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MORE RESULTS ON DEGREE DEVIATION AND DEGREE VARIANCE

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Abstract. This paper investigates degree deviation and variance in graph theory, with a specific focus on k-regular graphs and subdivision of graphs. These metrics are fundamental for characterizing graph irregularity and have significant applications in network analysis and the social sciences. Furthermore, we introduce novel geometric measures of irregularity, geometric degree deviation and geometric degree variance derived from the geometric mean of vertex degrees. By means of rigorous theorems and illustrative examples, we explore the relationships between graph structures and their degree properties. Our findings seek to advance the current understanding of graph irregularity and provide a solid foundation for future research in this area.

Keywords: Irregularity, Degree deviation, Degree variance.
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1. Introduction and preliminaries

Understanding irregularity measures of graphs, such as degree deviation, is vital for analyzing the structure and dynamics of complex networks. These measures help identify anomalies, assess robustness, and optimize network performance in fields like social sciences, biology, and computer science.

The new geometric irregularity metrics introduced in this paper offer valuable tools for studying network heterogeneity and improving network design and resilience. Additionally, examining how graph operations influence these measures may provide insights applicable to network control and evolution.

In graph theory, degree deviation and degree variance are two of the most important metrics used to quantify graph irregularity. These measures provide valuable insights into the distribution of vertex degrees within a graph, with significant implications across diverse domains, including network analysis, chemistry, and the social sciences.

Let G be a connected simple graph with vertex set $V(G)$, $|V(G)| = n$ and edge set $E(G)$, $|E(G)| = m$. We denote $\Delta = \Delta(G)$ as the maximum degree and $\delta = \delta(G)$ as the minimum degree of the vertices in G. A vertex adjacent to all other vertices in G is called a universal vertex and satisfies $\Delta(G) = n - 1$. The

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degree of a vertex u in G is represented as $d_G(u)$, and the number of vertices of degree i is denoted by N_i. A graph is termed r-regular if all its vertices have the same degree r. Conversely, an irregular graph is a connected graph that contains at least two vertices of different degrees.

If the degree multiset $D(G)$ of an irregular graph G consists of exactly k distinct degrees, then G is classified as a k-degree graph. Specifically, an irregular graph with exactly two different degrees is referred to as a bidegreed graph, denoted by $G(\Delta, \delta)$. The concept of irregularity was first introduced by Collatz and Sinogowitz [5] in 1957. According to Bell's definition [4], if $IT(G)$ is a topological invariant of a connected graph G such that $IT(G) \geq 0$, then $IT(G)$ is termed an irregularity measure if G is regular if and only if $IT(G) = 0$. Various irregularity measures have been defined, including the Collatz-Sinogowitz index [4], the Albertson index [2], total irregularity [1], and the Sigma index [6,8]. It is very easy fact that (Hand-shaking lemma) the sum of the degrees of vertices equals twice the number of edges, i.e., $2m = \sum_{v \in V(G)} deg(v)$ ([10]). Among these measures, degree deviation $S(G)$ and degree variance $\mathrm{Var}(G)$ are particularly prominent. Nikiforov ([9]) introduced the degree deviation for a connected graph G with n vertices and m edges as follows:

$$S(G) = \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right|.$$

Bell defined the degree variance as:

$$\mathrm{Var}(G) = \frac{1}{n} \sum_{i=1}^{n} \left(d_i - \frac{2m}{n} \right)^2.$$

Here, $\frac{2m}{n}$ represents the average degree of the vertices in graph G, which, in the case of unicyclic graphs, equals 2.

The k-subdivision of a graph G, presented as $G^{\frac{1}{k}}$, is formed by substituting each edge e of G with a path of length k (see [3]). In the subsequent sections, we will calculate the degree deviation and degree variance for various graphs, including the k-subdivision of graph G.

