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GENERALIZED EXTROPY OF k-RECORDS: PROPERTIES AND APPLICATIONS

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Abstract. In this paper, we introduce a generalized measure of extropy based on k-records and study its properties. We show that several existing extropies, such as survival, negative cumulative, past and weighted extropy are special cases of this generalized measure of extropy. We also propose a dynamic generalized measure of extropy based on k-records which includes residual extropy, dynamic survival extropy and weighted dynamic survival extropy. A generating function is discussed using this generalized extropy measure, using which we provide different extropy and entropy measures. Some important properties of generalized extropy of k-records and generating function are derived. We use simulation to assess the bias and mean squared error of the estimator of the generalized extropy and compute its values for real data.

Keywords: Generalized extropy, Generating function, Survival extropy, Weighted extropy.
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1. Introduction

Extropy, in a broad sense, is a term used to describe a theoretical measure of a system's intelligence, order, or complexity. It is often considered as the opposite or counterpoint to entropy, which measures the degree of disorder or randomness in a system. While entropy tends to increase over time in closed systems, the concept of extropy imagines a tendency toward increased order or organization. The idea of extropy is often associated with discussions about the future of intelligent life, technological progress, and the potential for systems to evolve towards higher levels of complexity and organization. It's used in speculative discussions about the trajectory of advanced civilizations and the possibilities for continued growth and improvement. Extropy of a non-negative random variable X that is absolutely continuous with probability density function (pdf) $f_X(x)$ is defined by [14] as the complementary point of entropy of [23]. This measure is defined as follows

$$J(X) = -\frac{1}{2}\int_0^{\infty} f_X^2(x)dx.$$

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[19] introduced extropy based on survival function $\bar{F}_X(x)$ called survival extropy and considered its dynamic form. Moreover, [20] defined the weighted extropy and the weighted residual extropy with weight x. [7] defined weighted survival extropy with weight $w(.) \geq 0$. [24] proposed negative cumulative extropy with cumulative distribution function (cdf) $F_X(x)$ and [7] defined its weighted state.

In many real situations, the uncertainty of a random phenomenon may be related to the past. Therefore [12] suggested a measure called past extropy as

$$\tilde{J}_t(X) = -\frac{1}{2F_X^2(t)} \int_0^t f_X^2(x)dx,$$

and [20] proposed its weighted state with weight x.

In many real life situations, the next highest or next lowest value of the variable under study is of great importance and this can be a basis for studying information measures such as record values and k-records arising from a random sample. The concept of k-records was introduced in [5].

Suppose $\{X_i, i = 1, 2, ..., n\}$ is a sequence of independent and identically distributed (iid) random variables and $X_{p:q}$ is the pth order statistic in a random sample of size q. For a positive integer k, $T^U_{n(k)}$ for $n = 1, 2, ...$ are denoted the times at which upper k-record values occur and are defined by $T^U_{1(k)} = k$ and, for $n \geq 2$, by $T^U_{n(k)} = \min\{j; j > T^U_{n-1(k)}, X_j > X_{T^U_{n-1(k)}-k+1:T^U_{n-1(k)}}\}$. Moreover, $\{U_{n(k)} = X_{T^U_{n(k)}-k+1:T^U_{n(k)}}\}$ are defined as the sequence of upper k-record values. If the parent distribution is absolutely continuous with survival function $\bar{F}$ and pdf f, then the pdf of the nth upper k-record value, $U_{n(k)}$, is given by (see [2])

$$f_{n(k)}(x) = \frac{k^n}{\Gamma(n)}[-\ln \bar{F}_X(x)]^{n-1}[\bar{F}_X(x)]^{k-1}f(x), \quad n = 1, 2, ... \tag{1}$$

Similarly, we can define the lower k-records. For a positive integer k, $T^L_{n(k)}$ for $n = 1, 2, ...$ are denoted the times at which lower k-record values occur and are defined by $T^L_{1(k)} = k$ and $T^L_{n(k)} = \min\{j; j > T^L_{n-1(k)}, X_j < X_{T^L_{n-1(k)}-k+1:T^L_{n-1(k)}}\}$. Then, we define the sequence of lower k-records denoted by $L_{n(k)}$ as $\{L_{n(k)} = X_{T^L_{n(k)}-k+1:T^L_{n(k)}}\}$. If the parent distribution is absolutely continuous with cdf F and pdf f, then the pdf of nth lower k-record value, $L_{n(k)}$, is given by (see [1])

$$f_{n(k)}(x) = \frac{k^n}{\Gamma(n)}[-\ln F_X(x)]^{n-1}[F_X(x)]^{k-1}f(x), \quad n = 1, 2, ... \tag{2}$$

[9, 10] proposed residual extropy and past extropy based on k-records. Also, they [11] gave a characterization result of symmetric distribution using extropy of nth upper k-record value and nth lower k-record value. [3] suggested weighted extropies of order statistics and k-record values. [8] defined weighted extropy of ranked set sampling. Generalized extropy in a general case based

on k-records that covers other extropies has not been introduced so far. We define a generalized extropy and study its properties based on k-records and show all of its special cases in Section 2. In Section 3, we introduce dynamic generalized extropy based on both upper and lower k-records. Then, we derive its properties and illustrate some special cases. In Section 4, we discuss a generating function using generalized extropy of k-records and obtain some of its properties. In Section 5 Based on a simulation study, we evaluated the performance of the estimator of the generalized extropy by examining its bias and mean squared error (MSE) for sample sizes of 20, 30, and 50. Furthermore, we applied the proposed estimator to real-world data to demonstrate its practical applicability and compute its. Finally, we make some concluding remarks in Section 6.

2. generalized extropy based on k-records

In this section, we introduce generalized extropy measure. Several extropy measures are special cases of this proposed measure.

Definition 2.1. Let $\{X_i, i = 1, 2, ..., n\}$ be a sequence of iid random variables with pdf f and cdf F and $\phi_{f_{n(k)}}$ be a weight function. Then, we define the generalized extropy (GEX) of nth upper k-record value, denoted by $\mathcal{G}J(U_{n(k)})$, as

$$\mathcal{G}J(U_{n(k)}) = -\frac{1}{2}\int_0^\infty \phi_{f_{n(k)}}(x) f_{n(k)}(x)dx, \tag{3}$$

where $f_{n(k)}(x)$ is pdf of the upper k-record values given in (1).

We note that the weight function $\phi_{f_{n(k)}}$ must satisfy measurability, integrability, and support alignment conditions to ensure the well-definedness and applicability of the proposed extropy measures. As a special case, put $k = 1$ and $n = 1$, then, the expression in (3) is

$$\mathcal{G}J(X) = -\frac{1}{2}\int_0^\infty \phi_f(x) f_X(x)dx.$$

We show that the GEX in (3) covers the above mentioned extropies based on k-records with special choices of $\phi_{f_{n(k)}}$.
First, [14] obtains with $\phi_{f_{n(k)}}(x) = f_{n(k)}(x)$, the survival extropy [19], with a choice of $\phi_{f_{n(k)}}(x) = \int_0^x \bar{F}_{n(k)}(t)dt$ where $\bar{F}_{n(k)}$ is the survival function of the upper k-record values. With a choice of $\phi_{f_{n(k)}}(x) = \int_0^x w(t)\bar{F}_{n(k)}(t)dt$, we have weighted survival extropy [7]. With a choice of $\phi_{f_{n(k)}}(x) = -\int_0^x (1 + F_{n(k)}(t))dt$, $\mathcal{G}J(U_{n(k)})$ reduces to the negative cumulative extropy of [24] based on k-records. With $\phi_{f_{n(k)}}(x) = -\int_0^x w(t)(1 + F_{n(k)}(t))dt$, we have introduced extropy [7]. Furthermore, the weighted extropy [20], with $\phi_{f_{n(k)}}(x) = x f_{n(k)}(x)$ is simply achieved. The special cases of $\mathcal{G}J(U_{n(k)})$ discussed here are all listed in Table 1.

Table 1. Special cases of GEX based on k-records

Extropy Measure	*authors*	$\phi_{f_{n(k)}}(x)$
Extropy	[14]	$f_{n(k)}(x)$
Survival extropy	[19]	$\int_0^x \bar{F}_{n(k)}(t)dt$
Weighted Survival extropy	[7]	$\int_0^x w(t)\bar{F}_{n(k)}(t)dt$
Negative cumulative extropy	[24]	$-\int_0^x (1+F_{n(k)}(t))dt$
Weighted Negative cumulative extropy	[7]	$-\int_0^x w(t)(1+F_{n(k)}(t))dt$
Weighted extropy	[20]	$xf_{n(k)}(x)$

In the following theorem, we derive an expression for the GEX of nth upper k-record value.

