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GENERALIZED MODULE ACTIONS AND TOPOLOGIES ON BANACH ALGEBRAS

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Abstract. This paper extends the notion of module actions on Banach algebras. It studies strict and quasi-strict topologies on the space of bilinear and separately continuous module actions defined over the dual space of a Banach algebra. The paper investigates how these topologies behave and their mathematical properties.

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1. Introduction

In [10, 11], Rieffel conducted a comprehensive study of the Banach module $\hom_A(A, W)$, which consists of the space of continuous homomorphisms from a topological algebra A into a topological left A-module W.

Let V and W be topological left A-modules, where both are topological vector spaces and A is a topological algebra. Denote by $\hom_A(V, W)$ the space of continuous linear A-module homomorphisms from V to W. When V is endowed with a bimodule structure over A, for each $T \in \hom_A(V, W)$, the operation $(a * T)(v) := T(v \cdot a)$ equips $\hom_A(V, W)$ with a left A-module structure. In particular, for any $b \in A$ and $v \in V$, the identity

$$(a * T)(b \cdot v) = T(b \cdot v) \cdot a = T(b \cdot (v \cdot a)) = b \cdot T(v \cdot a) = b \cdot (a * T)(v)$$

holds. This shows that $\hom_A(A, W)$ also inherits a left A-module structure under this operation. Furthermore, if A is commutative, the action $(T*a)(v) := T(a \cdot v)$ defines a right A-module structure on $\hom_A(V, W)$.

In the special case where $V = W = A$, the space $\hom_A(A, A)$ coincides with the multiplier algebra $M(A)$ of A. This naturally relates the notions of module homomorphisms, multipliers, and quasi-multipliers.

Let A be a Banach space. We denote its dual by A^*, and for each $a \in A$ and $\zeta \in A^*$, the pairing $\langle a, \zeta \rangle$ (also $\langle \zeta, a \rangle$) represents the canonical duality between A and A^*. We always view A as canonically embedded in A^{**} via the mapping π, defined by $\langle \pi(a), \zeta \rangle := \langle \zeta, a \rangle$ for all $a \in A$ and $\zeta \in A^*$.

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Assume now that A is a Banach algebra. The bidual A^{**} admits two natural extensions of multiplication: the first and second Arens products. This paper is concerned with the first Arens product. Let $a \in A$, $\zeta \in A^*$, and $F, G \in A^{**}$. Define the actions

$$\langle \zeta \cdot a, b\rangle := \langle \zeta, ab\rangle \quad \text{and} \quad \langle G \cdot \zeta, b\rangle := \langle G, \zeta \cdot b\rangle \quad \text{for all } b \in A.$$

Then, the first Arens product $F \circ G$ is given by

$$\langle F \circ G, \zeta\rangle := \langle F, G \cdot \zeta\rangle \quad \text{for all } \zeta \in A^*.$$

The second Arens product, denoted $\circ'$, is defined in a similar manner. With these multiplications, A^{**} becomes a Banach algebra containing A as a subalgebra.

A Banach algebra A is called factorable if $AA = A$. The subspaces $A^*A := \{\zeta \cdot a : \zeta \in A^*, a \in A\}$ and $AA^* := \{a \cdot \zeta : a \in A, \zeta \in A^*\}$ are often considered. We say that A^* factors on the left if $A^*A = A^*$ and on the right if $AA^* = A^*$.

In [1–4], we explored extensions of the theory of quasi-multipliers to settings beyond Banach algebras. In particular, we examined quasi-multipliers in the context of f-algebras, ℓ-algebras, duals of Banach algebras, and general topological algebras. Our analysis addressed how classical methods can be adapted to extend the theory to these broader frameworks.

In this paper, we further investigate module actions and introduce several notions of strict and quasi-strict topologies on the space $\mathcal{B}_{\text{mod}}(A^*)$, the collection of all bilinear module actions over A^*.