Additionally, we define the geometric mean of the degrees of the vertices in the graph G as:

$$\mathrm{GM}(G) = \left(\prod_{i=1}^{n} d(v_i) \right)^{\frac{1}{n}}.$$

While the traditional degree deviation and variance are based on the arithmetic mean $\frac{2m}{n}$, we now introduce two new irregularity measures based on the geometric mean: geometric degree deviation $S_{GM}(G)$ and geometric degree variance $\mathrm{Var}_{GM}(G)$, defined as follows:

$$S_{GM}(G) = \sum_{i=1}^{n} \left| d_i - \mathrm{GM}(G) \right|,$$

$$\mathrm{Var}_{GM}(G) = \frac{1}{n}\sum_{i=1}^{n}\left(d_i - \mathrm{GM}(G)\right)^2.$$

In Section 2, we study the effect of deletion (or addition) of a vertex on the degree deviation and degree variance. In Section 3, we study the effect of k-subdivision on degree deviation and degree variance. In Section 4, we introduce new measures of irregularity based on the geometric mean of vertex degrees in a graph and present several related results.

2. Results for some operations on graphs

It is a natural question to ask what happens to $S(G)$ and $Var(G)$ when we delete a vertex or an edge from a graph G and what happens to these parameters when the graph is subdivided. In this section, we try to answer some of these questions. Additionally, we will examine specific types of graphs, including trees and unicyclic graphs, to derive further insights into their degree properties.

Theorem 2.1. *If G is a connected graph with n vertices, then*

$$S(G) = 2\sum_{d_i > \frac{2m}{n}}\left|d_i - \frac{2m}{n}\right| = 2\sum_{d_i \leq \frac{2m}{n}}\left|d_i - \frac{2m}{n}\right|.$$

Proof. Consider $M = \frac{2m}{n}$ and $D_s(G)$ denote the multiset of all vertex degrees less than M in the graph G, such that $D_s(G)$ contains s elements. So,

$$\begin{aligned}
S(G) &= \sum_{i=1}^{n}\left|d_i - \frac{2m}{n}\right| \\
&= (M - d_1) + ... + (M - d_s) + (d_{s+1} - M) + ... + (d_n - M) \\
&= sM - (d_1 + d_2 + ... + d_s) + (d_{s+1} + ... + d_n) - (n-s)M \\
&= sM - (d_1 + d_2 + ... + d_n) + 2(d_{s+1} + ... + d_n) - (n-s)M \\
&= sM - nM + 2(d_{s+1} + ... + d_n) - (n-s)M \\
&= 2(d_{s+1} + ... + d_n) - 2(n-s)M \\
&= 2\sum_{d_i > \frac{2m}{n}}\left|d_i - \frac{2m}{n}\right|.
\end{aligned}$$

Also, clearly $2S(G) = 2(\sum_{d_i > \frac{2m}{n}}\left|d_i - \frac{2m}{n}\right| + \sum_{d_i \leq \frac{2m}{n}}\left|d_i - \frac{2m}{n}\right|)$. By the first part of the proof, $2S(G) = S(G) + 2\sum_{d_i \leq \frac{2m}{n}}\left|d_i - \frac{2m}{n}\right|$ and hence $S(G) = 2\sum_{d_i \leq \frac{2m}{n}}\left|d_i - \frac{2m}{n}\right|$. □

Corollary 2.2. (i) *If G is a bidegree graph $G(\delta, \Delta)$, then*

$$S(G) = 2N_\delta(M - \delta) = 2N_\Delta(\Delta - M).$$

(ii) *For the star graph $S_n = K_{1,n-1}$, $S(S_n) = 2(n - 3 + \frac{2}{n})$.*

(iii) *For the path graph* P_n, $S(P_n) = 4(1 - \frac{2}{n})$.

(iv) *If* $n \leq m$, *then* $S(K_{n,m}) = \frac{2nm(m-n)}{m+n}$.

Corollary 2.3. (i) *If* G *is a graph with* t *pendant vertices and* $1 \leq M = \frac{2m}{n} \leq 2$, *then* $S(G) = 2t(M-1) \leq 2t$.

(ii) *If* G *is a unicyclic graph with* t *pendant vertices, then* $S(G) = 2t$.

(iii) *If* T *is a tree with* n *vertices and* t *pendant vertices, then* $S(G) = 2t(1 - \frac{2}{n})$.