Theorem 2.2. *Let $\{X_i,\ i \geq 1\}$ be a sequence of iid random variables with cdf F and pdf f. Let $U_{n(k)}$ denote the nth upper k-record value of the sequence $\{X_i\}$. The GEX of $U_{n(k)}$ is given by*

$$\mathcal{G}J(U_{n(k)}) = -\frac{1}{2}E\Big[\phi_{f_{n(k)}}\big(F^{-1}(1-e^{-V_n})\big)\Big], \tag{4}$$

where V_n has the gamma distribution with parameters n and k.

Proof. Substituting (1) in (3), we get

$$\mathcal{G}J(U_{n(k)}) = -\frac{k^n}{2\Gamma(n)}\int_0^\infty \phi_{f_{n(k)}}(x)[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx.$$

Using the transformation $-\ln \bar{F}(x) = u$, we get

$$\begin{aligned}\mathcal{G}J(U_{n(k)}) &= -\frac{k^n}{2\Gamma(n)}\int_0^\infty \phi_{f_{n(k)}}\big(F^{-1}(1-e^{-u})\big)u^{n-1}e^{-uk}du \\ &= -\frac{1}{2}E\Big[\phi_{f_{n(k)}}\big(F^{-1}(1-e^{-V_n})\big)\Big].\end{aligned}$$

Hence the theorem. □

Example 2.3. *Let $\{X_i,\ i \geq 1\}$ be a sequence of iid random variables with a common distribution* U$(0,1)$*. Let $U^*_{n(k)}$ denote the nth upper k-record value* *arising from the sequence* $\{X_i,\ i \geq 1\}$. *Then, we obtain*

$$\begin{aligned}\mathcal{G}J(U^*_{n(k)}) &= -\frac{1}{2}\int_0^1 \phi_{f^*_{n(k)}}(x)\frac{k^n}{\Gamma(n)}[-\ln(1-x)]^{n-1}(1-x)^{k-1}dx \\ &= -\frac{1}{2}E\Big(\phi_{f^*_{n(k)}}(1-e^{-V_n})\Big),\end{aligned}$$

where V_n *has the gamma distribution with parameters* n *and* k.

Example 2.4. *Let* $\{X_i, i \geq 1\}$ *be a sequence of iid random variables having common exponential distribution with pdf* $f(x) = \theta e^{-\theta x},\ x > 0$ *and cdf* $F(x) = 1 - e^{-\theta x},\ x > 0$. *We have* $F^{-1}(x) = -\frac{1}{\theta}\ln(1-x)$ *and suppose* $\phi_{f_{n(k)}}(x) = xf_{n(k)}(x)$. *We have*

$$\phi_{f_{n(k)}}\big(F^{-1}(1-e^{-u})\big) = \frac{k^n}{\Gamma(n)}u^n e^{-ku}.$$

Then, we get

$$\mathcal{G}J(U_{n(k)}) = -\frac{1}{2}E\Big[\phi_{f_{n(k)}}\big(F^{-1}(1-e^{-V_n})\big)\Big] = -\frac{\Gamma(2n)}{2^{2n+1}\Gamma^2(n)}.$$

Further, we state the following theorem for the GEX of nth lower k-records. The proof is not included since it is similar to that of Theorem 2.2.

Theorem 2.5. *Within the framework of Theorem 2.1, Let* $L_{n(k)}$ *denote its* n*th lower* k*-record value. Then, the GEX of* $L_{n(k)}$ *is given by*

$$\mathcal{G}J(L_{n(k)}) = -\frac{1}{2}E\Big[\phi_{f_{n(k)}}\big(F^{-1}(e^{-V_n})\big)\Big], \tag{5}$$

where V_n *has the gamma distribution with parameters* n *and* k.

For deriving the properties of GEX of the nth upper and lower k-records, we state the definition of dispersive, usual stochastic and likelihood ratio order defined in [21].

Definition 2.6. Let X and Y be two non-negative random variables with distribution functions F and G, pdf f and g, survival functions $\bar{F}$ and $\bar{G}$, respectively. The random variable X is said to be less than or equal to Y,

(1) in the dispersive ordering, denoted by $X \leq_{disp} Y$, if $g(G^{-1}(\alpha)) \leq f(F^{-1}(\alpha))$ for all $\alpha \in (0,1)$,
(2) in usual stochastic (st) ordering, denoted by $X \leq_{st} Y$, if $\bar{F}(\alpha) \leq \bar{G}(\alpha)$ for every $\alpha \geq 0$,
(3) in the likelihood ratio order, denoted by $X \leq_{lr} Y$, if $\frac{f(\alpha)}{g(\alpha)}$ is decreasing in $\alpha \geq 0$.

It is well known that $X \leq_{lr} Y \Rightarrow X \leq_{st} Y$ and $X \leq_{st} Y$ if and only if $E[\varphi(X)] \leq E[\varphi(Y)]$ for all increasing functions φ.

Theorem 2.7. *Let X and Y be two non-negative random variables with distribution functions F and G, respectively and with probability density functions f and g, respectively.*
(a) If $\phi_{f_{n(k)}}$ is increasing in x, $\phi_{f_{n(k)}}(x) \geq \phi_{g_{n(k)}}(x)$ and $X \leq_{disp} Y$, then $\mathcal{G}J(U^X_{n(k)}) \leq \mathcal{G}J(U^Y_{n(k)})$.
(b) If $\phi_{f_{n(k)}}$ is increasing in x, $\phi_{f_{n(k)}}(x) \leq \phi_{g_{n(k)}}(x)$ and $X \geq_{disp} Y$, then $\mathcal{G}J(U^X_{n(k)}) \geq \mathcal{G}J(U^Y_{n(k)})$.

Proof. (a) Using Theorem $3.B.13(b)$ of [21], $X \leq_{disp} Y$ implies that $X \geq_{st} Y$. So we have $F^{-1}(1-e^{-V_n}) \geq G^{-1}(1-e^{-V_n})$ for $0 < 1-e^{-V_n} < 1$. Hence under the hypothesis of the theorem it follows that $\phi_{f_{n(k)}}(F^{-1}(1-e^{-V_n})) \geq \phi_{f_{n(k)}}(G^{-1}(1-e^{-V_n})) \geq \phi_{g_{n(k)}}(G^{-1}(1-e^{-V_n}))$. The proof then follows from (4).
(b) On similar arguments as in part (a), the result follows. □

Example 2.8. *Let X and Y be two non-negative random variables. For $n = k = 1$, $\phi_f(x) = \int_0^x \bar{F}(t)dt$, $\phi_g(x) = \int_0^x \bar{G}(t)dt$. $X \leq_{disp} Y$ implies that $X \geq_{st} Y$, so $\bar{F}(x) \geq \bar{G}(x)$ for every $x \geq 0$ and we have $\phi_f(x) \geq \phi_g(x)$. Thus, $\mathcal{G}J(X) \leq \mathcal{G}J(Y)$.*

Example 2.9. *Let X and Y be two non-negative random variables. For $n = k = 1$, $\phi_f(x) = xf(x)$, $\phi_g(x) = xg(x)$, $\mathcal{G}J(X)$ and $\mathcal{G}J(Y)$ are the weighted extropies and we have*

$$\mathcal{G}J(X) = -\frac{1}{2}\int_0^\infty x f^2(x)dx = -\frac{1}{2}\int_0^1 F^{-1}(u)f(F^{-1}(u))du.$$

Now, $X \leq_{disp} Y$ implies $f(F^{-1}(u)) \geq g(G^{-1}(u))$ and $F^{-1}(u) \geq G^{-1}(u)$ for $0 < u < 1$. Thus, $\mathcal{G}J(X) \leq \mathcal{G}J(Y)$ which is presented as a Corollary in [8].

First, in the following lemma we present a complete orthogonal system. (see [6])

Lemma 2.10. *A complete orthogonal system for the space $L_2(0,\infty)$ is given by a sequence of Laguerre functions*

$$\varphi_n(x) = \frac{1}{n!}e^{-\frac{x}{2}}L_n(x), \qquad n \geq 0,$$

where $L_n(x)$ is the Laguerre polynomial defined as the sum of coefficients of e^{-x} in the nth derivative of $x^n e^{-x}$, that is,

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^{ne^{-x}}).$$

The completeness of Laguerre functions in $L_2(0,\infty)$ means that if $f \in L_2(0,\infty)$ and $\forall n \geq 0$, $\int_0^\infty f(x)e^{-\frac{x}{2}}L_n(x)dx$ $= 0$ implies f is zero almost everywhere.

