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MULTIPLICATIVE DERIVATIONS IN $\vee$-HOOP ALGEBRAS

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Abstract. In this paper, first, while introducing multiplicative derivations, we examine some properties of these derivations and present properties of multiplicative derivations in $\vee$-hoop algebras. Then we show that the set of multiplicative derivations on $\vee$-hoop algebras forms a distributive lattice under certain conditions. Also, while examining the relationship between the square root and the derivation on $\vee$-hoop algebras and introducing the critical point by using the composition of them, we present some characteristics of the critical point. Finally, we show that the set of critical points forms a distributive lattice.

Keywords: Hoop algebra, Derivation, Multiplicative derivation, Square root, Fixed point, Critical point.
2020 MSC: 06F99, 06B99, 16W25, 03G25.

1. Introduction

Hoop algebras are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach [6, 7]. In a manuscript by Büchi and Owens [8] devoted to a study of Bosbach's algebras, written in the mid-seventies, the commutative members of this equational class were given the name *hoops*. Over the past 50 years, many scientists and researchers have studied the properties of these algebras, resulting in numerous published articles. The notion of derivation, introduced from the analytic theory, is useful for studying the structure and properties of algebraic systems. Several authors studied derivations in rings and near-rings [3, 10, 21]. Jun and Xin [14] applied the notion of derivation to BCI-algebras. In [22], Szász introduced the concept of derivation for lattices and investigated some of its properties. Also, in [25], Xin et al. improved derivation for a lattice and discussed some related properties. In 2004, Jun and Xin [14], introduced the notion of derivation on BCI-algebras, which is defined in a way similar to the notion in ring theory, and investigated some properties related to this concept. In 2010, Alshehri [2] applied the notions of (additive) derivations to MV-algebras and discussed some related properties. She also proved that an additive derivation of a linearly ordered MV-algebra is isotone. After the work of Alshehri, many research articles have appeared on the derivations of MV-algebras in different aspects. For example, in 2013,

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Yazarli et al. [26] further investigated several kinds of generalized derivations on MV-algebras and obtained some interesting results. Ghorbani et al. [12], generalized the concept of derivation on MV-algebras to $(\odot, \oplus)$-derivations and $(\ominus, \odot)$-derivations, presenting other results. In the same year, Torkzadeh and Abbasian [23], also presented results regarding the effect of the derivation on BL-algebras. In 2019, Wang et al. [24] gave some representations of MV-algebras in terms of derivations. In [27], Zhang and Xin introduced the notion of derivation in semihoops algebras and, while investigating some properties, especially regarding the fixed point, they introduced the concepts of differential semihoops and differential filters and then studied some of their properties. In 2020, Zhu et al. [28] introduced the notion of generalized derivations on residuated lattices. Also in [15], Yemene et al. presented some types of derivations in bounded commutative residuated lattices, and in 2025, Nasr Esfahani et al. [19] investigated some properties of ideals and derivations in BL-algebras.

In this paper, we first define the concept of derivation on hoop algebras and study some of its properties. Then, we show that the set of fixed points of each hoop algebra, under certain conditions, forms a hoop algebra. Following this, we demonstrate that the set of derivations of a hoop algebra, under certain conditions, forms a bounded lattice. In Section 4, we examine the relationship between square root and derivation on hoop algebras. Additionally, we introduce the concept of the critical point, provide some examples, and present some properties of this point. Finally, we show that the set of all critical points forms a distributive lattice (see Proposition 4.15 below).

2. **Preliminaries**

In this section, we introduce the definitions and some features of hoop algebras, as well as the types of hoop algebras, including bounded hoop algebras, cancellative hoop algebras, etc. Then we present some features of the square root of hoop algebras.

Definition 2.1. [1, 16] A hoop algebra is an algebra $(H; \odot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that

(H1) $(H; \odot, 1)$ is a commutative monoid with a unit element 1;
(H2) $x \rightarrow x = 1$ for all $x \in H$;
(H3) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$ for all $x, y \in H$;
(H4) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ for all $x, y, z \in H$.