Corollary 2.3.(ii) shows that, among all n-vertex trees, the tree with the maximum degree deviation is the one that has the greatest number of pendant vertices, while the tree with the fewest pendant vertices has the minimum degree deviation. Therefore, among all n-vertex trees, the path P_n attains the minimum degree deviation, while the star S_n attains the maximum, i.e., for each tree T with n vertices, $S(P_n) \leq S(T) \leq S(S_n)$. Note also that the only graphs satisfying the condition $1 \leq M \leq 2$ are trees and unicyclic graphs, as also stated in parts (ii) and (iii) of Corollary 2.3.

The following example shows that there is no direct relationship between $S(G)$ and $S(G-v)$.

Example 2.4. (i) *Let* $G = C_4$, *the cyclic graph of order* 4, *and* $v \in V(G)$. *Then* $S(G) = 0$ *and* $S(G-v) = \frac{4}{3}$. *So* $S(G) \lneqq S(G-v)$.

(ii) *Let* G *be a graph obtained by joining a pendant vertex* v *to the cycle* C_4. *Then* $S(G) = 2$ *and* $S(G-v) = 0$. *Hence* $S(G-v) \lneqq S(G)$.

For regular graphs, we can obtain the exact values of $S(G-v)$ and $Var(G-v)$.

Theorem 2.5. *If* G *is a* k*-regular graph with* n *vertices and* $v \in V(G)$, *then*

(i) $S(G-v) = \frac{2k(n-k-1)}{n-1}$. (ii) $Var(G-v) = \frac{k(n-k-1)}{(n-1)^2}$.

Proof. (i) The graph $G-v$ has $n-1$ vertices and $\frac{k(n-2)}{2}$ edges such that $(n-1)-k$ vertices are of degree k and k vertices are of degree $k-1$. So,

$$\begin{aligned} S(G-v) &= (n-k-1)[k - \frac{k(n-2)}{n-1}] + k[\frac{k(n-2)}{n-1} - (k-1)] \\ &= (n-k-1)\frac{k}{n-1} + k\frac{n-k-1}{n-1} \\ &= \frac{2k(n-k-1)}{n-1}. \end{aligned}$$

$$\begin{aligned} \text{(ii) } Var(G-v) &= \frac{1}{n-1}\big((n-k-1)[k - \frac{k(n-2)}{n-1}]^2 + k[\frac{k(n-2)}{n-1} - (k-1)]^2\big) \\ &= (n-k-1)\frac{k^2}{(n-1)^3} + k\frac{(n-k-1)^2}{(n-1)^3} \\ &= \frac{k(n-k-1)}{(n-1)^3}\big(k+n-k-1\big) = \frac{k(n-k-1)}{(n-1)^2}. \end{aligned}$$

□

Theorem 2.6. *Let G be a k-regular graph of order n. If the graph $G+v$ is obtained by connecting a pendant vertex v to one of the vertices of the graph G, then*

(i) $S(G+v) = \frac{2(k-2)n+2}{n+1}$.
(ii) $Var(G+v) = \frac{(k^2-2k+2)n^2+k(kn+2)-2}{(n+1)^3}$.

Proof. (i) Let G be a k-regular graph. It is not difficult to check that $\bar{d} = \frac{kn+2}{n+1}$. So there exist $(n-1)$ vertices of degree k, one vertex of degree $k+1$ and one vertex of degree 1. Then,

$$\begin{aligned} S(G+v) &= (n-1)[k - \frac{kn+2}{n+1}] + (k+1-\frac{kn+2}{n+1}) + (\frac{kn+2}{n+1} - 1) \\ &= (n-1)\frac{k-2}{n+1} + \frac{n+k-1}{n+1} + \frac{(k-1)n+1}{n+1} \\ &= \frac{2(k-2)n+2}{n+1}, \end{aligned}$$

(ii)

$$\begin{aligned} Var(G+v) &= \frac{1}{n+1}[(n-1)(k-\frac{kn+2}{n+1})^2 + (k+1-\frac{kn+2}{n+1})^2 + (\frac{kn+2}{n+1}-1)^2] \\ &= (n-1)\frac{(k-2)^2}{(n+1)^3} + \frac{(n+k-1)^2}{(n+1)^3} + \frac{((k-1)n+1)^2}{(n+1)^3} \\ &= \frac{(k^2-2k+2)n^2+k(kn+2)-2}{(n+1)^3}. \end{aligned}$$

□

3. Results for k-subdivision of graphs

This section is devoted to the study of the degree deviation and degree variance for the k-subdivision ($k \in \mathbb{N}$) of a graph. First, recall the following definition.

