes \sigma(b) = \sigma(d), \quad \forall d \in \sigma(a) \cdot \sigma(b).$$

So, $\forall a, b \in R$ and $A \subset \sigma(a), B \subset \sigma(b)$,

$$\sigma(a) \oplus \sigma(b) = \sigma(a + b) = \sigma(A + B), \quad \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B).$$

By induction, extend the relations to finite sums and products. So, $\forall u \in U$ and $\forall x \in u$, we have $\sigma(x) = \sigma(u)$. Consequently, $x \in \gamma(a)$ implies $x \in \sigma(a), \forall x \in R$. But σ is transitively closed, so $x \in \underline{\gamma}(a)$ implies $x \in \sigma(a)$.

Therefore, $\underline{\gamma}$ is the smallest equivalence such that $R/\underline{\gamma}$ is a ring and $\underline{\gamma} = \gamma^*$.

Now we can give the following general definition:

Definition 2.4 An H_v-ring $(R, +, \cdot)$ is called an H_v-field if R/γ^* is a field. The elements of an H_v-field are called hyper-numbers or H_v-numbers.

Definition 2.5 The H_v-semigroup $(H, \cdot)$ is called an h/v-group if H/β^* is a group.

The h/v-group is a generalization of the H_v-group, where the new axiom reproductive of classes is valid.

The general definition of an H_v-Lie algebra was given as follows [10], [14]:

Definition 2.6 Let $(L, +)$ be an H_v-vector space over $(F, +, \cdot)$, $\varphi : F \to F/\gamma^*$ the canonical map, $\omega_F = \{x \in F : \varphi(x) = 0\}$, where 0 is zero of F/γ^*. Let ω_L be the core of $\varphi' : L \to L/\varepsilon^*$. Consider the bracket hyper-operation

$$[,] : L \times L \to \mathcal{P}(L) : (x, y) \mapsto [x, y]$$

then L is called an H_v-Lie algebra over F if the following axioms are valid:

(L1) The bracket hyper-operation is bilinear:

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \emptyset,$$
$$[\text{x}, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \emptyset,$$

$\forall x, x_1, x_2, y, y_1, y_2 \in L$ and $\forall \lambda_1, \lambda_2 \in F$

(L2) $[x, x] \cap \omega_L \neq \emptyset, \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \forall x, y \in L.$

The fundamental relations give simple, but very useful, results:

Theorem 2.7 Let $(H, \cdot)$ be an H_v-group and H/β^* its fundamental group. If H/β^* is not commutative or not cyclic, then $(H, \cdot)$ is not weakly commutative or cyclic, respectively.

Definition 2.8 We call a raised very-thin H_v-field one that is obtained from a ring by enlarging only one result and adding only one element, such that the fundamental structure is a field.

3. Classes of $H_v - structures$

Several hyper-operations based on ordinary operations are defined and studied deeply. Interesting classes are the following [17], [22], [27]:

Definition 3.1 Let (G,.) be a groupoid, then $\forall P \subset G$, $P \neq \phi$, the P-*hyper-product* is defined:

<u>P</u>: x.<u>P</u>.y=$(x.P).y \cup x.(P.y)$, $\forall x, y, G$.

Generalization: Let (G,.) be an abelian group, take $P \subset G$, $\mid P \mid > 1$. Define a hyper-product $(\times_P)$ by

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

We call this a P_e-hyper-product. $(G, \times_P)$ is an abelian H_v-group.

Definition 3.2 Let $(G, \cdot)$ be a groupoid, then for any map $f : G \to G$, the ∂-hyper-product is defined by

$x \partial y = \{f(x) \cdot y, x \cdot f(y)\}, \quad \forall x, y \in G.$

If $(\cdot)$ is commutative, then (∂) is weak commutative.

The motivation for this definition is the map derivative where only the product of functions can be used.

Let $(A, +, \cdot)$ be an algebra on F and $f : A \to A$, a map. The ∂-hyper-product on the Lie bracket $[x, y] = x \cdot y - y \cdot x$ is defined by

$x \partial y = \{f(x) \cdot y - f(y) \cdot x, f(x) \cdot y - y \cdot f(x), x \cdot f(y) - f(y) \cdot x, x \cdot f(y) - y \cdot f(x)\}.$

Enlargements of structures are considered in the sense that we add elements in some results. Useful enlargements in applications are those with known fundamental structures. This can be achieved since, in the weak case, we use the Little Theorem [25], [29], [30].

Definitions 3.3 We enlarge an operation if we add more elements to at least one result. The obtained H_v-structure is called enlarged. The special case is the enlarged very-thin.

The e-construction: Let $(G, \cdot)$ be a group, e the unit, we define hyper-products $(\otimes)$ by:

$x \otimes y = \{x \cdot y, g_1, g_2, \dots\}, \quad \forall x, y \in G - \{e\}$, and $g_1, g_2, \dots \in G - \{e\}$.