Theorem 2.11. *Assuming that the function ϕ is one-to-one, the variables X and Y with the survival functions $\bar{F}$ and $\bar{G}$ belong to the same location family of distributions if and only if*

$$\mathcal{G}J(U^Y_{n(k)}) = \mathcal{G}J(U^X_{n(k)}) \tag{6}$$

Proof. The necessity is obvious. For the sufficient part, from $\mathcal{G}J(U^Y_{n(k)}) = \mathcal{G}J(U^X_{n(k)})$, we have

$$\int_0^\infty \phi_{f_{n(k)}}\big(F^{-1}(1-e^{-u})\big)u^{n-1}e^{-uk}du$$
$$= \int_0^\infty \phi_{g_{n(k)}}\big(G^{-1}(1-e^{-u})\big)u^{n-1}e^{-uk}du.$$

Thus we have

$$\int_0^\infty \Big[\phi_{f_{n(k)}}\big(F^{-1}(1-e^{-u})\big) - \phi_{g_{n(k)}}\big(G^{-1}(1-e^{-u})\big)\Big]u^{n-1}e^{-uk}du = 0.$$

The above expression can be rewritten as

$$\int_0^\infty \Big[\phi_{f_{n(k)}}\big(F^{-1}(1-e^{-u})\big) - \phi_{g_{n(k)}}\big(G^{-1}(1-e^{-u})\big)\Big](e^{\frac{u}{2}-uk})e^{-\frac{u}{2}}L_n(u)du = 0.$$

Using Lemma (2.10), we get

$$\phi_{f_{n(k)}}\big(F^{-1}(1-e^{-u})\big) = \phi_{g_{n(k)}}\big(G^{-1}(1-e^{-u})\big).$$

By setting $v = 1 - e^{-u}$, we have

$$\phi_{f_{n(k)}}\big(F^{-1}(v)\big) = \phi_{g_{n(k)}}\big(G^{-1}(v)\big).$$

Hence, we conclude $F^{-1}(v) = G^{-1}(v)$, because of the common support of the variables X and Y, the desired result follows. □

The following theorem shows the effect of monotone transformations on the GEX of k-record values. Let the variable X have its cdf and pdf as F and f, respectively.

Theorem 2.12. *Let X be a non-negative random variable and let $Y = \Phi(X)$ where Φ is a strictly increasing function with $\Phi(0) = 0$ and $\Phi(\infty) = \infty$, with pdf $g(y) = \frac{f(\Phi^{-1}(y))}{\Phi'(\Phi^{-1}(y))}$ and cdf $G(y) = F(\Phi^{-1}(y))$, where Φ' is the derivative of function Φ. Then, the GEX of the nth upper k-record value corresponding to Y is given by*

$$\mathcal{G}J(U^Y_{n(k)}) = -\frac{1}{2}E\Big[\phi_{g_{n(k)}}\big(\Phi(F^{-1}(1-e^{-V_n}))\big)\Big]. \tag{7}$$

Here, V_n has the gamma distribution with parameters n, k and $g_{n(k)}(y)$ is as

$$g_{n(k)}(y) = \frac{k^n}{\Gamma(n)}[-\ln \bar{F}(\Phi^{-1}(y))]^{n-1}\bar{F}(\Phi^{-1}(y))^{k-1}\frac{f(\Phi^{-1}(y))}{\Phi'(\Phi^{-1}(y))}.$$

Proof. Using the transformations $x = \Phi^{-1}(y)$ and $-\ln \bar{F}(x) = u$, it follows easily from the proof of Theorem 2.2. □

Example 2.13. *Let non-negative random variable X have an exponential distribution with the pdf and cdf*

$$f(x) = \lambda e^{-\lambda x}, \qquad F(x) = 1 - e^{-\lambda x}.$$

If $Y = X^2$ with pdf and cdf respectively $g(y) = \frac{\lambda e^{-\lambda\sqrt{y}}}{2\sqrt{y}}$ and $G(y) = 1 - e^{-\lambda\sqrt{y}}$, we obtain the GEX of Y for $n, k = 1$, $\phi_g(y) = g(y)$ as

$$\mathcal{G}J(Y) = -\frac{\lambda^2}{4}.$$

3. Dynamic GEX based on k-records

This section deals with the definition of dynamic generalized extropy in two parts including generalized residual extropy (GREX) and generalized past extropy (GPEX) based on k-records. We will express some of the properties of dynamic generalized extropy.

Definition 3.1. Under the assumptions of Theorem 2.2, we define the GREX of nth upper k-record value, denoted by $\mathcal{G}J_t(U_{n(k)})$, as

$$\mathcal{G}J_t(U_{n(k)}) = -\frac{1}{2\bar{F}^2_{n(k)}(t)} \int_t^\infty {\phi^t}_{f_{n(k)}}(x) f_{n(k)}(x)dx. \tag{8}$$

If $n = 1$ and $k = 1$, the equation (8) is as follows

$$\mathcal{G}J_t(X) = -\frac{1}{2\bar{F}^2(t)} \int_t^\infty {\phi^t}_f(x) f(x)dx.$$

We show (8) reduces to the residual extropy of [16], the dynamic survival extropy and weighted dynamic survival extropy of [18, 19] based on k-records for some specific choices of $\phi^t_{f_{n(k)}}(.)$. With ${\phi^t}_{f_{n(k)}}(x) = f_{n(k)}(x)$, we have $\mathcal{G}J_t(U_{n(k)}) = -\frac{1}{2\bar{F}^2_{n(k)}(t)} \int_t^\infty f^2_{n(k)}(x)dx$ that it is the same as the residual extropy. With a choice of ${\phi^t}_{f_{n(k)}}(x) = \int_t^x \bar{F}_{n(k)}(u)du$, we have the dynamic survival extropy of [19]. For the weighted dynamic survival extropy of [18], we choose ${\phi^t}_{f_{n(k)}}(x) = \int_t^x u\bar{F}_{n(k)}(u)du$.

Definition 3.2. Assuming the conditions stated in Theorem 2.2, we define the generalized past extropy (gpex) of nth upper k-record value, denoted by $\mathcal{G}\tilde{J}_t(U_{n(k)})$, as

$$\mathcal{G}\tilde{J}_t(U_{n(k)}) = -\frac{1}{2F^2_{n(k)}(t)} \int_0^t {\phi^t}_{f_{n(k)}}(x) f_{n(k)}(x)dx. \tag{9}$$

The gpex in (9) covers some extropies based on k-records with special choices of ${\phi^t}_{f_{n(k)}}(.)$. The past extropy proposed by [12] that for k-records presented by [10], with a choice of ${\phi^t}_{f_{n(k)}}(x) = f_{n(k)}(x)$, we get $\mathcal{G}\tilde{J}_t(U_{n(k)}) = -\frac{1}{2F^2_{n(k)}(t)}\int_0^t f^2_{n(k)}(x)dx$. With choice of ${\phi^t}_{f_{n(k)}}(x) = \int_x^t F_{n(k)}(u)du$, we get the dynamic cumulative extropy introduced by [13]. The weighted dynamic cumulative extropy defined by [18] based on k-records, with ${\phi^t}_{f_{n(k)}}(x) = \int_x^t uF_{n(k)}(u)du$, we have $\mathcal{G}\tilde{J}_t(U_{n(k)}) = -\frac{1}{2F^2_{n(k)}(t)}\int_0^t uF^2_{n(k)}(u)du$. The results are listed in Table 2.

Now, we have expressed the GREX of k-record values as the product of the

Table 2. Special cases of dynamic generalized extropy based on k-records

Extropy Measure	authors	${\phi^t}_{f_{n(k)}}(x)$
residual extropy	[16]	$f_{n(k)}(x)$
dynamic survival extropy	[19]	$\int_t^x \bar{F}_{n(k)}(u)du$
weighted dynamic survival extropy	[18]	$\int_t^x uF_{n(k)}(u)du$
past extropy	[12]	$f_{n(k)}(x)$
dynamic cumulative extropy	[13]	$\int_x^t F_{n(k)}(u)du$
weighted dynamic cumulative extropy	[18]	$\int_x^t uF_{n(k)}(u)du$

GREX extropy of k-records arising from uniform distribution and the ratio of two expectations of a truncated gamma distributed random variable.

Lemma 3.3. *Let $\{X_i,\ i \geq 1\}$ be a sequence of iid random variables with a common distribution* U(0, 1). *Let $U^*_{n(k)}$ denote the nth upper k-record value arising from the sequence $\{X_i,\ i \geq 1\}$. Then the GREX of $U^*_{n(k)}$ is given by*

$$\mathcal{G}J_t(U^*_{n(k)}) = -\frac{\Gamma(n)}{2\Gamma(n, -k\ln(1-t))}E\big[{\phi^t}_{f^*_{n(k)}}(1 - e^{-V_n})\big], \tag{10}$$

where $f^*_{n(k)}$ is the pdf of $U^*_{n(k)}$ and $V_n \sim \gamma_{-\ln(1-t)}(n, k)$ where $\gamma_t(\alpha, \lambda)$ is the left truncated gamma distribution with pdf

$$h_t(x) = \frac{\lambda^\alpha}{\Gamma(\alpha, \lambda t)}x^{\alpha-1}e^{-\lambda x}, \qquad x > t > 0, \quad \alpha, \lambda > 0,$$

and $\Gamma(.,.)$ is the upper incomplete gamma function given by
$\Gamma(a, x) = \int_x^\infty u^{a-1}e^{-u}du,\ \ a, x > 0.$

Proof. Substituting (1) in (8), we have $\mathcal{G}J_t(U^*_{n(k)}) =$

$$= -\frac{k^n\Gamma^2(n)}{2\Gamma^2(n,-k\ln(1-t))\Gamma(n)}\int_t^1 {\phi^t}_{f^*_{n(k)}}(x)[-\ln(1-x)]^{n-1}[1-x]^{k-1}dx$$
$$= -\frac{k^n\Gamma(n)}{2\Gamma^2(n,-k\ln(1-t))}\int_{-\ln(1-t)}^{\infty} {\phi^t}_{f^*_{n(k)}}(1-e^{-u})u^{n-1}e^{-uk}du.$$

The desired result follows by integrating the above equation. □

In the following theorem, we formulate the GREX of upper k-records arising from any continuous distribution in terms of the GREX of upper k-records arising from U$(0,1)$.