Definition 3.1. The k-subdivision of a graph G, presented as $G^{\frac{1}{k}}$, is formed by substituting each edge e of G with a path of length k.

For example, since $V(G^{\frac{1}{k}}) = n+(k-1)m, P_n^{\frac{1}{k}} = P_{n+(k-1)m}$ and $C_n^{\frac{1}{k}} = C_{n+(k-1)m}$.

Theorem 3.2. *Let G be a graph and $k \in \mathbb{N}$. Then $S(G) \leq S(G^{\frac{1}{k}})$.*

Proof. Let G be of order n and size m. Two cases may occur:

Case 1: If G is a tree. It is easy to observe that the number of vertices in $G^{\frac{1}{k}}$ is greater than the number of vertices in G, and the number of pendant vertices in both is equal. Therefore, using Corollary 2.3 (iii), we conclude that $S(G) \leq S(G^{\frac{1}{k}})$.

Case 2: If G is not a tree. The number of vertices and edges in graph $G^{\frac{1}{k}}$ is equal to $n_1 = n + (k-1)m$ and $m_1 = km$, respectively. Therefore, the average degree of the vertices in $G^{\frac{1}{k}}$ are $\frac{2km}{n+(k-1)m}$, and it can be seen that $2 \leq \frac{2km}{n+(k-1)m} \leq \frac{2m}{n}$. So, the number of vertices with a degree greater than $\frac{2m}{n}$ remains unchanged. Now, the conclusion follows from Theorem 2.1. □

The following theorem will be used to derive further results

Theorem 3.3. [7] *If T is a tree of order n, then*

$$Var(T) = \frac{2(n-2)}{n^2} + \frac{1}{n}\sum_{i=3}^{\Delta} N_i(i-1)(i-2).$$

By Theorem 3.3, we observe that for the path P_n and the star S_n, $Var(P_n) = \frac{2(n-2)}{n^2}$ and $Var(S_n) = \frac{(n-1)(n-2)^2}{n^2}$. So the path P_n and the star S_n attain the minimum and maximum degree variance. Thus for each tree T with n vertices,

$$\frac{2}{(n-1)(n-2)} \leq \frac{Var(T)}{Var(S_n)} \leq 1.$$

Corollary 3.4. *If T is a tree of order n and $k \in \mathbb{N}$, then,*

$$Var(T^{\frac{1}{k}}) = \frac{2\big((n-1)k-1\big)}{\big((n-1)k+1\big)^2} + \frac{1}{(n-1)k+1}\sum_{i=3}^{\Delta} N_i(i-1)(i-2).$$

Here, we compare $Var(T)$ with $Var(T^{\frac{1}{k}})$.

Theorem 3.5. *If T is a tree of order n and $k \in \mathbb{N}$, then*

$$Var(T) \leq (\frac{(n-1)k+1}{n})Var(T^{\frac{1}{k}}).$$

Proof. By Theorem 3.3 and Corollary 3.4, the value of $\sum_{i=3}^{\Delta} N_i(i-1)(i-2)$ in $Var(T)$ and $Var(T^{\frac{1}{k}})$ are equal. Since $\frac{(n-2)}{n} \leq \frac{\big((n-1)k-1\big)}{(n-1)k+1}$, so

$$\begin{aligned}(\frac{(n-1)k+1}{n})Var(T^{\frac{1}{k}}) &= \frac{2\big((n-1)k-1\big)}{n\big((n-1)k+1\big)} + \frac{1}{n}\sum_{i=3}^{\Delta} N_i(i-1)(i-2)\\ &\geq \frac{2(n-2)}{n^2} + \frac{1}{n}\sum_{i=3}^{\Delta} N_i(i-1)(i-2) = Var(T).\end{aligned}$$

□

The following theorem provides the degree deviation and degree variance of the k-subdivision of unicyclic graphs.