Then $(G, \otimes)$ is an H_v-group containing $(G, \cdot)$, called an e-H_v-group.

Example. In quaternions

$Q = \{1, -1, i, -i, j, -j, k, -k\}$, with $i^2 = j^2 = -1, i \cdot j = -j \cdot i = k$, denote $\underline{\mathbf{i}} = \{i, -i\}$, $\underline{\mathbf{j}} = \{j, -j\}$, $\underline{\mathbf{k}} = \{k, -k\}$.

We can define hyper-products $(*)$ by enlarging some results but remaining in the same class. Thus, we can take results like $(-1) * k = \underline{k}$, $k * i = \underline{j}$, $i * j = \underline{k}$. Then, in all those cases we obtain that $(Q, *)$ is an e-H_v-group.

Remark. In a field $(F, +, \cdot)$, if we enlarge any product $a \cdot b$, by $a \otimes b = \{a \cdot b, c\}$, $c \neq a \cdot b$, or any sum $a + b$, by $a \oplus b = \{a + b, c\}$, $c \neq a + b$, then we obtain the fundamental fields, respectively, $(F, +, \otimes)/\gamma^* \cong \{0\}$ or $(F, \oplus, \cdot)/\gamma^* \cong \{0\}$.

Therefore, there does not exist any non-degenerate enlarged H_v-field obtained from a field.

The interesting cases come if we enlarge a ring in order to obtain an H_v-field [29]:

Theorems 3.4 In the ring $(\mathbb{Z}_n, +, \cdot)$ with $n = m \cdot s$, if we enlarge $0 \cdot m$ by $0 \otimes m = \{0, m\}$, then we obtain $(\mathbb{Z}_n, +, \otimes)/\gamma^* \cong (\mathbb{Z}_m, +, \cdot)$. If we enlarge products as $2 \cdot m$ by setting $2 \otimes m = \{2 \cdot m, 3 \cdot m\}$, then 0, 1 are scalars. If $n = p \cdot s$, where p is prime, and we enlarge $0 \cdot p$ by $0 \otimes p = \{0, p\}$, then we obtain the very-thin H_v-field $(\mathbb{Z}_n, +, \otimes)$.

In applications, small H_v-fields which satisfy additional axioms are used. Thus, appropriate lists of all small H_v-fields derived from ordinary rings are obtained. For example, we present a list of multiplicative H_v-fields which are very-thin minimal, weak commutative and have scalar 0 and 1 [29], [30], [33]:

Constructions 3.5

(a) On $(\mathbb{Z}_4, +, \cdot)$, the H_v-fields are isomorphic to $2 \otimes 3 = \{0, 2\}$ or $3 \otimes 2 = \{0, 2\}$, we have $(\mathbb{Z}_4, +, \otimes)/\gamma^* \cong (\mathbb{Z}_2, +, \cdot)$, and fundamental classes: $[0] = \{0, 2\}$, $[1] = \{1, 3\}$.

(b) On $(\mathbb{Z}_6, +, \cdot)$ the H_v-fields are isomorphic to $2 \otimes 3 = \{0, 3\}$, $2 \otimes 4 = \{2, 5\}$, $3 \otimes 4 = \{0, 3\}$, $3 \otimes 5 = \{0, 3\}$, $4 \otimes 5 = \{2, 5\}$, we have $(\mathbb{Z}_6, +, \otimes)/\gamma^* \cong (\mathbb{Z}_3, +, \cdot)$, and fundamental classes: $[0] = \{0, 3\}$, $[1] = \{1, 4\}$, $[2] = \{2, 5\}$.

(c) On $(\mathbb{Z}_6, +, \cdot)$ the H_v-fields are isomorphic to $2 \otimes 3 = \{0, 2\}$ or $\{0, 4\}$, $2 \otimes 4 = \{0, 2\}$ or $\{2, 4\}$, $2 \otimes 5 = \{0, 4\}$ or $\{2, 4\}$, $3 \otimes 4 = \{0, 2\}$ or $\{0, 4\}$,

$3 \otimes 5 = \{3, 5\}$, $4 \otimes 5 = \{0, 2\}$ or $\{2, 4\}$, we have $(\mathbb{Z}_6, +, \otimes)/\gamma^* \cong (\mathbb{Z}_2, +, \cdot)$, and fundamental classes: $[0] = \{0, 2, 4\}$, $[1] = \{1, 3, 5\}$.

The helix-operations are weak hyper-operations defined on non-square matrices on classical rings or fields [9], [15], [16], [35], [36].