Theorem 3.4. *In accordance with the assumptions of Theorem 2.2, the GREX of $U_{n(k)}$ is given by*

$$\mathcal{G}J_t(U_{n(k)}) = \mathcal{G}J_{F(t)}(U^*_{n(k)})\frac{E\Big[{\phi^t}_{f_{n(k)}}\big(F^{-1}(1-e^{-V_n})\big)\Big]}{E\big[{\phi^t}_{f^*_{n(k)}}(1-e^{-V_n})\big]}. \tag{11}$$

Here, $U^*_{n(k)}$ denote the nth upper k-record value arising from U$(0,1)$ *and $V_n \sim \gamma_{-\ln\bar{F}(t)}(n,k)$.*

Proof. Substituting (1) in (8), we get

$$\mathcal{G}J_t(U_{n(k)}) = -\frac{k^n\Gamma(n)}{2\Gamma^2(n,-k\ln\bar{F}(t))}\int_t^\infty {\phi^t}_{f_{n(k)}}(x)[-\ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)dx$$

Using the transformation $-\ln\bar{F}(x) = u$, we get

$$\mathcal{G}J_t(U_{n(k)}) = -\frac{k^n\Gamma(n)}{2\Gamma^2(n,-k\ln\bar{F}(t))}\int_{-\ln\bar{F}(t)}^\infty {\phi^t}_{f_{n(k)}}\big(F^{-1}(1-e^{-u})\big)u^{n-1}e^{-uk}du,$$

$$= -\frac{\Gamma(n)}{2\Gamma(n,-k\ln\bar{F}(t))}E\Big[{\phi^t}_{f_{n(k)}}\big(F^{-1}(1-e^{-V_n})\big)\Big]. \tag{12}$$

According to Lemma 3.3, the result follows. □

In the following examples, we apply Theorem 3.4 and Lemma 3.3 to obtain GREX with various ${\phi^t}_{f_{n(k)}}(x)$.

Example 3.5. *Under the assumptions of Theorem 2.2, let ${\phi^t}_{f_{n(k)}}(x) = f_{n(k)}(x)$, hence, using Lemma 3.3, we obtain $\mathcal{G}J_t(U^*_{n(k)}) = -\frac{k^{2n}}{2(2k-1)^{2n-1}}\frac{\Gamma(2n-1,-(2k-1)\ln(1-t))}{\Gamma^2(n,-k\ln(1-t))}$ and using Theorem 3.4, we get*

$$\mathcal{G}J_t(U_{n(k)}) = -\frac{k^{2n}}{2(2k-1)^{2n-1}}\frac{\Gamma(2n-1,-(2k-1)\ln\bar{F}(t))}{\Gamma^2(n,-k\ln\bar{F}(t))}E\Big[f\big(F^{-1}(1-e^{-V_n})\big)\Big]$$
$$= \mathcal{G}J_{F(t)}(U^*_{n(k)})E\Big[f\big(F^{-1}(1-e^{-V_n})\big)\Big],$$

where $V_n \sim \gamma_{-\ln\bar{F}(t)}(2n-1,2k-1)$.

Example 3.6. *Subject to the assumptions of Theorem 2.2, let* $\phi^t{}_{f_{n(k)}}(x) = \int_t^x \bar{F}_{n(k)}(u)du$, *we obtain* $\phi^t{}_{f_{n(k)}}\big(F^{-1}(1-e^{-v})\big)$ *in* (11) *as*

$$\phi^t{}_{f_{n(k)}}\big(F^{-1}(1-e^{-v})\big) = \int_{-\ln \bar{F}(t)}^{v} \frac{\Gamma(n,kz)}{\Gamma(n)} e^{-z} f^{-1}\big(F^{-1}(1-e^{-z})\big)dz.$$

We get

$$\mathcal{G}J_t(U^*_{n(k)}) = -\frac{\Gamma(1,-\ln(1-t))}{2\Gamma^2(n,-k\ln(1-t))} E\Big[\Gamma^2(n,Zk)\Big],$$

where $Z \sim \gamma_{-\ln(1-t)}(1,1)$ *and we have*

$$\begin{aligned}\mathcal{G}J_t(U_{n(k)}) &= -\frac{1}{2\Gamma^2(n,-k\ln\bar{F}(t))}\int_{-\ln\bar{F}(t)}^{\infty} \Gamma^2(n,zk)e^{-z}f^{-1}\big(F^{-1}(1-e^{-z})\big)dz \\ &= -\frac{\Gamma(1,-\ln\bar{F}(t))}{2\Gamma^2(n,-k\ln\bar{F}(t))}E\Big[\Gamma^2(n,Zk)f^{-1}\big(F^{-1}(1-e^{-Z})\big)\Big],\end{aligned}$$

where $Z \sim \gamma_{-\ln\bar{F}(t)}(1,1)$, *Thus*

$$\mathcal{G}J_t(U_{n(k)}) = \mathcal{G}J_{F(t)}(U^*_{n(k)})\frac{E\Big[\Gamma^2(n,Zk)f^{-1}\big(F^{-1}(1-e^{-Z})\big)\Big]}{E\Big[\Gamma^2(n,Zk)\Big]}.$$

Further, we state the following theorem for the GREX of nth lower k-records. The proof is not included since it is similar to that of Theorem 3.4.

Lemma 3.7. *Let* $\{X_i,\ i \geq 1\}$ *be a sequence of iid random variables with a common distribution* U(0, 1). *Let* $L^*_{n(k)}$ *denote the nth lower k-record value arising from the sequence* $\{X_i,\ i \geq 1\}$. *Then the GREX of* $L^*_{n(k)}$ *is given by*

$$\mathcal{G}J_t(L^*_{n(k)}) = -\frac{\Gamma(n)}{2\rho(n,-k\ln t)}E\big[\phi^t{}_{f^*_{n(k)}}(e^{-V_n})\big],$$

where $\rho(a,x)$ *is the lower incomplete gamma function given by* $\rho(a,x) = \int_0^x u^{a-1}e^{-u}du;\quad a,x>0.$

Theorem 3.8. *Assuming the conditions stated in Theorem 2.5, the GREX of* $L_{n(k)}$ *is given by*

$$\mathcal{G}J_t(L_{n(k)}) = \mathcal{G}J_{F(t)}(L^*_{n(k)})\frac{E\Big[\phi^t{}_{f_{n(k)}}\big(F^{-1}(e^{-V_n})\big)\Big]}{E\big[\phi^t{}_{f^*_{n(k)}}(e^{-V_n})\big]}. \tag{13}$$

Here, $L^*_{n(k)}$ *denote the nth lower k-record value arising from* U(0, 1) *and* $V_n \sim \varrho_{-\ln F(t)}(n,k)$ *that* $\varrho_t(\alpha,\lambda)$ *is the right truncated gamma distribution with pdf*

$$h_t(x) = \frac{\lambda^\alpha}{\rho(\alpha,\lambda t)}x^{\alpha-1}e^{-\lambda x}, \qquad t > x > 0, \quad \alpha,\lambda > 0.$$

Definition 3.9. A random variable X is said to have increasing (decreasing) generalized residual extropy, if and only if $\mathcal{G}J_t(U_{n(k)})$ is increasing (decreasing) in t.

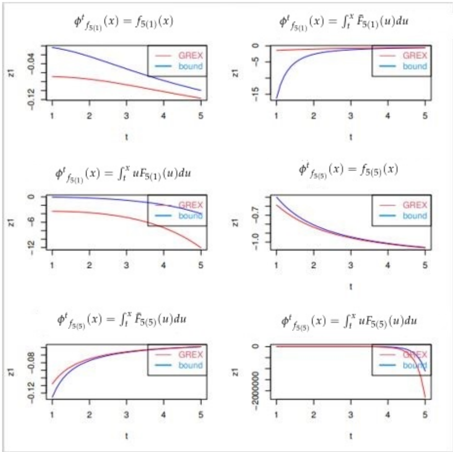

Figure 1. Comparison lower (upper) bound and GREX for various cases of $\phi^t{}_{f_{n(k)}}$

The following theorem provides a lower (upper) bound for GREX of records.