Theorem 3.6. *Let G be a graph and $k \in \mathbb{N}$, then $S(G) = S(G^{\frac{1}{k}})$ if and only if G is unicyclic graph.*

Proof. If G is a unicyclic graph, then the graph $G^{\frac{1}{k}}$ is also a unicyclic. In the k-subdivision, only the number of vertices of degree two increases and the degrees of the remaining vertices remain unchanged. Additionally, we have,

$$|V(G^{\frac{1}{k}})| = n + (k-1)m = n + (k-1)n = kn$$

and $|E(G^{\frac{1}{k}})| = km$. Since the average degree is 2, it follows by definition that

$$S(G) = S(G^{\frac{1}{k}}).$$

Conversly: assume thet $S(G) = S(G^{\frac{1}{k}})$. Let $M = \frac{2m}{n}$ be the average degree of the vertices of the graph G, and $M_k = \frac{2km}{n+(k-1)m}$ be the average degree of the vertices of the graph $G^{\frac{1}{k}}$. If $M = M_k$ then $m = n$, which shows that graph G is unicyclic. If $M \leq M_k$, then we have,

$$\begin{aligned}
\sum_{d_i<M_k}(M_k - d_i) &= \sum_{d_i<M}(M_k - d_i) + \sum_{M<d_i<M_k}(M_k - d_i)\\
&= \sum_{d_i<M}(M_k - M + M - d_i) + \sum_{M<d_i<M_k}(M_k - d_i)\\
&= \sum_{d_i<M}(M_k - M) + \sum_{d_i<M}(M - d_i) + \sum_{M<d_i<M_k}(M_k - d_i)\\
&= \sum_{d_i<M}(M_k - M) + \sum_{d_i<M\&d_i\in D_s(G)}(M - d_i)\\
&+ \sum_{d_i<M\&d_i\notin D_s(G)}(M - d_i) + \sum_{M<d_i<M_k}(M_k - d_i).
\end{aligned}$$

By Theorem 2.1, $S(G) = 2\sum_{d_i<M}(M - d_i)$ and $S(G^{\frac{1}{k}}) = 2\sum_{d_i<M_k}(M_k - d_i)$, and based on the assumpthion $\sum_{d_i<M}(M-d_i) = \sum_{d_i<M_k}(M_k-d_i)$. Therefore, we have

$$\textstyle\sum_{d_i<M}(M_k - M) + \sum_{d_i<M\&d_i\notin D_s(G)}(M - d_i) + \sum_{M<d_i<M_k}(M_k - d_i) = 0.$$

This equality holds only when $M = M_k$, which means that G is an unicyclic graph. □

Similarly, it can be shown that G is also an unicyclic graph when $M_k \leq M$. Similar to the proof of Theorem 3.6, we have the following theorem.

Theorem 3.7. *If G is a unicyclic graph and $k \in \mathbb{N}$, then $Var(G) = kVar(G^{\frac{1}{k}})$.*

4. Geometric mean irregularity

In this section, we introduce new measures of irregularity based on the geometric mean of vertex degrees in a graph. The geometric mean provides a distinct perspective on the distribution of vertex degrees, complementing traditional metrics such as degree deviation and degree variance. By defining the geometric mean of the degrees of the vertices in a graph G, we derive two new irregularity measures: geometric degree deviation and geometric degree variance.

These measures allow us to analyze the irregularity of graphs in a novel way, emphasizing the relationships between the geometric mean and the degree distribution. We will explore the properties of these geometric measures, demonstrate their applicability to various graph types, and establish their significance in understanding graph irregularity.