Definitions 3.6 On a matrix $A = (a_{ij}) \in M_{m\times n}$, $s, t \in \mathbb{N}$, $1 \leq s \leq m$, $1 \leq t \leq n$, define the map helix-$\underline{st}$

$\underline{st} : M_{m\times n} \to M_{s\times t} : A \to A_{\underline{st}} = (\underline{a_{ij}})$,

where $\underline{a_{ij}} = \{a_{i+\kappa s, j+\lambda t} \mid 1 \leq i \leq s, 1 \leq j \leq t, \kappa, \lambda \in \mathbb{N}, i+\kappa\cdot s \leq m, j+\lambda\cdot t \leq n\}$.

(a) Let $A = (a_{ij}) \in M_{m\times n}$, $B = (b_{ij}) \in M_{u\times v}$, and $s = \min(m, u)$, $t = \min(n, v)$. The helix-sum is

$\oplus : M_{m\times n} \times M_{u\times v} \to \mathcal{P}(M_{s\times t}) : (A, B) \to A \oplus B = A_{\underline{st}} + B_{\underline{st}} = (\underline{a_{ij}}) + (\underline{b_{ij}}) \subset M_{s\times t}$,

where $(\underline{a_{ij}}) + (\underline{b_{ij}}) = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in \underline{a_{ij}}$ and $b_{ij} \in \underline{b_{ij}}\}$.

(b) Let $A = (a_{ij}) \in M_{m\times n}$, $B = (b_{ij}) \in M_{u\times v}$, and $s = \min(n, u)$. The helix-product is

$\otimes : M_{m\times n} \times M_{u\times v} \to \mathcal{P}(M_{m\times v}) : (A, B) \to A \otimes B = A_{\underline{ms}} \cdot B_{\underline{sv}} = (\underline{a_{ij}}) \cdot (\underline{b_{ij}}) \subset M_{m\times v}$,

where $(\underline{a_{ij}}) \cdot (\underline{b_{ij}}) = \{(c_{ij}) = (\sum a_{it} \cdot b_{tj}) \mid a_{ij} \in \underline{a_{ij}}$ and $b_{ij} \in \underline{b_{ij}}\}$.

The helix-sum is commutative and the helix-product is weak associative.

We study only matrices $M_{m\times n}$ where $m < n$, since there are analogous results when $m > n$. Several sets of matrices are interesting for the helix-product: A matrix $A = (a_{ij}) \in M_{m\times n}$ is an S-helix, if $A_{\underline{mm}}$ is a set of upper triangular matrices with diagonal entries singletons. An S-helix matrix is S_0-helix if the condition $a_{11} \cdot \cdots \cdot a_{mm} \neq 0$ is valid. The set of S_0-helix matrices of type $m \times n$ is closed under the helix-product where the S_0-helix matrix X has inverses X^{-1}, i.e., if I_c is the unit matrix, $I_c \in X \otimes X^{-1} \cap X^{-1} \otimes X$.

Example 3.7 Consider the matrices of the type 3×5 on real or complex numbers or on any finite field. We take, in the 7th dimension case,

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{11} & x_{15} \\ 0 & x_{22} & x_{23} & 0 & x_{22} \\ 0 & 0 & x_{33} & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{15} \\ 0 & y_{22} & y_{23} & 0 & y_{22} \\ 0 & 0 & y_{33} & 0 & 0 \end{pmatrix},$$

so, we have

$$X \otimes Y = \begin{pmatrix} x_{11} & \{x_{12}, x_{15}\} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{15} \\ 0 & y_{22} & y_{23} & 0 & y_{22} \\ 0 & 0 & y_{33} & 0 & 0 \end{pmatrix}.$$

Then, denoting C_{ij} the ij entry of the result, we have the non-zero results:

$C_{11} = C_{14} = \{x_{11} \cdot y_{11}\}$,
$C_{12} = \{x_{11} \cdot y_{12} + x_{12} \cdot y_{22}, x_{11} \cdot y_{12} + x_{15} \cdot y_{22}\}$,
$C_{13} = \{x_{11} \cdot y_{13} + x_{12} \cdot y_{23} + x_{13} \cdot y_{33}, x_{11} \cdot y_{13} + x_{15} \cdot y_{23} + x_{13} \cdot y_{33}\}$,
$C_{15} = \{x_{11} \cdot y_{15} + x_{12} \cdot y_{22}, x_{11} \cdot y_{15} + x_{15} \cdot y_{22}\}$,
$C_{22} = C_{25} = \{x_{22} \cdot y_{22}\}$,
$C_{23} = \{x_{22} \cdot y_{23} + x_{23} \cdot y_{33}\}$,
$C_{33} = \{x_{33} \cdot y_{33}\}$,

Therefore, the helix product is a set with cardinality up to 2^3.