In a hoop algebra $(H; \odot, \rightarrow, 1)$, we define $x \leq y$ if and only if $x \rightarrow y = 1$. It is easily seen that $\leq$ defines a partial order relation on H (see also [1]). We define $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for every $n \in \mathbb{N}$. An element $x \in H$ is called idempotent if $x \odot x = x$. The set of idempotent elements in H is denoted by $\mathrm{Id}(H)$. H is called idempotent if each of its elements is idempotent. H is called bounded if it has a least element denoted by 0. Let H be a bounded hoop algebra. We define the unary operation " $'$ " for every $x \in H$ as $x' := x \rightarrow 0$.

We also set $(x')' = x''$. An element $x \in H$ is called regular if $x'' = x$. The set of regular elements in H is denoted by reg(H). If for every $x \in H$, $x'' = x$, then we say that the bounded hoop algebra H has the "double negation property", or simply (DNP).

Let H and G be two hoop algebras. A map $f : H \to G$ is called a *hoop homomorphism* if for every $x, y \in H$, $f(x \odot y) = f(x) \odot f(y)$, $f(x \to y) = f(x) \to f(y)$. Then one can prove that it is order-preserving and $f(1) = 1$. In addition, if f is one to one, then it is an order embedding. Indeed, for $x, y \in H$ we have $x \leq y$ iff $x \to y = 1$ iff $f(x \to y) = f(1) = 1$ iff $f(x) \to f(y) = 1$ iff $f(x) \leq f(y)$. If, in addition, the hoop algebras are bounded, then we also assume $f(0) = 0$.

Proposition 2.2. [17, Proposition 2.3] *If $(H; \odot_H, \to_H, 1_H)$ and $(G; \odot_G, \to_G, 1_G)$ are two hoop algebras, then $(H \times G; \odot_{H\times G}, \to_{H\times G}, (1_H, 1_G))$ becomes a hoop algebra with the operations $(x_1, x_2) \odot_{H\times G} (y_1, y_2) = (x_1 \odot_H y_1, x_2 \odot_G y_2)$, and $(x_1, x_2) \to_{H\times G} (y_1, y_2) = (x_1 \to_H y_1, x_2 \to_G y_2)$, for every $x_1, y_1 \in H$ and $x_2, y_2 \in G$.*

Theorem 2.3. [6, 7, 11] *Let $(H; \odot, \to, 1)$ be a hoop algebra. Then, for every $x, y, z \in H$,*
(i) *$(H; \leq)$ is a $\wedge$-semilattice with $x \wedge y = x \odot (x \to y)$,*
(ii) *$x \odot y \leq z$ if and only if $x \leq y \to z$,*
(iii) *$x \odot y \leq x, y$,*
(iv) *$x \odot (x \to y) \leq x, y$,*
(v) *$x \leq y \to x$,*
(vi) *$1 \to x = x$,*
(vii) *$x \to 1 = 1$,*
(viii) *$x \leq y$ implies $x \odot z \leq y \odot z$,*
(ix) *$x \leq y$ implies $z \to x \leq z \to y$,*
(x) *$x \leq y$ implies $y \to z \leq x \to z$.*

Proposition 2.4. *[11] For any hoop algebra H, define $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. Then, the following conditions are equivalent:*
(i) *The operation $\vee$ is associative.*
(ii) *$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$, for all $x, y, z \in H$.*
(iii) *$x \leq y$ implies $x \vee z \leq y \vee z$, for all $x, y, z \in H$.*
(iv) *The operation $\vee$ is a join operation on H.*

A hoop algebra H is called $\vee$-hoop, if $\vee$ is a join operation on H (see [11, Remark 2.4]).

Proposition 2.5. (See [11, Lemma 2.9]) *Let H be a hoop algebra, $x, y \in H$. If arbitrary joins exist, then*
(i) *$x \odot (\bigvee_{i\in I} y_i) = \bigvee_{i\in I}(x \odot y_i)$, for every $\{y_i \mid i \in I\} \subseteq H$,*
(ii) *$x \wedge (\bigvee_{i\in I} y_i) = \bigvee_{i\in I}(x \wedge y_i)$ for every $\{y_i \mid i \in I\} \subseteq H$.*

Remark 2.6. [11, Proposition 4.6] Let H be a $\vee$-hoop algebra. Then, $(H; \vee, \wedge)$ is a distributive lattice.