Theorem 3.10. *If $U_{n(k)}$ have increasing (decreasing) GREX, then $\mathcal{G}J_t(U_{n(k)}) \geq (\leq) \frac{A(t)}{4\bar{F}^2_{n(k)}(t) r_{n(k)}(t)} - \frac{\phi^t{}_{f_{n(k)}}(t)}{4\bar{F}_{n(k)}(t)}$ for all $t > 0$, where $r_{n(k)}(t)$ is the hazard rate function and $A(t) = \int_{-\ln \bar{F}(t)}^{\infty} \acute{\phi}^t{}_{f_{n(k)}}\left(F^{-1}(1 - e^{-u})\right) \frac{k^n}{\Gamma(n)} u^{n-1} e^{-uk} du$.*

Proof. Differentiating $\mathcal{G}J_t(U_{n(k)})$ with respect to t, we get

$$\frac{d}{dt}\mathcal{G}J_t(U_{n(k)}) = \frac{\bar{F}^2_{n(k)}(t)\Big(\frac{1}{2}\phi^t{}_{f_{n(k)}}(t) f_{n(k)}(t) - \frac{1}{2}A(t)\Big) + 2\bar{F}^3_{n(k)}(t) f_{n(k)}(t)\mathcal{G}J_t(U_{n(k)})}{\bar{F}^4_{n(k)}(t)}$$
$$= \frac{\frac{1}{2}\phi^t{}_{f_{n(k)}}(t) r_{n(k)}(t)}{\bar{F}_{n(k)}(t)} - \frac{\frac{1}{2}A(t)}{\bar{F}^2_{n(k)}(t)} + 2\mathcal{G}J_t(U_{n(k)}) r_{n(k)}(t),$$

and the result follows. □

For the exponential distribution with $\lambda = 0.5$, $n = 5$ and various cases of $\phi^t{}_{f_{n(k)}}$, the lower (upper) bound and GREX are obtained and is displayed in Figure 1.

Lemma 3.11. *If $Z = aX + b$, where a and b are constants, then $\mathcal{G}J_t(U^Z_{n(k)}) = \mathcal{G}J_{\frac{t-b}{a}}(U^X_{n(k)})$.*

Proof. We get $\bar{F}^{aX+b}_{n(k)}(t) = \bar{F}^X_{n(k)}(\frac{t-b}{a})$ and using (8), the result follows. □

Theorem 3.12. *Let X and Y be two non-negative random variables with distribution functions F and G respectively and with probability density functions*

f and g respectively.
(a) If ${\phi^t}_{f_{n(k)}}$ is increasing in x, ${\phi^t}_{f_{n(k)}}(x) \geq {\phi^t}_{g_{n(k)}}(x)$ and $X \leq_{disp} Y$, then $\mathcal{G}J_t(U^X_{n(k)}) \leq \mathcal{G}J_t(U^Y_{n(k)})$.
(b) If ${\phi^t}_{f_{n(k)}}$ is increasing in x, ${\phi^t}_{f_{n(k)}}(x) \leq {\phi^t}_{g_{n(k)}}(x)$ and $X \geq_{disp} Y$, then $\mathcal{G}J_t(U^X_{n(k)}) \geq \mathcal{G}J_t(U^Y_{n(k)})$.

Proof. Since $X \leq_{disp} Y$, we have $X \geq_{st} Y$, then $F^{-1}(1 - e^{-V_n}) \geq G^{-1}(1 - e^{-V_n})$ for $0 < F(t) < 1 - e^{-V_n} < 1$ and we have ${\phi^t}_{f_{n(k)}}(F^{-1}(1 - e^{-V_n})) \geq {\phi^t}_{g_{n(k)}}(G^{-1}(1 - e^{-V_n}))$, hence,

$$\begin{aligned}\mathcal{G}J_t(U^X_{n(k)}) &= -\frac{\Gamma(n)}{2\Gamma(n, -k\ln \bar{F}(t))} E\Big[{\phi^t}_{f_{n(k)}}\big(F^{-1}(1 - e^{-V_n})\big)\Big] \\ &\leq -\frac{\Gamma(n)}{2\Gamma(n, -k\ln \bar{G}(t))} E\Big[{\phi^t}_{g_{n(k)}}\big(G^{-1}(1 - e^{-V_n})\big)\Big] \\ &= \mathcal{G}J_t(U^Y_{n(k)}).\end{aligned}$$

Similarly, we can prove (b). □

Theorem 3.13. *Let X be a non-negative random variable and let $Y = \Phi(X)$ where Φ is a strictly increasing function with $\Phi(t) = t$ and $\Phi(\infty) = \infty$, with probability density function $g(y) = \frac{f(\Phi^{-1}(y))}{\Phi'(\Phi^{-1}(y))}$ and cumulative distribution function $G(y) = F(\Phi^{-1}(y))$. Then, the generalized residual extropy of the nth upper k-record value corresponding to Y is given by*

$$\mathcal{G}J_t(U^Y_{n(k)}) = -\frac{\Gamma(n)}{2\Gamma(n, -k\ln \bar{F}(t))} E\Big[{\phi^t}_{g_{n(k)}}\big(\Phi(F^{-1}(1 - e^{-V_n}))\big)\Big].$$

Here, $V_n \sim \gamma_{-\ln \bar{F}(t)}(n, k)$ and $g_{n(k)}(y)$ is as

$$g_{n(k)}(y) = \frac{k^n}{\Gamma(n)}[-\ln \bar{F}(\Phi^{-1}(y))]^{n-1} \bar{F}(\Phi^{-1}(y))^{k-1} \frac{f(\Phi^{-1}(y))}{\Phi'(\Phi^{-1}(y))}.$$

4. Generating Function

Now, We define the generating function related to GEX measures discussed in the previous sections.

Definition 4.1. Provided that the assumptions of Theorem 2.2 hold, we define a generating function for the GEX of $U_{n(k)}$ as

$$GF_{GEX}(U_{n(k)}) = -\frac{1}{2}\int_0^\infty e^{\alpha\phi_{f_{n(k)}}(x)} f_{n(k)}(x)dx. \tag{14}$$

the above function is a function in terms of α.

Differentiating (14) with respect to α, we get

$$\frac{d}{d\alpha} GF_{GEX}(U_{n(k)}) = -\frac{1}{2}\int_0^\infty \phi_{f_{n(k)}}(x) e^{\alpha\phi_{f_{n(k)}}(x)} f_{n(k)}(x)dx.$$

Now, by setting $\alpha = 0$ in the above expression, we get the GEX measure in (3). Therefore, We refer to it as GEX of order 1. Higher-order derivatives of the generating function with respect to the shape parameter α naturally lead to higher-order generalized extropies, thus capturing progressively deeper layers of informational complexity within the distribution of k-records. Through this differentiation process, we introduce a generalized extropy (GEX) framework that, even in the particular case when $n = k = 1$, subsumes several well-established entropy measures previously proposed in the literature. This demonstrates a meaningful theoretical connection between the GEX model and the entropy structure, revealing the versatility of the generating function in encoding both distributional and informational characteristics. Furthermore, as supported by developments in information theory, generating functions have been effectively employed for probability densities to derive various information measures. This reinforces the role of the generating function not only as a probabilistic tool but also as a foundational device for entropy analysis. The generalized extropies of order 2 are as follows

$$\frac{d^2}{d\alpha^2} GF_{GEX}(U_{n(k)})|_{\alpha=0} = -\frac{1}{2}\int_0^{\infty} [\phi_{f_{n(k)}}(x)]^2 f_{n(k)}(x)dx.$$

Therefore, The generalized extropies of order β is as follows

$$\frac{d^\beta}{d\alpha^\beta} GF_{GEX}(U_{n(k)})|_{\alpha=0} = -\frac{1}{2}\int_0^{\infty} [\phi_{f_{n(k)}}(x)]^\beta f_{n(k)}(x)dx. \tag{15}$$

We define (15) as the information generating function $IGF_\beta(U_{n(k)})$. Differentiating $IGF_\beta(U_{n(k)})$ with respect to β and setting $\beta = 0$, we obtain

$$\frac{d}{d\beta}(IGF_\beta(U_{n(k)}))|_{\beta=0} = -\frac{1}{2}\int_0^{\infty} \ln \phi_{f_{n(k)}}(x) f_{n(k)}(x)dx. \tag{16}$$

For the choice of $n = 1$, $k = 1$ and $\phi_f(x) = f(x)^2$, (16) reduces to the Shannon entropy [23]. If $n = 1$, $k = 1$, $\phi_f(x) = e^{\sqrt[\alpha-1]{\frac{1}{(1-f^{\alpha-1}(x))^2}}}$, (16) reduces to the generalized Tsallis entropy of order α [25] and if $\phi_f(x) = \bar{F}(x)^{\frac{2\bar{F}(x)}{f(x)}}$, we have cumulative residual entropy of [17]. We define (16) as generalized entropy (GEN).