4. Representations of H_v-groups

The Representation Theory of hypergroups was started by T. Vougiouklis in the middle of the '80s. At that time, it was a hard problem since there was not a standard definition even for hyper-rings. The representations of H_v-groups began immediately in 1990, together with the weak hyperstructures, again by T. Vougiouklis [10], [22], [23], [26], [33]. The representations can be achieved either by H_v-matrices or by generalized permutations [21], [23]. The representation problem by H_v-matrices is the following [26], [33]:

Definitions 4.1 A matrix is called an H_v-matrix if it has entries from an H_v-field $(F, +, \cdot)$. The hyper-product of H_v-matrices $A = (a_{ij})$, $B = (b_{ij})$, of type $m \times n$, $n \times r$, respectively, is defined as usual but it is a set of $m \times r$ H_v-matrices:

$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{C = (c_{ij}) \mid c_{ij} \in \bigoplus \sum a_{ik} \cdot b_{kj}\}$,

where $(\bigoplus)$ is the n-ary circle hyper-operation, which means the union of all possible patterns of parentheses put for elements, on the hyper-sum.

Let $(H, \cdot)$ be an H_v-group. An H_v-matrix representation on $M_F = \{(a_{ij}) \mid a_{ij} \in F\}$ is called any map

$T : H \to M_F : h \mapsto T(h)$ such that $T(h_1 \cdot h_2) \cap T(h_1) \cdot T(h_2) \neq \emptyset, \quad \forall h_1, h_2 \in H$.

If $T(h_1 \cdot h_2) \subseteq T(h_1) \cdot T(h_2)$ is valid, then T is called an inclusion representation, and is called faithful if it is one-to-one and $T(h_1 \cdot h_2) = T(h_1) \cdot T(h_2) = \{T(h) \mid h \in h_1 \cdot h_2\}, \quad \forall h_1, h_2 \in H$.

The main theorem of the representations, which in fact connects structures with hyperstructures, is the following [18], [22], [23].

Theorem 4.2 A necessary condition to have an inclusion representation T of an H_v-group $(H, \cdot)$ by $n \times n$, H_v-matrices over the H_v-ring $(R, +, \cdot)$ is the following:

$\forall \beta^*(x), x \in H$ there must exist $a_{ij} \in H, i, j \in \{1, \ldots, n\}$ such that
$T(\beta^*(a)) \subseteq \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, \ldots, n\}\}$.

Every inclusion representation $T : H \to M_R : a \mapsto T(a) = (a_{ij})$ induces a representation T^* of the fundamental group H/β^* over the fundamental ring R/γ^*, by setting

$T^*(\beta^*(a)) = [\gamma^*(a_{ij})], \quad \forall \beta^*(a) \in H/\beta^*$,

where $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$.

The representation theory of H_v-structures needs special classes of H_v-groups, H_v-fields, and H_v-vector spaces. On the other side, new classes and objects in mathematics itself appeared which are interesting in the research.

On H_v-structures one can define several hyper-operations: Let $M = M_{m\times n}$ be a module of $m \times n$ matrices over a ring R and $P = \{P_i : i \in I\} \subseteq M$. We define a kind of a P-hyper-product $\underline{P}$ on M as follows

$\underline{P} : M \times M \to \mathcal{P}(M) : (A, B) \mapsto A \cdot P \cdot B = \{A \cdot P_i^t \cdot B : i \in I\} \subseteq M$

where P^t is the transpose of P. P is a bilinear Rees' product where, instead of one sandwich matrix, a set is used. P is strongly associative and inclusion distributive to addition:

$A \cdot \underline{P} \cdot (B + C) \subseteq A \cdot \underline{P} \cdot B + A \cdot \underline{P} \cdot C, \quad \forall A, B, C \in M$

The Lie-Santilli admissible can be defined in H_v-structures, as well [8], [10]:

Definition 4.3 Let L be an H_v-vector space on $(F, +, \cdot)$, $\varphi : F \to F/\gamma^*$ the canonical map with core $\omega_F = \{x \in F : \varphi(x) = 0\}$, 0 the zero of F/γ^*, ω_L the core of $\varphi' : L \to L/\varepsilon^*$. Take subsets $R, S \in L$, then a *Lie-Santilli admissible* hyper-algebra is obtained by taking the Lie-bracket $[,]_{RS} : L \times L \to \mathcal{P}(L)$: $[x, y]_{RS} = (x \cdot R) \cdot y - (y \cdot S) \cdot x$

Special cases:

(a) Take only S, then $[x, y]_S = x \cdot y - y \cdot S \cdot x$,
(b) Take only R, then $[x, y]_R = x \cdot R \cdot y - y \cdot x$.

The importance in the hyper case comes from Santilli admissible, since we transfer this theory to representations in two ways: using either ordinary matrices with a hyper-product, or hyper-matrices with an ordinary product.