2. Generalized module actions

Definition 2.1. Let $m : A^* \times A \to A^*$ be a bilinear map. We say that m defines a *right module action* over A^* if it satisfies the following conditions:

$$m(a \cdot \zeta, b) = a \cdot m(\zeta, b) \quad \text{and} \quad m(\zeta, ba) = m(\zeta, b) \cdot a, \tag{1}$$

for all $\zeta \in A^*$ and $a, b \in A$.

Similarly, a bilinear map $m' : A \times A^* \to A^*$ is called a *left module action* over A^* if the following equalities hold:

$$m'(ab, \zeta) = a \cdot m'(b, \zeta) \quad \text{and} \quad m'(b, \zeta \cdot a) = m'(b, \zeta) \cdot a,$$

for every $\zeta \in A^*$ and $a, b \in A$.

Let $\mathcal{B}_{\text{mod}}(A^*)$ be the set of all bilinear and separately continuous right module actions over A^*. It is clear that $\mathcal{B}_{\text{mod}}(A^*)$ forms a linear space. Moreover, it becomes a Banach space when equipped with the norm

$$\|m\| = \sup\{\|m(\zeta, a)\| : \zeta \in A^*,\ a \in A,\ \|\zeta\| \le 1,\ \|a\| \le 1\}.$$

Let A be a Banach algebra. A mapping $T\colon A^* \to A^*$ is called a *right module action* over A^* if

$$T(a \cdot \zeta) = a \cdot T(\zeta), \quad \text{for all } \zeta \in A^*,\ a \in A.$$

We denote by $\mathcal{M}_{\rm mod}(A^*)$ the space of all bounded linear right module actions over A^*. It is clear that for each $a \in A$, the right multiplication operator R_a defined by

$$R_a(\zeta) = \zeta \cdot a$$

is a right module action over A^*.

Definition 2.2. A bounded approximate identity $\{e_\lambda : \lambda \in I\}$ in a Banach algebra A is called an *ultra***-approximate identity* if, for every $m \in \mathcal{B}_{\rm mod}(A^*)$ and every $\zeta \in A^*$, the net $\{m(\zeta, e_\lambda) : \lambda \in I\}$ is Cauchy in A^*.

Theorem 2.3. *Let A be a factorable Banach algebra equipped with an ultra*-approximate identity $\{e_\lambda\}$. Then the mapping*

$$\rho\colon \mathcal{M}_{\rm mod}(A^*) \to \mathcal{B}_{\rm mod}(A^*), \quad \rho_T(\zeta, a) = (T\zeta) \cdot a$$

for all $T \in \mathcal{M}_{\rm mod}(A^*)$, $\zeta \in A^*$, and $a \in A$, is bijective with $\|\rho\| \leq 1$. Furthermore, if $\{e_\lambda\}$ is an approximate identity of norm one, then ρ is an isometry.

Proof. Let $T \in \mathcal{M}_{\rm mod}(A^*)$ be arbitrary. It is clear that ρ_T defines a bilinear mapping from $A^* \times A$ into A^* and that it is bounded with norm at most $\|T\|$. For $\zeta \in A^*$ and $a, b \in A$, we compute:

$$\rho_T(a \cdot \zeta, b) = T(a \cdot \zeta) \cdot b = (a \cdot T\zeta) \cdot b = a \cdot (T\zeta \cdot b) = a \cdot \rho_T(\zeta, b),$$

and

$$\rho_T(\zeta, ba) = (T\zeta) \cdot (ba) = ((T\zeta) \cdot b) \cdot a = \rho_T(\zeta, b) \cdot a.$$

Clearly, ρ_T is bilinear and separately continuous, hence $\rho_T \in \mathcal{B}_{\rm mod}(A^*)$. Therefore, the mapping $\rho\colon \mathcal{M}_{\rm mod}(A^*) \to \mathcal{B}_{\rm mod}(A^*)$ is linear and satisfies $\|\rho_T\| \leq \|T\|$, which implies $\|\rho\| \leq 1$.

If $\rho_T = 0$, then $(T\zeta) \cdot a = 0$ for all $\zeta \in A^*$ and $a \in A$. Thus, for any $b \in A$,

$$\langle T\zeta \cdot a, b\rangle = \langle T\zeta, ab\rangle = 0.$$

Since A is factorable, it follows that $T = 0$.