In the following, we show that the underlying distributions can be uniquely determined by a location change using the equality in generating function for the GEX of k-record values and study some properties of the generating function for the GEX. The proof is not included since it is similar to the previous sections.

Theorem 4.2. *Assuming that the function ϕ is one-to-one, the variables X and Y with the survival functions $\bar{F}$ and $\bar{G}$ belong to the same location family of distributions if and only if*

$$GF_{GEX}(U^Y_{n(k)}) = GF_{GEX}(U^X_{n(k)})$$

Proof. The proof is similar to Theorem 2.11. □

Theorem 4.3. *Let X be a non-negative random variable and let $Y = \Phi(X)$ where Φ is a strictly increasing function with $\Phi(0) = 0$ and $\Phi(\infty) = \infty$, with pdf $g(y) = \frac{f(\Phi^{-1}(y))}{\Phi'(\Phi^{-1}(y))}$ and cdf $G(y) = F(\Phi^{-1}(y))$. Then, the generating function for the GEX of the nth upper k-record value corresponding to Y is given by*

$$GF_{GEX}(U^Y_{n(k)}) = -\frac{1}{2}E\Big[e^{\alpha\phi_{g_{n(k)}}\left(\Phi(F^{-1}(1-e^{-V_n}))\right)}\Big]. \tag{17}$$

Here, V_n has the gamma distribution with parameters n, k.

Theorem 4.4. *Let X and Y be two non-negative random variables with distribution functions F and G respectively and with probability density functions f and g respectively.*
a) If $\phi_{f_{n(k)}}$ is increasing in x, $\phi_{f_{n(k)}}(x) \geq \phi_{g_{n(k)}}(x)$ and $X \leq_{disp} Y$, then $GF_{GEX}(U^X_{n(k)}) \leq GF_{GEX}(U^Y_{n(k)})$. b) If $\phi_{f_{n(k)}}$ is increasing in x, $\phi_{f_{n(k)}}(x) \leq \phi_{g_{n(k)}}(x)$ and $X \geq_{disp} Y$, then $GF_{GEX}(U^X_{n(k)}) \geq GF_{GEX}(U^Y_{n(k)})$.

Proof. Substituting (1) in (14), we obtain
$GF_{GEX}(U_{n(k)}) = -\frac{1}{2}E\big[e^{\alpha\phi_{f_{n(k)}}(F^{-1}(1-e^{-V_n})}\big]$. Since $\alpha > 0$ and the exponential function is increasing, the proof follows from Theorem 2.7. □

In the following, we define generating function related to the GREX measures.

Definition 4.5. Given the premises outlined in Theorem 2.2, we define a generating function for the GREX of nth upper k-record value as

$$GF^t{}_{GREX}(U_{n(k)}) = -\frac{1}{2\bar{F}^2_{n(k)}(t)}\int_t^\infty e^{\alpha\phi^t{}_{f_{n(k)}}(x)} f_{n(k)}(x)dx.$$

Differentiating the above equation with respect to α and considering $\alpha = 0$, we get the GREX measure.

Theorem 4.6. *In accordance with the assumptions of Theorem 2.2, we obtain a generating function for the GREX of $U_{n(k)}$ as*

$$GF^t{}_{GREX}(U_{n(k)}) = -\frac{\Gamma(n)}{2\Gamma(n, -k\ln\bar{F}(t))}E\Big[e^{\alpha\phi^t{}_{f_{n(k)}}(F^{-1}(1-e^{-V_n}))}\Big],$$

where $V_n \sim \gamma_{-\ln\bar{F}(t)}(n, k)$.

Proof. The proof is similar to the Theorem 3.4. □

Similarly, the information generating function of $U_{n(k)}$ is as

$$IGF^t_\beta(U_{n(k)}) = -\frac{\Gamma(n)}{2\Gamma(n, -k\ln\bar{F}(t))}E\Big[[\phi^t{}_{f_{n(k)}}(F^{-1}(1-e^{-V_n}))]^\beta\Big]. \tag{18}$$

Differentiating $IGF^t_\beta(U_{n(k)})$ with respect to β and setting $\beta = 0$, the generalized residual entropy (GREN) obtains as

$$(19)\quad \mathcal{GEN}^t(U_{n(k)}) = -\frac{\Gamma(n)}{2\Gamma(n, -k\ln\bar{F}(t))} E\Big[\ln\Big(\phi^t{}_{f_{n(k)}}(F^{-1}(1-e^{-V_n}))\Big)\Big].$$

In the following examples, we illustrate (18) and (19) for various cases.

Example 4.7. *Let* $\{X_i,\ i \geq 1\}$ *be a sequence of iid random variables with a common distribution* U(0, 1). *Then, we get*

$$IGF^t_\beta(U_{n(k)}) = -\frac{\Gamma(n)}{2\Gamma(n, -k\ln(1-t))} E\Big[\phi^t{}_{f_{n(k)}}(1-e^{-V_n})]^\beta\Big].$$

Example 4.8. *Let* $\phi^t{}_{f_{n(k)}}(x) = f_{n(k)}(x)$*, we have:* $IGF^t_\beta(U_{n(k)}) =$

$$= -\frac{\Gamma^{1-\beta}(n)k^{n(\beta+1)}\Gamma(n(\beta+1)-\beta, -(k\beta+k-\beta)\ln\bar{F}(t))}{2\Gamma^2(n, -k\ln\bar{F}(t))(k\beta+k-\beta)^{n(\beta+1)-\beta}} E\Big[f^\beta(F^{-1}(1-e^{-V_n}))\Big],$$

where $V_n \sim \gamma_{-\ln\bar{F}(t)}(n(\beta+1)-\beta, k\beta+k-\beta)$.

Example 4.9. *Let* $n = 1$, $k = 1$ *and* $\phi^t{}_f(x) = f(x)$*, we obtain*

$$\mathcal{GREN}^t(X) = -\frac{1}{2\bar{F}^2(t)}\int_t^\infty f(x)\ln(f(x))dx.$$

Example 4.10. *Let* $\phi^t{}_{f_{n(k)}}(x) = f_{n(k)}(x)$*, we get:* $\mathcal{GREN}^t(U_{n(k)}) =$

$$= -\frac{\Gamma(n)}{2\Gamma(n, -k\ln\bar{F}(t))}$$
$$\times\Big[n\ln k + (n-1)E(\ln V_n) - (k-1)E(V_n) + E(\ln f(F^{-1}(1-e^{-V_n}))) - \ln\Gamma(n)\Big],$$

where $V_n \sim \gamma_{-\ln\bar{F}(t)}(n, k)$ *and* $E(V_n) = \frac{\Gamma(n+1, -k\ln\bar{F}(t))}{k\Gamma(n, -k\ln\bar{F}(t))}$.

5. Application with real data

The estimation of generalized extropy plays a crucial role in quantifying uncertainty and information content, especially in situations where the classical entropy or extropy measures may not fully capture the distributional nuances. Generalized extropy extends the original concept by incorporating flexible weighting schemes or power parameters, thereby allowing a broader class of distributions and behaviors to be analyzed. Accurate estimation of this measure enables researchers to study its mathematical properties such as non-negativity, continuity, and maximality and their implications in practical applications like model selection, goodness-of-fit testing, and statistical learning. According to Table 1, the estimates of the generalized extropy (GEX) in equation (3) for each specific case of the function $\phi_{f_{n(k)}}(x)$ are presented based on the approach proposed by [26].