The admissible on non-square matrices is defined as follows:

Definition 4.4 Let $(L = M_{m\times n}, +)$ be the H_v-vector space of $m \times n$ hyper-matrices on $(F, +, \cdot)$, $\varphi : F \to F/\gamma^*$, the canonical map, $\omega_F = \{x \in F : \varphi(x) = 0\}$, and ω_L the core of $\varphi' : L \to L/\varepsilon^*$. Take $R, S \subseteq L$, then we obtain a *Lie-Santilli admissible hyper-algebra* with Lie bracket

$[,]_{RS} : L \times L \to \mathcal{P}(L) : [x, y]_{RS} = x \cdot R^t \cdot y - y \cdot S^t \cdot x$.

Notice that $[x, y]_{RS} = x \cdot R^t \cdot y - y \cdot S^t \cdot x = \{x \cdot r^t \cdot y - y \cdot s^t \cdot x \mid r \in R \text{ and } s \in S\}$.

In hyper-matrix representations, results with small cardinality are needed. Thus, small H_v-fields are used, obtained from structures enlarging one operation [10], [25], [29], [30], [33].

Example 4.5 Take the H_v-field $(\mathbb{Z}_6, +, \cdot)$, with $2 \otimes 4 = \{2, 5\}$. Fundamental classes $[0] = \{0, 3\}$, $[1] = \{1, 4\}$, $[2] = \{2, 5\}$, and $(\mathbb{Z}_6, +, \otimes)/\gamma^* \cong (\mathbb{Z}_3, +, \cdot)$. Take the 2×2 H_v-matrices on $(\mathbb{Z}_6, +, \otimes)$

$$\begin{pmatrix} 1 & \mathbb{Z}_6 \\ 0 & 4 \end{pmatrix}.$$

Thus, we have 6 elements:
$c_0 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, $c_1 = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}$, $c_2 = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$, $c_3 = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$, $c_4 = \begin{pmatrix} 1 & 4 \\ 0 & 4 \end{pmatrix}$, $c_5 = \begin{pmatrix} 1 & 5 \\ 0 & 4 \end{pmatrix}$.

We obtain the table:

$\otimes$	$\mathbf{c_0}$	$\mathbf{c_1}$	$\mathbf{c_2}$	$\mathbf{c_3}$	$\mathbf{c_4}$	$\mathbf{c_5}$
$\mathbf{c_0}$	c_0	c_1	c_2	c_3	c_4	c_5
$\mathbf{c_1}$	c_4	c_5	c_0	c_1	c_2	c_3
$\mathbf{c_2}$	c_2, c_5	c_0, c_3	c_1, c_4	c_2, c_5	c_0, c_3	c_1, c_4
$\mathbf{c_3}$	c_0	c_1	c_2	c_3	c_4	c_5
$\mathbf{c_4}$	c_4	c_5	c_0	c_1	c_2	c_3
$\mathbf{c_5}$	c_2	c_3	c_4	c_5	c_0	c_1

The small H_v-structure $(C, \otimes)$, $C = \{b_0, b_1, b_2, b_3, b_4, b_5\}$, is an H_v-group, where the reproductive on fundamental classes is valid. $(\otimes)$ is weak associative as we can see

$(c_1 \otimes c_4) \otimes c_2 = c_2 \otimes c_2 = \{c_1, c_4\}$ and $c_1 \otimes (c_4 \otimes c_2) = c_1 \otimes c_0 = c_4$.

Moreover, it is an interesting hyper-structure as we see some of its properties: It is cyclic with generators c_2 and c_4; it has subgroups with one element $\{c_0\}$, $\{c_3\}$; c_1, c_3 are generators of the subgroup with 3 elements, given by the table

$\otimes$	$\mathbf{c_1}$	$\mathbf{c_3}$	$\mathbf{c_5}$
$\mathbf{c_1}$	c_5	c_1	c_3
$\mathbf{c_3}$	c_1	c_3	c_5
$\mathbf{c_5}$	c_3	c_5	c_1

5. **Applications in Hadronic Mechanics**

The most important application of H_v-structures is on the isotopy Lie-Santilli theory, defined in the 1960s to solve Hadronic Mechanics problems. The main object is the H_v-field corresponding to the isofield, introduced by Santilli & Vougiouklis, which is described in the following [8], [10], [12], [13], [14], [28], [33].

The Lie-Santilli *isotopy* is a *lifting* of the n-dimensional unit matrix of a normal theory into a nowhere singular, symmetric, positive-defined, n-dimensional new matrix. The original theory is reconstructed to admit the new matrix as a unit.

Definitions 5.1 $(H, \cdot)$ is an *e-hyperstructure* if it contains a unique scalar unit e, and all elements x have an inverse x^{-1}: $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

$(F, +, \cdot)$, where $(+)$ is an operation, and $(\cdot)$ is a hyper-operation, is an e-hyper-field if: $(F, +)$ is an abelian group, $(\cdot)$ is weakly associative, $(\cdot)$ is weakly distributive to $(+)$, 0 is absorbing, 1 is scalar, and every element has a unique inverse. The elements of $(F, +, \cdot)$ are called e-hyper-numbers.