Now let $m \in \mathcal{B}_{\rm mod}(A^*)$ be arbitrary. Define a mapping $T\colon A^* \to A^*$ by

$$T\zeta = \lim_\lambda m(\zeta, e_\lambda),$$

which exists in A^* by the assumption that $\{e_\lambda\}$ is an ultra*-approximate identity. Then $T \in \mathcal{M}_{\rm mod}(A^*)$ and

$$\rho_T(\zeta, a) = (T\zeta) \cdot a = \lim_\lambda m(\zeta, e_\lambda) \cdot a = m(\zeta, a),$$

so ρ is surjective.

Finally, suppose that $\{e_\lambda\}$ is an approximate identity for A of norm one. Let $T \in \mathcal{M}_{\rm mod}(A^*)$ and $\varepsilon > 0$ be arbitrary. Choose $\zeta \in A^*$ such that $\|\zeta\| \leq 1$ and $\|T\zeta\| > \|T\| - \varepsilon$. Since for any $a \in A$,

$$\lim_\lambda \langle T\zeta \cdot e_\lambda, a\rangle = \lim_\lambda \langle T\zeta, e_\lambda \cdot a\rangle = \langle T\zeta, a\rangle,$$

it follows that

$$\|\rho_T\| \geq \lim_{\lambda} \|\rho_T(\zeta, e_\lambda)\| = \lim_{\lambda} \|T\zeta \cdot e_\lambda\| > \|T\| - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\|\rho_T\| = \|T\|$, so ρ is an isometry. □

Let A be a factorable Banach algebra with an ultra*-approximate identity. Using Theorem 2.3, we can define a multiplication operation on $\mathcal{B}_{\text{mod}}(A^*)$ that turns it into a Banach algebra. The procedure is as follows:

Let $m_1, m_2 \in \mathcal{B}_{\text{mod}}(A^*)$. By Theorem 2.3, there exist $T_1, T_2 \in \mathcal{M}_{\text{mod}}(A^*)$ such that $m_1 = \rho_{T_1}$ and $m_2 = \rho_{T_2}$. We define their product by

$$m_1 \circ_\rho m_2 := \rho_{T_2 T_1},$$

which is a well-defined multiplication in $\mathcal{B}_{\text{mod}}(A^*)$.

3. Strict topology and quasi-strict topology on $\mathcal{B}_{mod}(A^*)$

This section is devoted to the definition of the strict and quasi-strict topologies on the space $\mathcal{B}_{\text{mod}}(A^*)$, and to extending certain related results from [5,7,9] to the setting of bilinear module actions.

Note that $\mathcal{B}_{\text{mod}}(A^*)$ can be endowed with an A-bimodule structure as follows. For $m \in \mathcal{B}_{\text{mod}}(A^*)$ and $a \in A$, define the left and right actions $a * m$ and $m * a$ as functions from $A^* \times A$ to A^*, given by

$$\begin{aligned} (a * m)(\zeta, b) &= m(\zeta \cdot a, b), \\ (m * a)(\zeta, b) &= m(\zeta, ab), \end{aligned} \qquad (\zeta \in A^*,\ b \in A).$$

It is straightforward to verify that $a * m$ and $m * a$ both belong to $\mathcal{B}_{\text{mod}}(A^*)$, thus confirming that $\mathcal{B}_{\text{mod}}(A^*)$ indeed forms an A-bimodule.

Let τ denote the norm topology on $\mathcal{B}_{\text{mod}}(A^*)$.