- **case 1.** If $\phi_{1,f_{n(k)}}(x) = f_{n(k)}(x)$,

$$\widehat{\mathcal{G}J}(U_{n(k)}) = -\frac{1}{2N}\frac{k^{2n}}{\Gamma(n)^2}\sum_{i=1}^{N}\left[-\ln\left(1-\frac{i}{N+1}\right)\right]^{2(n-1)} \left[1-\frac{i}{N+1}\right]^{2(k-1)} \frac{2m}{N(X_{(i+m)}-X_{(i-m)})}.$$

- **case 2.** If $\phi_{f_{n(k)}}(x) = \int_0^x \bar{F}_{n(k)}(t)dt$,

$$\widehat{\mathcal{G}J}(U_{n(k)}) = -\frac{1}{2N}\sum_{j=1}^{N}\left\{(1-\frac{j}{N+1})^k\sum_{i=0}^{n-1}\frac{(-k\ln(1-\frac{j}{N+1}))^i}{i!}\right\}^2 \frac{N(X_{(j+m)}-X_{(j-m)})}{2m}.$$

- **case 3.** If $\phi_{f_{n(k)}}(x) = \int_0^x w(t)\bar{F}_{n(k)}(t)dt$ and $w(t)=t$,

$$\widehat{\mathcal{G}J}(U_{n(k)}) = -\frac{1}{2N}\sum_{j=1}^{N}X_j\left\{(1-\frac{j}{N+1})^k\sum_{i=0}^{n-1}\frac{(-k\ln(1-\frac{j}{N+1}))^i}{i!}\right\}^2 \frac{N(X_{(j+m)}-X_{(j-m)})}{2m}.$$

- **case 4.** If $\phi_{f_{n(k)}}(x) = -\int_0^x (1+F_{n(k)}(t))dt$,

$$\widehat{\mathcal{G}J}(U_{n(k)}) = \frac{1}{2N}\sum_{j=1}^{N}\left\{1-G^2[-\ln(1-\frac{j}{N+1})]\right\}\frac{N(X_{(j+m)}-X_{(j-m)})}{2m},$$

  where G is the gamma cdf with parameters n, k.
- **case 5.** If $\phi_{f_{n(k)}}(x) = -\int_0^x w(t)(1+F_{n(k)}(t))dt$ and $w(t)=t$,

$$\widehat{\mathcal{G}J}(U_{n(k)}) = \frac{1}{2N}\sum_{j=1}^{N}X_j\left\{1-G^2[-\ln(1-\frac{j}{N+1})]\right\}\frac{N(X_{(j+m)}-X_{(j-m)})}{2m}.$$

- **case 6.** If $\phi_{f_{n(k)}}(x) = xf_{n(k)}(x)$,

$$\widehat{\mathcal{G}J}(U_{n(k)}) = -\frac{k^{2n}}{2N\Gamma^2(n)}\sum_{j=1}^{N}X_j(1-\frac{j}{N+1})^{2(k-1)}\left(-\ln(1-\frac{j}{N+1})\right)^{2(n-1)} \frac{2m}{N(X_{(j+m)}-X_{(j-m)})}.$$

There is a window size m which is less than $\sqrt{N}$ and $X_{(1)}, X_{(2)}, \ldots, X_{(N)}$ are order statistics based on $X_1, \ldots, X_N$. When $i < m$, then $X_{(i-m)} = X_{(1)}$, and if $i > N-m$, then $X_{(i+m)} = X_{(N)}$. According to the lines of the proof provided by [26] , we find that the Vasicek-type estimator of extropy, $\widehat{\mathcal{G}J}(U_{n(k)})$, converges in probability to the true extropy, as $n, m \to \infty$, and $\frac{m}{n} \to 0$. This limit relation provides a guideline for selecting appropriate values of m. To evaluate the statistical properties of the proposed estimator, a Monte Carlo simulation study with 10,000 replications was conducted. In each replication, samples of sizes $N = 20, 30, 50$ and $n = 3, k = 2$ were generated from the distributions provided below. The estimator of interest was then computed for each sample. Finally, to assess the accuracy and efficiency of the estimator, the bias and mean squared error (MSE) were calculated based on the results from the replications. Among the values of m ranging from 1 to $\sqrt{N}$, the value that results in the smallest bias and MSE is selected. The results are presented in Tables 3-5. The results indicate that, as the sample size increases, both the bias and the MSE decrease across all estimators. This confirms the consistency and improved performance of the estimators with a larger sample size. For the different values of ϕ, the optimal window size m tends to increase with the sample size for $\phi_{1,f_{3(2)}}(x), \phi_{3,f_{3(2)}}(x), \phi_{5,f_{3(2)}}(x), \phi_{6,f_{3(2)}}(x)$, and it is approximately proportional to $\sqrt{N}$. In contrast, for $\phi_{2,f_{3(2)}}(x)$ and $\phi_{4,f_{3(2)}}(x)$, the optimal window size remains nearly constant at a value of 1 across all sample sizes. This behavior is consistent across all distributions considered, where the optimal values of m exhibit little variation, especially for $\phi_{2,f_{3(2)}}(x)$ and $\phi_{4,f_{3(2)}}(x)$.

(1) **Exponential distribution:** $f(x) = \theta e^{-\theta x}, \theta > 0, x > 0$. Denoted by $Exp(\theta)$.
(2) **Gamma distribution:** $f(x) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x}, \theta > 0, x > 0$. Denoted by $\Gamma(\theta)$.
(3) **Weibull distribution:** $f(x) = \theta x^{\theta-1} e^{-x^{\theta}}, \theta > 0, x > 0$. Denoted by $W(\theta)$.
(4) **Uniform distribution:** $f(x) = 1, 0 \leq x \leq 1$. Denoted by U(0, 1).
(5) **Half-Normal distribution:** $f(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}, x > 0$. Denoted by HN.
(6) **Chen distribution:** $F(x) = 1 - e^{2(1-e^{x^{\theta}})}, \theta > 0, x > 0$. Denoted by $CH(\theta)$.
(7) **Modified Extreme Value distribution:** $F(x) = 1 - e^{\frac{1}{\theta}(1-e^{x})}, \theta > 0, x > 0$. Denoted by $MEV(\theta)$.
(8) **Log-normal distribution:** $f(x) = \frac{1}{\theta x\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2\theta^2}}, \theta > 0, x > 0$. Denoted by $LN(\theta)$.

The use of real data in estimating extropy is essential for evaluating the practical performance, robustness, and applicability of the proposed estimators under realistic conditions. We now provide the GEX estimates computed using the observed real-world data. Dataset 1 is taken from [4]. This dataset represents the vinyl chloride data obtained from clean-up gradient monitoring wells. This dataset has been fitted very well by exponential distribution (see [4], [22])

TABLE 3. Bias and MSE of the GEX estimator for $N = 20, n = 3$ and $k = 2$

distribution	$Exp(3)$		$Exp(0.5)$		$\Gamma(2)$		$\Gamma(1)$		$LN(0.5)$		$\chi^2(1)$	
	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)
$\phi_{1,f_{3(2)}}(x)$	0.1381	0.0805(4)	0.0230	0.0024(4)	0.0353	0.0046(4)	0.0482	0.0091(4)	0.0772	0.0272(3)	0.0355	0.0068(4)
$\phi_{2,f_{3(2)}}(x)$	0.0023	0.0017(2)	0.0130	0.0611(1)	0.1210	0.0495(1)	0.0081	0.0156(1)	0.1866	0.0415(1)	0.0512	0.0320(1)
$\phi_{3,f_{3(2)}}(x)$	0.0084	0.0005(4)	0.3037	0.6810(4)	0.1248	0.2413(2)	0.0769	0.0427(4)	0.0061	0.0108(4)	0.1515	0.1260(4)
$\phi_{4,f_{3(2)}}(x)$	0.0097	0.0073(2)	0.0671	0.2600(2)	0.0785	0.1123(2)	0.0202	0.0648(2)	0.1608	0.0483(1)	0.1183	0.1839(2)
$\phi_{5,f_{3(2)}}(x)$	0.0054	0.0046(4)	0.2573	6.2761(4)	0.0526	1.4817(4)	0.0275	0.3597(4)	0.0095	0.0829(4)	0.0659	1.5374(4)
$\phi_{6,f_{3(2)}}(x)$	0.3115	0.1122(3)	0.2276	0.3682(4)	0.1615	0.0569(4)	0.1000	0.0230(3)	0.4391	2.5662(3)	0.0675	0.0120(4)

distribution	$W(1.5)$		HN		U(0, 1)		$CH(1.6)$		$MEV(0.6)$	
	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)
$\phi_{1,f_{3(2)}}(x)$	0.0907	0.0249(4)	0.0956	0.0235(4)	0.4423	0.3472(4)	0.3503	0.2349(4)	0.0474	0.0254(4)
$\phi_{2,f_{3(2)}}(x)$	0.0481	0.0091(1)	0.0177	0.0062(1)	0.0235	0.0021(1)	0.0367	0.0030(1)	0.2084	0.0479(1)
$\phi_{3,f_{3(2)}}(x)$	0.0220	0.0089(4)	0.0223	0.0074(4)	0.0003	0.0005(4)	0.0065	0.0004(4)	0.2208	0.0543(4)
$\phi_{4,f_{3(2)}}(x)$	0.0329	0.0178(1)	0.0064	0.0142(1)	0.0252	0.0019(1)	0.0408	0.0038(1)	0.2842	0.0883(2)
$\phi_{5,f_{3(2)}}(x)$	0.0013	0.0446(4)	0.0021	0.0327(4)	0.0013	0.0005(4)	0.0004	0.0008(3)	0.3916	0.1694(4)
$\phi_{6,f_{3(2)}}(x)$	0.3716	0.1684(3)	0.3469	0.1484(3)	0.9489	1.0535(4)	0.7651	0.6607(3)	0.5789	0.3869(3)