Moreover, we focus on H_v-fields on $(\mathbb{Z}_n, +, \cdot)$, satisfying in iso-theory the conditions:

(1) very-thin minimal, (2) weak commutative, (3) 0, 1 scalars, (4) unique inverses.

Thus, we enlarge the product putting one more element where we cannot enlarge the result if it is 1 and we do not put 1 in the enlargement. It is clear that this theory can be faced in H_v-structures because there are many axioms to be valid.

According to Santilli iso-theory, on a field $F = (F, +, \cdot)$, an iso-field $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ is defined, with *iso-numbers*: $\hat{a} = a \times \hat{1}$, $a \in F$, $\hat{1}$ positive-defined outside F, with the sum $\hat{+}$, with unit 0, and $\hat{\times}$ is a new product:

$\hat{a}\hat{\times}\hat{b} := \hat{a} \times \hat{T} \times \hat{b}, \quad \text{with} \quad \hat{1} = \hat{T}^{-1}, \quad \forall \hat{a}, \hat{b} \in \hat{F}$

called *iso-product*, for which $\hat{1}$ is the left and right unit of $\hat{F}$,

$\hat{1}\hat{\times}\hat{a} = \hat{a} \times \hat{1} = \hat{a}, \quad \forall \hat{a} \in \hat{F}$

called *iso-unit*. The rest of the properties of a field are reformulated analogously.

This theory is transferred to hyperstructures by replacing only the product $\hat{\times}$ by a hyper-product including the old one. Two generalizations are used:

Definition 5.2 On an iso-field $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ we replace in the results of the iso-product

$\hat{a}\hat{\times}\hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \quad \text{where} \quad \hat{1} = \hat{T}^{-1}$

of $\hat{T}$ by a set $\hat{H}_{ab} = \{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\}, \quad \hat{x}_1, \hat{x}_2, \dots \in \hat{F} - \{\hat{0}, \hat{1}\}$,

$\forall \hat{a}\hat{\times}\hat{b}$ with $\hat{a}, \hat{b} \notin \{\hat{0}, \hat{1}\}$.

If $\hat{a}$ or $\hat{b}$ is equal to $\hat{0}$ or $\hat{1}$, then take $\hat{H}_{ab} = \{\hat{T}\}$.

So, the new iso-hyper-product is

$\hat{a}\hat{\times}\hat{b} = \hat{a} \times \hat{H}_{ab} \times \hat{b} = \hat{a} \times \{\hat{T}, \hat{x}_1, \hat{x}_2, \dots\} \times \hat{b}, \quad \forall \hat{a}, \hat{b} \in \hat{F}$,

and $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})b$ is an iso-H_v-field. The elements of $\hat{F}$ are called *iso*$-H_v$-numbers.

The most important case is the *very-thin iso-*H_v-field, where $\hat{H}_{ab} = \{\hat{T}, \hat{x}\}$.

Definition 5.3 Take iso-field $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$, $\hat{a} = a \times \hat{1}$, $a \in F$, with sum $\hat{+}$ and iso-product $\hat{\times}$:

$\hat{a}\hat{\times}\hat{b} := \hat{a} \times \hat{T} \times \hat{b}, \quad \text{with } \hat{1} = \hat{T}^{-1}, \quad \forall \hat{a}, \hat{b} \in \hat{F}$.

Let $\hat{P} = \{\hat{T}, \hat{p}_1, \dots, \hat{p}_s\}$, $\hat{p}_1, \dots, \hat{p}_s \in \hat{F} - \{\hat{0}, \hat{1}\}$, define isoP-$H_v$-field, $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times}_P)$, with $\hat{\times}_P$:

$$\hat{a}\hat{\times}_P\hat{b} := \begin{cases} \hat{a} \times \hat{P} \times \hat{b} = \{\hat{a} \times \hat{h} \times \hat{b} \mid \hat{h} \in \hat{P}\} & \text{if } \hat{a} \neq \hat{1} \text{ and } \hat{b} \neq \hat{1} \\ \hat{a} \times \hat{T} \times \hat{b} & \text{if } \hat{a} = \hat{1} \text{ or } \hat{b} = \hat{1}. \end{cases}$$

The elements of $\hat{F}$ are called *isoP-*H_v*-numbers*. Remark that if $\hat{P} = \{\hat{T}, \hat{p}\}$, the inverses in isoP-$H_v$-fields are not unique.

In the next example, we apply the P-construction, with $\hat{P} = \{\hat{T}, \hat{p}\}$, to obtain an H_v-field.