Definition 4.1. Let G be a graph with n vertices. The geometric mean of the degrees of the vertices of G is defined as follows:

$$GM(G) = \left(\prod_{i=1}^{n} d(v_i)\right)^{\frac{1}{n}}$$

Definition 4.2. Let G be a graph and let $GM(G)$ denote the geometric mean of the degrees of its vertices. The geometric degree deviation and geometric degree variance, denoted by $S_{GM}(G)$ and $Var_{GM}(G)$, respectively, are defined as follows:

$$S_{GM}(G) = \sum_{i=1}^{n} |d_i - GM(G)|$$
$$Var_{GM}(G) = \frac{1}{n} \sum_{i=1}^{n} (d_i - GM(G))^2$$

Theorem 4.3. *$S_{GM}(G)$ and $Var_{GM}(G)$ are irregularity measures.*

Theorem 4.4. *Let G be a bidegreed graph with n vertices. Then,*

(i) $S_{GM}(G) = N_\Delta \Delta - N_\delta \delta + (N_\delta - N_\Delta)\delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}}$.

(ii) $Var_{GM}(G) = \frac{1}{n}\left(N_\Delta \Delta^2 + N_\delta \delta^2 + n\delta^{\frac{2N_\delta}{n}} \Delta^{\frac{2N_\Delta}{n}} - 2(N_\delta \delta + N_\Delta \Delta)\delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}}\right)$.

Proof. It is not to check that $GM(G) = \delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}}$. So

(i)

$$\begin{aligned} S_{GM}(G) &= N_\delta |\delta - \delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}}| + N_\Delta |\Delta - \delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}}| \\ &= N_\Delta \Delta - N_\delta \delta + (N_\delta - N_\Delta)\delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}} \end{aligned}$$

(ii)

$$\begin{aligned} Var_{GM}(G) &= \frac{1}{n}\left(N_\delta(\delta - \delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}})^2 + N_\Delta(\Delta - \delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}})^2\right) \\ &= \frac{1}{n}\left(N_\Delta \Delta^2 + N_\delta \delta^2 + (N_\delta + N_\Delta)\delta^{\frac{2N_\delta}{n}} \Delta^{\frac{2N_\Delta}{n}} - 2(N_\delta \delta + N_\Delta \Delta)\delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}}\right). \end{aligned}$$

Now we are done using the fact that $N_\delta + N_\Delta = n$.

□

By Theorem 4.4 we have the following examples:

Example 4.5. *If P_n is a path with n vertices, then*

(i) $S_{GM}(P_n) = 2(n-3) - (n-4)\sqrt[n]{2^{n-2}}$.
(ii) $Var_{GM}(P_n) = \frac{1}{n}\Big(n(\sqrt[n]{2^{n-2}}-2)^2 + 4\sqrt[n]{2^{n-2}} - 6\Big)$.

Example 4.6. *If S_n is a star graph, then*

(i) $S_{GM}(S_n) = (n-2)\sqrt[n]{n-1}$.
(ii) $Var_{GM}(S_n) = (\sqrt[n]{n-1})^2 - 4\frac{n-1}{n}\sqrt[n]{n-1} + n - 1$.

Theorem 4.7. *If G is a k-regular graph of order n, then*

(i) $S_{GM}(G+v) = kn - (n-1)\sqrt[n+1]{(k+1)k^{n-1}}$.
(ii) $Var_{GM}(G+v) =$
$\frac{1}{n+1}\Big((n+1)(\sqrt[n+1]{(k+1)k^{n-1}})^2 - 2(kn+2)\sqrt[n+1]{(k+1)k^{n-1}} + k^2n + 2k + 2\Big)$.

Proof. (i) Since G is a k-regular graph,
$S_{GM}(G+v) = (n-1)[k - \sqrt[n+1]{(k+1)k^{n-1}}] + (k+1-\sqrt[n+1]{(k+1)k^{n-1}}) + (\sqrt[n+1]{(k+1)k^{n-1}} - 1) = kn - (n-1)\sqrt[n+1]{(k+1)k^{n-1}}$.
(ii) $Var_{GM}(G+v) = \frac{1}{n+1}[(n-1)(k - \sqrt[n+1]{(k+1)k^{n-1}})^2 + (k+1-\sqrt[n+1]{(k+1)k^{n-1}})^2 + (\sqrt[n+1]{(k+1)k^{n-1}} - 1)^2] = \frac{1}{n+1}[(n+1)(\sqrt[n+1]{(k+1)k^{n-1}})^2 - 2(kn+2)\sqrt[n+1]{(k+1)k^{n-1}} + k^2n + 2k + 2]$.