Definition 3.1. The *strict topology* β on $\mathcal{B}_{\text{mod}}(A^*)$ is defined by the family of seminorms

$$m \mapsto \|m * a\|, \qquad (a \in A).$$

Definition 3.2. The *quasi-strict topology* γ on $\mathcal{B}_{\text{mod}}(A^*)$ is defined by the family of seminorms

$$m \mapsto \|m(\zeta, a)\|, \qquad (\zeta \in A^*,\ a \in A).$$

Lemma 3.3. *If A is a factorable Banach algebra, then $\gamma \subseteq \beta \subseteq \tau$.*

Proof. Let $\{m_\alpha\}_{\alpha \in I}$ be a net in $\mathcal{B}_{\text{mod}}(A^*)$ converging to $m \in \mathcal{B}_{\text{mod}}(A^*)$ in the β-topology. Fix an arbitrary $\zeta \in A^*$. Since A is factorable, for every $a \in A$ there exist $b, c \in A$ such that $a = bc$. By the definition of the β-topology, we have $\|m_\alpha * b - m * b\| \to 0$, and hence

$$\begin{aligned} \|m_\alpha(\zeta, a) - m(\zeta, a)\| &= \|m_\alpha(\zeta, bc) - m(\zeta, bc)\| \\ &= \|(m_\alpha * b)(\zeta, c) - (m * b)(\zeta, c)\| \to 0. \end{aligned}$$

This proves that $\{m_\alpha\}$ converges to m in the γ-topology. The inclusion $\beta \subseteq \tau$ is immediate from the definition. $\square$

Proposition 3.4. *Let A be a factorable Banach algebra equipped with an ultra*-approximate identity. Suppose that A^* factors on the right. Then the map*

$$\phi_A : (A, \tau) \to (\mathcal{B}_{\text{mod}}(A^*), \beta)$$

defined by

$$(\phi_A(a))(\zeta, b) = \zeta \cdot ab$$

is a continuous injective homomorphism.

Proof. Let $\{a_\alpha\}$ be a net in A converging to $a \in A$ in the topology τ. For any $\zeta \in A^*$ and $b, c \in A$, we compute

$$\begin{aligned}\|(\phi_A(a_\alpha) * b)(\zeta, c) - (\phi_A(a) * b)(\zeta, c)\| &= \|(\phi_A(a_\alpha))(\zeta, bc) - (\phi_A(a))(\zeta, bc)\| \\ &= \|\zeta \cdot a_\alpha bc - \zeta \cdot abc\| \to 0.\end{aligned}$$

Therefore, $\phi_A(a_\alpha) \to^\beta \phi_A(a)$. This establishes continuity.

We now show that ϕ_A is multiplicative. Let $a_1, a_2 \in A$. By Theorem 2.3, there exist $T_1, T_2 \in \mathcal{M}_{\text{mod}}(A^*)$ such that

$$\phi_A(a_1) = \rho_{T_1}, \quad \phi_A(a_2) = \rho_{T_2}.$$

For all $\zeta \in A^*$ and $b \in A$, we have

$$T_1(\zeta) \cdot b = \zeta \cdot a_1 b, \quad T_2(\zeta) \cdot b = \zeta \cdot a_2 b.$$

Thus,

$$\begin{aligned}(\phi_A(a_1) \circ_\rho \phi_A(a_2))(\zeta, b) &= \rho_{T_2 T_1}(\zeta, b) = T_2(T_1(\zeta)) \cdot b = T_1(\zeta) \cdot (a_2 b) \\ &= \zeta \cdot a_1 a_2 b = \phi_A(a_1 a_2)(\zeta, b).\end{aligned}$$

To prove injectivity, suppose $\phi_A(a) = 0$ for some $a \in A$. Then for all $\zeta \in A^*$ and $b \in A$, we have

$$\langle \zeta, ab \rangle = 0 \quad \Rightarrow \quad \langle b \cdot \zeta, a \rangle = 0.$$

Since A^* factors on the right, the functionals $b \cdot \zeta$ span a dense subset of A^*, so it follows that $a = 0$. Hence, ϕ_A is injective. $\square$

Proposition 3.5. *Let A be a factorable Banach algebra equipped with an ultra*-approximate identity. If A^* factors on the right, then the image $\varphi_A(A)$ is closed in the β-topology.*

Proof. Let $m \in \overline{\varphi_A(A)}^\beta$, so there exists a net $\{a_\alpha\} \subseteq A$ such that $\varphi_A(a_\alpha) \xrightarrow{\beta} m$.