Example 5.4 For $\hat{\mathbf{Z}}_{10} = \mathbf{Z}_{10}(\underline{\hat{a}}, \hat{+}, \hat{\times})$, and if we take $\hat{P} = \{\underline{\hat{2}}, \underline{\hat{7}}\}$, we have the table:

$\hat{\times}$	$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{\mathbf{6}}}$	$\underline{\hat{\mathbf{7}}}$	$\underline{\hat{\mathbf{8}}}$	$\underline{\hat{\mathbf{9}}}$
$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$
$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{9}}$
$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$
$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{0}}$	$\underline{\hat{3}}$	$\underline{\hat{2}}$	$\underline{\hat{3}},\underline{\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{2}},\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{4}},\underline{\hat{9}}$
$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$
$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{0}}$	$\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0}},\underline{\hat{5}}$
$\underline{\hat{\mathbf{6}}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$
$\underline{\hat{\mathbf{7}}}$	$\underline{\hat{0}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{2}},\underline{\hat{7}}$	$\underline{\hat{6}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{4}}$	$\underline{\hat{3}},\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{1}},\underline{\hat{6}}$
$\underline{\hat{\mathbf{8}}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$
$\underline{\hat{\mathbf{9}}}$	$\underline{\hat{0}}$	$\underline{\hat{9}}$	$\underline{\hat{6}}$	$\underline{\hat{4}},\underline{\hat{9}}$	$\underline{\hat{2}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{8}}$	$\underline{\hat{1}},\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}},\underline{\hat{7}}$

Then the fundamental classes are
$[\hat{0}] = \{\underline{\hat{0}}, \underline{\hat{5}}\}$, $[\hat{1}] = \{\underline{\hat{1}}, \underline{\hat{6}}\}$, $[\hat{2}] = \{\underline{\hat{2}}, \underline{\hat{7}}\}$, $[\hat{3}] = \{\underline{\hat{3}}, \underline{\hat{8}}\}$, $[\hat{4}] = \{\underline{\hat{4}}, \underline{\hat{9}}\}$,
and the multiplicative table, excluding the zero class [0], is the following:

$\times$	$[\mathbf{1}]$	$[\mathbf{2}]$	$[\mathbf{3}]$	$[\mathbf{4}]$
$[\mathbf{1}]$	$[1],[2]$	$[2],[4]$	$[3],[1]$	$[4],[3]$
$[\mathbf{2}]$	$[2],[4]$	$[3]$	$[4]$	$[1]$
$[\mathbf{3}]$	$[3],[1]$	$[4]$	$[1]$	$[2]$
$[\mathbf{4}]$	$[4],[3]$	$[1]$	$[2]$	$[3]$

Thus, $\hat{\mathbf{Z}}_{10} = \mathbf{Z}_{10}(\underline{\hat{a}}, \hat{+}, \hat{\times})$ is an H_v-field.

6. Applications in Education and Arts

H_v-structures have many applications in other branches of mathematics and sciences. These applications range from hadronic physics, iso-theory, leptons, bio-mathematics, cryptography, and linguistics, to mention but a few. They are related to fuzzy theory; so, they can be applicable in industry and production, too. In books and papers [1], [2], [4], [5], [6], [7], [10], [11], [28], [34], [37], one can find numerous applications.

An application of H_v-structures in physics, leptons, was faced by Davvaz. The Standard Model is a theory to describe the elementary particles and the interacting forces between them. On the set of Leptons, using their interactions as a hyper-operation, we form the appropriate table from which we obtain an H_v-group. In cases such as the multiplicative table of Leptons, where we have a hyper-product, it is a hard job to check the associative or weak-associative properties. To do this, we use computers and special programs. Such a program was used to obtain that the set of Leptons is an H_v-group.

H_v-structures have applications in Biology, as one can see in the book [10]. These can be achieved in collaboration with biologists who point out the objects, as well as the reactions between them. The next step is to find appropriate H_v-structures, and the final step is to find which of the properties these special H_v-structures have can be applied. Among these applications, we present the following:

Example 6.1 Take the set of all blood types people may have: $H = \{O, A, B, AB\}$. Then take the operation ($\otimes$), where 'in the blood types of parents correspond to the types their child may have'. The ($\otimes$) is a hyperproduct, as the research states, given by the table:

$\otimes$	**O**	**A**	**B**	**AB**
O	O	O, A	O, B	A, B
A	O, A	O, A	O, A, B, AB	A, B, AB
B	O, B	O, A, B, AB	O, B	A, B, AB
AB	A, B	A, B, AB	A, B, AB	A, B, AB

The weak associativity, not strong, is valid, as we can see in, for example,
$(O \otimes B) \otimes AB = \{O, B\} \otimes AB = \{A, B, AB\}$, $\quad O \otimes (B \otimes AB) = O \otimes \{A, B, AB\} = \{O, A, B\}$.