□

The following corollary examines the effect of vertex removal on the geometric degree deviation and the geometric degree variance of regular graphs.

Corollary 4.8. *If G is a k-regular graph of order n, then*

(i) $S_{GM}(G-v) = k(n-2k) + (2k-n+1)(k-1)^{\frac{k}{n-1}}k^{\frac{n-k-1}{n-1}}$.
(ii) $Var_{GM}(G-v) =$
$\frac{1}{n+1}\Big(k^2(n-3) + k + (n-1)(k-1)^{\frac{2k}{n-1}}k^{\frac{2n-2k-2}{n-1}} - 2k(n-2)(k-1)^{\frac{k}{n-1}}k^{\frac{n-k-1}{n-1}}\Big)$.

Theorem 4.9. *Let G be a graph and for every $d_i \in D(G), d_i \geq \frac{2m}{n}$ or $d_i \leq GM$.*

(i) *If $\sum_{d_i \leq GM} N_{d_i} > \sum_{d_i > GM} N_{d_i}$, then $S(G) > S_{GM}(G)$,*
(ii) *If $\sum_{d_i \leq GM} N_{d_i} < \sum_{d_i > GM} N_{d_i}$, then $S(G) < S_{GM}(G)$.*

Proof. Let $d_1, d_2, \cdots d_n$ be the sequence of degrees of vertices of the graph G. Then,

$$\begin{aligned}
S_{GM}(G) &= \sum_{i=1}^{n} |d_i - GM| \\
&= \sum_{d_i \le GM} (GM - d_i) + \sum_{d_i > GM} (d_i - GM) \\
&= \sum_{d_i \le GM} (GM - d_i + \frac{2m}{n} - \frac{2m}{n}) + \sum_{d_i > GM} (d_i - GM + \frac{2m}{n} - \frac{2m}{n}) \\
&= \sum_{d_i \le GM} (\frac{2m}{n} - d_i) + \sum_{d_i > GM} (d_i - \frac{2m}{n}) + \sum_{d_i \le GM} (GM - \frac{2m}{n}) \\
&+ \sum_{d_i > GM} (\frac{2m}{n} - GM) \\
&= S(G) + \sum_{d_i \le GM} (GM - \frac{2m}{n}) + \sum_{d_i > GM} (\frac{2m}{n} - GM)
\end{aligned}$$

Because the mean value of GM is always less than $\frac{2m}{n}$, if $\sum_{d_i \le GM} N_{d_i} > \sum_{d_i > GM} N_{d_i}$, then $S(G) > S_{GM}(G)$, and if $\sum_{d_i \le GM} N_{d_i} < \sum_{d_i > GM} N_{d_i}$, then $S(G) < S_{GM}(G)$. □

Theorem 4.10. *Let $G(\Delta, \delta)$ be a bidegreed graph.*

(i) *If $N_\delta > N_\Delta$, then $S(G) > S_{GM}(G)$,*
(ii) *If $N_\delta < N_\Delta$, then $S(G) < S_{GM}(G)$.*

Proof. According to the definition of $S(G)$ and $S_{GM}(G)$, we have

$$\begin{aligned}
S(G) &= \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right| \\
&= N_\delta(\frac{2m}{n} - \delta) + N_\Delta(\Delta - \frac{2m}{n}) \\
&= N_\delta(\frac{2m}{n} - \delta + GM - GM) + N_\Delta(\Delta - \frac{2m}{n} + GM - GM) \\
&= N_\delta(GM - \delta) + N_\Delta(\Delta - GM) + (\frac{2m}{n} - GM)(N_\delta - N_\Delta) \\
&= S_{GM}(G) + (\frac{2m}{n} - GM)(N_\delta - N_\Delta)
\end{aligned}$$

because the mean value of GM is always less than $\frac{2m}{n}$, if $N_\delta > N_\Delta$ then $S(G) > S_{GM}(G)$ and if $N_\delta < N_\Delta$, then $S(G) < S_{GM}(G)$. □