By the definition of the β-topology, for each $b \in A$, we have

$$\varphi_A(a_\alpha) * b \to m * b \quad \text{in norm.}$$

Since

$$(\varphi_A(a_\alpha) * b)(\zeta, c) = \zeta \cdot a_\alpha bc \quad \text{for all } \zeta \in A^*, \ c \in A,$$

we deduce that $\zeta \cdot a_\alpha bc \to (m * b)(\zeta, c)$ for all ζ, b, c.

This implies that $\{a_\alpha\}$ is Cauchy in the τ-topology of A. Since A is complete, there exists $a \in A$ such that $a_\alpha \to_\tau a$.

By Proposition 3.4, φ_A is continuous, hence

$$\varphi_A(a_\alpha) \xrightarrow{\beta} \varphi_A(a).$$

Because limits in Hausdorff spaces are unique and $\varphi_A(a_\alpha) \xrightarrow{\beta} m$, it follows that $m = \varphi_A(a)$. Therefore, $m \in \varphi_A(A)$, and so the image is β-closed. □

Theorem 3.6. *Let A be a factorable Banach algebra equipped with an ultra***-approximate identity. Then the topological vector spaces $(\mathcal{B}_{\mathrm{mod}}(A^*), \gamma)$, $(\mathcal{B}_{\mathrm{mod}}(A^*), \tau)$, and $(\mathcal{B}_{\mathrm{mod}}(A^*), \beta)$ have identical collections of bounded subsets.*

Proof. Since $\gamma \subseteq \tau$, every τ-bounded set is necessarily γ-bounded.

Conversely, suppose that $K \subseteq \mathcal{B}_{\mathrm{mod}}(A^*)$ is γ-bounded. Then for each $a \in A$ and $\zeta \in A^*$, there exists a constant $r = r(a, \zeta) > 0$ such that

$$\|m(a,\zeta)\| \le r, \quad \text{for all } m \in K. \tag{2}$$

For each $m \in K$ and $\zeta \in A^*$, define the operator $M_\zeta : A \to A^*$ by

$$M_\zeta(a) := m(a,\zeta), \quad (a \in A).$$

Let $\mathcal{K} := \{M_\zeta : m \in K\}$. By inequality (2), we have

$$\|M_\zeta(b)\| = \|m(b,\zeta)\| \le r(b,\zeta), \quad \text{for all } m \in K,\ b \in A.$$

Thus, $\mathcal{K}$ is pointwise bounded. By the principle of uniform boundedness, there exists a constant $c = c(\zeta) > 0$ such that

$$\|M_\zeta\| \le c, \quad \text{for all } m \in K. \tag{3}$$

Now, for each $m \in K$, define a seminorm λ_m on A^* by

$$\lambda_m(\zeta) := \|M_\zeta\| = \sup_{\|a\|\le 1} \|m(a,\zeta)\|, \quad (\zeta \in A^*).$$

To verify continuity of λ_m, let $\{\zeta_n\} \subseteq A^*$ converge to $\zeta_0 \in A^*$. Then

$$|\lambda_m(\zeta_n) - \lambda_m(\zeta_0)| \le \lambda_m(\zeta_n - \zeta_0) = \sup_{\|a\|\le 1} \|m(a, \zeta_n - \zeta_0)\| \to 0,$$

so λ_m is continuous.

Since the family $\Lambda := \{\lambda_m : m \in K\}$ is pointwise bounded and each λ_m is continuous, the Banach–Steinhaus theorem (see [6, p. 142]) implies that there exist a ball

$$B(\zeta_0, r) := \{\zeta \in A^* : \|\zeta - \zeta_0\| \le r\}$$

and a constant $\alpha_0 > 0$ such that

$$\lambda_m(\zeta) \le \alpha_0, \quad \text{for all } \zeta \in B(\zeta_0, r),\ m \in K.$$

Now, for any $\zeta \in A^*$ with $\|\zeta\| \le 1$, we estimate

$$\lambda_m(\zeta) = \frac{\lambda_m(r\zeta + \zeta_0 - \zeta_0)}{r} \le \frac{1}{r}\left(\lambda_m(r\zeta + \zeta_0) + \lambda_m(\zeta_0)\right) \le \frac{2\alpha_0}{r}.$$

Hence,

$$\|m\| = \sup_{\|a\|\leq 1, \|\zeta\|\leq 1} \|m(a,\zeta)\| = \sup_{\|\zeta\|\leq 1} \lambda_m(\zeta) \leq \frac{2\alpha_0}{r},$$

which shows that K is τ-bounded. This establishes that the bounded sets in $(\mathcal{B}_{\text{mod}}(A^*), \gamma)$ and $(\mathcal{B}_{\text{mod}}(A^*), \tau)$ coincide.