Thus, $(H, \otimes)$ is an H_v-semigroup, and $(\{O, A\}, \otimes)$, $(\{O, B\}, \otimes)$ are hypergroups.

If we extend the blood types by using (+) and (-), then we have the set
$H = \{O+, O-, A+, A-, B+, B-, AB+, AB-\}$,
on which we obtain again that $(H, \otimes)$ is an H_v-semigroup.

In teaching, education, art, and philosophy, there are several applications of hyperstructures, which come from properties, mainly the weak ones, they have [16], [32], [34].

In arts, we can see how a sculptor, taking a piece of marble, constructs a statue. By using the chisel, he throws away all the useless pieces of marble. Thus, he does exactly what mathematicians do in weak hyperstructures, with the axioms, expel the useless hyperstructures. This is the same method which we use in Lie-Santilli admissible theory.

We cannot find the fundamental classes in an analytic way since they depend on all results of all hyper-operations. Thus, we need new proofs and special elements. We need a special proof in order to discover the 'reason' why we have these results. Any relation uses even the last one result to determine its classes. If there are special elements, as the singles, which are strictly formed and carry inside them the relation, then these elements form the classes. We call this procedure 'judging from the results proof' and it looks like the 'reductio ad absurdum' proof [31], [32].

The main question in helix-operations is the following: Can we use the real meaning of them in other sciences under similar circumstances? The helix-product acts as follows: It replaces and shifts elements together with the corresponding ones, treating them in the same way. This is a modulo-like procedure and reminds us of the repetition in teaching or the motivo in music compositions [9], [15].

In applications, especially in physics, researchers need strict and complicated hyperstructures; thus, several wide classes were introduced even from examples.

Denote E_{ij} the matrix with 1 in the ij entry and 0 elsewhere.

Example 6.2 Take the 2×2 upper triangular H_v-matrices on the H_v-field $(\mathbb{Z}_4, +, \otimes)$ with $3 \otimes 2 = \{0, 2\}$.
In the set $X = \{a, a_1, a_2, a_3, b, b_1, b_2, b_3, c, c_1, c_2, c_3, d, d_1, d_2, d_3\}$ with:

$$\begin{aligned}
a = E_{11} + E_{22}, \quad & a_1 = E_{11} + E_{12} + E_{22}, \quad a_2 = E_{11} + 2E_{12} + E_{22},\\
& a_3 = E_{11} + 3E_{12} + E_{22},\\
b = E_{11} + 3E_{22}, \quad & b_1 = E_{11} + E_{12} + 3E_{22}, \quad b_2 = E_{11} + 2E_{12} + 3E_{22},\\
& b_3 = E_{11} + 3E_{12} + 3E_{22},\\
c = 3E_{11} + E_{22}, \quad & c_1 = 3E_{11} + E_{12} + E_{22}, \quad c_2 = 3E_{11} + 2E_{12} + E_{22},\\
& c_3 = 3E_{11} + 3E_{12} + E_{22},\\
d = 3E_{11} + 3E_{22}, \quad & d_1 = 3E_{11} + E_{12} + 3E_{22}, \quad d_2 = 3E_{11} + 2E_{12} + 3E_{22},\\
& d_3 = 3E_{11} + 3E_{12} + 3E_{22},
\end{aligned}$$

we have: $(X, \otimes)$ is a weak commutative H_v-group, fundamental classes $\underline{a} = \{a, a_2\}$, $\underline{a_1} = \{a_1, a_3\}$, $\underline{b} = \{b, b_2\}$, $\underline{b_1} = \{b_1, b_3\}$, $\underline{c} = \{c, c_2\}$, $\underline{c_1} = \{c_1, c_3\}$, $\underline{d} = \{d, d_2\}$, $\underline{d_1} = \{d_1, d_3\}$, and there is the unit a, and every element has a unique double inverse. The c_2 is the right inverse to c and c_2, and the d_2 is the right inverse to d and d_2. $(X, \otimes)$ is not cyclic, since $(\underline{X}, \underline{\otimes})$ is not cyclic.

7. Conclusions

On the way from structures to hyperstructures, the fundamental relations are always used. The generalization of axioms to the corresponding weak ones leads to the largest class of hyperstructures, the H_v-structures. The number of H_v-structures defined on a set is extremely big; therefore, they admit a lot of applications in applied sciences and in pure mathematics as well. In order to find, in applied sciences, an appropriate model from weak hyperstructures, one needs more axioms. This is the reason that the H_v-structures can give models to strict and complex sciences such as hadronic mechanics and biology.

A numerous of sciences, ranging from physics up to medicine and DNA, ask from H_v-structures mathematical models. Some of them are very hard projects and need the collaboration of several scientists. On the other side, new concepts and structures appeared in mathematics themselves, such as the e-hyperstructures, which are very interesting to be studied.

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