Theorem 4.11. *Let G be a graph and $k \in \mathbb{N}$. Then $GM(G) \geq 2$ if and only if $GM(G) \geq GM(G^{\frac{1}{k}})$.*

Proof. Let $\prod_{i=1}^{n} d(v_i) = A$. So $GM(G) = A^{\frac{1}{n}}$ and $GM(G^{\frac{1}{k}}) = (A2^{(k-1)m)})^{\frac{1}{n+(k-1)m}}$. Then, $GM(G) = A^{\frac{1}{n}} \geq 2$ if and only if $A^{(k-1)m} \geq 2^{(k-1)mn}$ if and only if

$$A^{n+(k-1)m} > A^{n}2^{(k-1)mn}$$

and this holds if and only if

$$GM(G) = A^{\frac{1}{n}} \geq (A2^{(k-1)m)})^{\frac{1}{n+(k-1)m}} = GM(G^{\frac{1}{k}}).$$

□

Theorem 4.12. *If G is a graph with n vertices, then*

$$2 \sum_{d_i \leq GM} |d_i - GM| \leq S_{GM}(G) \leq 2 \sum_{d_i > GM} |d_i - GM| .$$

Proof. Consider $D_s(G) = \{d_i \in D(G) | d_i \leq GM\}$, such that $D_s(G)$ contains s elements. So,

$$\begin{aligned} S_{GM}(G) &= \sum_{i=1}^{n} |d_i - GM| \\ &= (GM - d_1) + ... + (GM - d_s) + (d_{s+1} - GM) + ... + (d_n - GM) \\ &= sGM - (d_1 + d_2 + ... + d_s) + (d_{s+1} + ... + d_n) - (n-s)GM \\ &= sGM - (d_1 + d_2 + ... + d_n) + 2(d_{s+1} + ... + d_n) - (n-s)GM \\ &= sGM - n(\frac{2m}{n}) + 2(d_{s+1} + ... + d_n) - (n-s)GM \\ &= 2(d_{s+1} + ... + d_n) - 2(n-s)G + n(GM - \frac{2m}{n}) \\ &= 2 \sum_{d_i > GM} |d_i - GM| + n(GM - \frac{2m}{n}). \end{aligned}$$

Because the mean value of GM is always less than $\frac{2m}{n}$, then $S_{GM}(G) \leq 2\sum_{d_i>GM} |d_i - GM|$. Similarly, $S_{GM}(G) \geq 2\sum_{d_i \leq GM} |d_i - GM|$. □

Example 4.13. (i) *If G is a bidegreed graph, then by Theorem 4.12,*

$$2N_\delta(\delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}} - \delta) \leq S_{GM}(G) \leq 2N_\Delta(\Delta - \delta^{\frac{N_\delta}{n}} \Delta^{\frac{N_\Delta}{n}}).$$

(ii) *If S_n is a star graph, then by Theorem 4.12, we have*

$$2(n-1)(\sqrt[n]{n-1} - 1) \leq S_{GM}(S_n) \leq 2(n - 1 - \sqrt[n]{n-1}).$$

(iii) *If P_n is a path graph, then by Theorem 4.12, we have*

$$4(\sqrt[n]{2^{n-2}} - 1) \leq S_{GM}(P_n) \leq 2(n-2)(2 - \sqrt[n]{2^{n-2}}).$$

5. Conclusion

This paper investigated degree deviation and variance in k-regular graphs and graph subdivisions. By introducing novel geometric irregularity measures, we expanded the toolkit for analyzing graph irregularity. Our results reveal relationships between graph structure and degree properties while analyzing how these measures behave under various graph operations, such as vertex deletion, addition, and k-subdivision. Key findings include the characterization of degree deviation and variance for specific graph classes, such as trees, unicyclic graphs, and bidegreed graphs, as well as the establishment of inequalities and exact formulas for these measures. The introduction of geometric mean-based irregularity measures has further enriched the understanding of graph heterogeneity, with applications in network analysis and design. This work provides a foundation for future studies of complex networks and practical applications.

Data Availability

The data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this article.

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