Finally, by Lemma 3.3, we conclude that the bounded sets in all three topologies γ, τ, and β on $\mathcal{B}_{\text{mod}}(A^*)$ coincide. □

Theorem 3.7. *Let A be a factorable Banach algebra equipped with an ultra*-approximate identity.*

(i) *The space $(\mathcal{B}_{mod}(A^*), \gamma)$ is complete.*
(ii) *If A admits a norm-one approximate identity, then the space $(\mathcal{B}_{mod}(A^*), \beta)$ is also complete.*

Proof. (i) Let $\{m_\alpha\}_{\alpha\in I}$ be a γ-Cauchy net in $\mathcal{B}_{mod}(A^*)$. By the definition of the γ-topology, for any $\zeta \in A^*$ and $a \in A$, the net $\{m_\alpha(\zeta, a)\}_{\alpha\in I}$ is Cauchy in the norm topology of A^*. Define

$$m(\zeta, a) = \lim_{\alpha} m_\alpha(\zeta, a).$$

Clearly, m is a bilinear mapping on $A^* \times A$ satisfying condition (1). Moreover, by the Uniform Boundedness Principle [8, p. 172], m is separately continuous, and thus $m \in \mathcal{B}_{mod}(A^*)$.

(ii) Suppose now that $\{m_\alpha\}_{\alpha\in I}$ is a β-Cauchy net in $\mathcal{B}_{mod}(A^*)$. For each $a \in A$, define the operator

$$T_a^\alpha : A^* \to A^*, \quad T_a^\alpha(\zeta) = m_\alpha(\zeta, a),$$

which belongs to $\mathcal{M}_{mod}(A^*)$. It follows that $\rho_{T_a^\alpha} = m_\alpha * a$. By the definition of the β-topology, the net $\{\rho_{T_a^\alpha}\}_{\alpha\in I}$ is Cauchy in the norm of $\mathcal{B}_{mod}(A^*)$.

Using Theorem 2.3, we know that the map ρ is an isometry, so the net $\{T_a^\alpha\}$ is Cauchy in the norm of $\mathcal{M}_{mod}(A^*)$. Since $\mathcal{M}_{mod}(A^*)$ is a Banach space, there exists $T_a \in \mathcal{M}_{mod}(A^*)$ such that $\|T_a^\alpha - T_a\| \to 0$.

By Lemma 3.3, the net $\{m_\alpha\}$ is γ-Cauchy. From part (i), we conclude that $\mathcal{B}_{mod}(A^*)$ is γ-complete. Hence, there exists $m \in \mathcal{B}_{mod}(A^*)$ such that

$$\lim_{\alpha} m_\alpha(\zeta, a) = m(\zeta, a) \quad \text{for all } \zeta \in A^*,\ a \in A.$$

Now, for each $b \in A$, we compute:

$$\begin{aligned} \rho_{T_a}(\zeta, b) &= \lim_{\alpha} \rho_{T_a^\alpha}(\zeta, b) = \lim_{\alpha} (m_\alpha * a)(\zeta, b) \\ &= \lim_{\alpha} m_\alpha(\zeta, ab) = m(\zeta, ab) = (m * a)(\zeta, b). \end{aligned}$$

Therefore,

$$\|m_\alpha * a - m * a\| = \|\rho_{T_a^\alpha} - \rho_{T_a}\| = \|T_a^\alpha - T_a\| \to 0,$$

which implies that m is the β-limit of the net $\{m_\alpha\}_{\alpha\in I}$. Hence, $\mathcal{B}_{mod}(A^*)$ is complete with respect to the β-topology. □

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