TABLE 4. Bias and MSE of the GEX estimator for $N = 30, n = 3$ and $k = 2$

distribution	$Exp(3)$		$Exp(0.5)$		$\Gamma(2)$		$\Gamma(1)$		$LN(0.5)$		$\chi^2(1)$	
	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)
$\phi_{1,f_{3(2)}}(x)$	0.0585	0.0298(5)	0.0104	0.0009(5)	0.0161	0.0016(5)	0.0201	0.0033(5)	0.0327	0.0100(5)	0.0155	0.0027(5)
$\phi_{2,f_{3(2)}}(x)$	0.0017	0.0011(1)	0.0083	0.0406(1)	0.1012	0.0338(1)	0.0026	0.0100(1)	0.1753	0.0354(1)	0.0362	0.0201(1)
$\phi_{3,f_{3(2)}}(x)$	0.0059	0.0003(5)	0.2168	0.4063(5)	0.0868	0.1495(5)	0.0525	0.0246(5)	0.0081	0.0069(5)	0.1053	0.0682(5)
$\phi_{4,f_{3(2)}}(x)$	0.0136	0.0048(1)	0.0422	0.1766(1)	0.0588	0.0709(2)	0.0388	0.0435(1)	0.1511	0.0375(1)	0.1071	0.1217(1)
$\phi_{5,f_{3(2)}}(x)$	0.0110	0.0035(4)	0.2323	4.5876(5)	0.0516	1.0991(5)	0.0248	0.2730(5)	0.0021	0.0617(5)	0.1495	1.1740(5)
$\phi_{6,f_{3(2)}}(x)$	0.2590	0.0721(4)	0.1363	0.3304(5)	0.0991	0.0562(5)	0.0516	0.0078(5)	0.3613	0.0459(5)	0.0432	0.0113(5)

distribution	$W(1.5)$		HN		U(0, 1)		$CH(1.6)$		$MEV(0.6)$	
	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)
$\phi_{1,f_{3(2)}}(x)$	0.0434	0.0085(5)	0.0484	0.0084(5)	0.3019	0.1595(5)	0.2142	0.0869(4)	0.0451	0.0242(2)
$\phi_{2,f_{3(2)}}(x)$	0.0391	0.0063(1)	0.0109	0.0042(1)	0.0162	0.0014(1)	0.0260	0.0018(1)	0.2004	0.0418(5)
$\phi_{3,f_{3(2)}}(x)$	0.0150	0.0058(5)	0.0160	0.0048(4)	0.0002	0.0003(5)	0.0074	0.0003(5)	0.2209	0.0528(4)
$\phi_{4,f_{3(2)}}(x)$	0.0237	0.0114(2)	0.0009	0.0093(2)	0.0165	0.0011(1)	0.0297	0.0023(1)	0.2076	0.0877(1)
$\phi_{5,f_{3(2)}}(x)$	0.0036	0.0319(5)	0.0065	0.0241(5)	0.0000	0.0004(5)	0.0039	0.0006(5)	0.3088	0.1686(5)
$\phi_{6,f_{3(2)}}(x)$	0.2967	0.0988(4)	0.2790	0.0879(4)	0.8279	0.7652(5)	0.6539	0.4537(4)	0.4903	0.2577(4)

Dataset 1: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2. Dataset 2, originally published by [15], consists of 50 observations representing the number of thousands of cycles to failure for electrical appliances in a life testing experiment. [27] demonstrated that the Chen distribution provides a good fit for modeling this dataset.

Dataset 2: 0.014, 0.034, 0.059, 0.061, 0.069, 0.08, 0.123, 0.142, 0.165, 0.21, 0.381, 0.464, 0.479, 0.556, 0.574, 0.839, 0.917, 0.969, 0.991, 1.064, 1.088, 1.091, 1.174, 1.27, 1.275, 1.355, 1.397, 1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001,

Table 5. Bias and MSE of the GEX estimator for $N = 50, n = 3$ and $k = 2$

distribution	$Exp(3)$		$Exp(0.5)$		$\Gamma(2)$		$\Gamma(1)$		$LN(0.5)$		$\chi^2(1)$	
	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)
$\phi_{1,f_{3(2)}}(x)$	0.0126	0.0122(7)	0.0019	0.0003(7)	0.0039	0.0006(7)	0.0050	0.0014 (7)	0.0035	0.0037(7)	0.0026	0.0012(7)
$\phi_{2,f_{3(2)}}(x)$	0.0008	0.0007(1)	0.0028	0.0240(2)	0.0802	0.0209(1)	0.0007	0.0059(1)	0.1602	0.0288(1)	0.0203	0.0112(1)
$\phi_{3,f_{3(2)}}(x)$	0.0035	0.0002(7)	0.1262	0.2072(7)	0.0563	0.0849(7)	0.0310	0.0128(7)	0.0094	0.0041(7)	0.0604	0.0314(7)
$\phi_{4,f_{3(2)}}(x)$	0.0112	0.0027(1)	0.0402	0.0997(2)	0.0420	0.0423(2)	0.0322	0.0246(1)	0.1409	0.0288(1)	0.0747	0.0681(1)
$\phi_{5,f_{3(2)}}(x)$	0.0108	0.0024(5)	0.2403	3.0207(7)	0.0384	0.7319(7)	0.0181	0.1930(7)	0.0011	0.0373(7)	0.1431	0.8225(7)
$\phi_{6,f_{3(2)}}(x)$	0.2225	0.0514(7)	0.1624	0.2939(7)	0.0158	0.0540(7)	0.0153	0.0022(7)	0.3052	0.0288(7)	0.0278	0.0111(7)

distribution	$W(1.5)$		HN		U$(0,1)$		$CH(1.6)$		$MEV(0.6)$	
	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)	Bias	MSE(m)
$\phi_{1,f_{3(2)}}(x)$	0.0140	0.0029(7)	0.0175	0.0028(7)	0.1897	0.0633(7)	0.1260	0.0321(7)	0.0377	0.0183(2)
$\phi_{2,f_{3(2)}}(x)$	0.0273	0.0035(1)	0.0075	0.0026(2)	0.0098	0.0008(1)	0.0157	0.0009(1)	0.2014	0.0445(7)
$\phi_{3,f_{3(2)}}(x)$	0.0093	0.0033(7)	0.0099	0.0028(7)	0.0001	0.0002(7)	0.0084	0.0002(7)	0.2177	0.0499(7)
$\phi_{4,f_{3(2)}}(x)$	0.0149	0.0066(1)	0.0004	0.0054(2)	0.0100	0.0006(1)	0.0202	0.0013(1)	0.1240	0.0780(7)
$\phi_{5,f_{3(2)}}(x)$	0.0350	0.0207(5)	0.0090	0.0158(7)	0.0001	0.0002(7)	0.0024	0.0004(7)	0.3172	0.1611(7)
$\phi_{6,f_{3(2)}}(x)$	0.2488	0.0662(7)	0.2320	0.0577(6)	0.7290	0.5671(7)	0.5833	0.3507(6)	0.4330	0.1941(6)

Table 6. Estmation of the GEX for Dataset 1

$N = 34$	$\phi_{1,f_{3(2)}}(x)$	$\phi_{2,f_{3(2)}}(x)$	$\phi_{3,f_{3(2)}}(x)$	$\phi_{4,f_{3(2)}}(x)$	$\phi_{5,f_{3(2)}}(x)$	$\phi_{6,f_{3(2)}}(x)$
Estimation of the GEX	-0.0897(5)	-0.9199(1)	-1.3780 (5)	1.9951(1)	4.6632(5)	-0.2363(5)

2.161, 2.292, 2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248, 3.715, 3.79, 3.857, 3.912, 4.1.

Table 7. Estimation of the GEX for Dataset 2

$N = 50$	$\phi_{1,f_{3(2)}}(x)$	$\phi_{2,f_{3(2)}}(x)$	$\phi_{3,f_{3(2)}}(x)$	$\phi_{4,f_{3(2)}}(x)$	$\phi_{5,f_{3(2)}}(x)$	$\phi_{6,f_{3(2)}}(x)$
Estimation of the GEX	-0.1592(7)	-0.9104(1)	-1.0077(7)	1.4732(1)	2.2510(7)	-0.3979(6)

6. **Conclusion**

In this paper, we presented a generalized measure of extropy and a dynamic generalized measure of extropy based on k-records. Several theorems and properties are also presented. We have shown that several existing extropy measures, such as survival, negative cumulative, dynamic survival extropy and weighted dynamic survival extropy are spacial cases of this generalized extropy measures. Further, we have presented generating function to obtain generalized measures of higher order. Using this generating function, we established different extropy and entropy measures. Finally, we conducted a Monte Carlo simulation study to evaluate the bias and MSE of the estimator of the GEX. Additionally, the estimator values for six specific cases of the $\phi_{f_{n(k)}}(x)$ were calculated using real data.

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