Recall [4, Definition 1.12] that a *Wajsberg hoop* is a hoop algebra that satisfies the following condition of antipode (T): $(x \to y) \to y = (y \to x) \to x$, $\forall x, y \in H$. Any hoop that satisfies (T) is in fact a lattice, and the operation $\vee$ is equal to $x \vee y = (x \to y) \to y$ for every $x, y \in H$ (As mentioned already $x \wedge y = x \odot (x \to y)$ for all $x, y \in H$).

Proposition 2.7. [6, 7] *Let H be a bounded hoop algebra. Then, for every $x, y \in H$,*
(i) $x \leq x''$, $x \odot x' = 0$, $x''' = x'$, $x' \leq x \to y$,
(ii) *If* $x \leq y$, *then* $y' \leq x'$,
(iii) $x \odot y = 0$ *if and only if* $x \leq y'$.

Definition 2.8. [9] Let L be a bounded lattice with 0 and 1. For $a \in L$, we say $b \in L$ is a *complement* of a if $a \vee b = 1$ and $a \wedge b = 0$. If a has a unique complement, we denote this complement by a^*. The set of all complemented elements in L, denoted by $B(L)$, is called the *Boolean center* of L.

Lemma 2.9. [20, Lemmas 1.13 and 1.14] *Let H be a bounded $\vee$-hoop algebra, $x \in B(H)$, and x^* be the complement of x. Then*
(i) $x' = x^*$ *and* $x'' = x$,
(ii) $x \odot a = x \wedge a$, *for all* $a \in H$.

Lemma 2.10. [20, Proposition 1.17] *Let H be a bounded $\vee$-hoop algebra and $x \in B(H)$. In this case, the following statements hold:*
(i) $x^2 = x$ *and* $x'' = x$,
(ii) $x^2 = x$ *and* $x' \to x = x$,
(iii) $(x \to a) \to x = x$, *for all* $a \in H$,
(iv) $x' \wedge x = 0$.

A hoop algebra H is called *cancellative* if $(H; \odot, 1)$ is a cancellative monoid (see [11]).

Proposition 2.11. [11, Proposition 4.2] *Let H be a cancellative hoop algebra. Then, for any $x, y, z \in H$:*
(i) $z \to x = (z \odot y) \to (x \odot y)$,
(ii) $x \odot y \leq z \odot y$ *if and only if* $x \leq z$.

Definition 2.12. [5] Let H be a hoop algebra. A subset F of H is called a *filter* if it satisfies the following conditions:
(F1) $1 \in F$,
(F2) $x \odot y \in F$, for all $x, y \in F$,
(F3) If $x \leq y$ and $x \in F$, then $y \in F$, for all $x, y \in H$.

Let H be a hoop algebra and $\emptyset \neq X \subseteq H$. The intersection of all filters of H containing X is denoted by $\langle X \rangle$ and it is equal to

$$\langle X \rangle = \Big\{ a \in H : \exists n \in \mathbb{N}, x_1, x_2, \cdots, x_n \in X, \quad x_1 \odot x_2 \odot \cdots \odot x_n \leq a \Big\}.$$

In particular, for each $x \in H$ we have $\langle x \rangle = \{a \in H : \exists n \in \mathbb{N}, x^n \leq a\}$.

A filter F of H is called proper if $F \neq H$. It is clear that H and $\{1\}$ are trivial filters of H. It can be seen that if H is a bounded hoop algebra, then a filter is proper if and only if it does not contain 0. A proper filter P of a $\vee$-hoop algebra H is *prime* if for every $x, y \in H$, $x \vee y \in P$ implies $x \in P$ or $y \in P$.

Definition 2.13. [13] Let H be a hoop algebra. A mapping $s : H \to H$ is called a *square root* on H if it satisfies the following conditions:
(S1) For every $x \in H$, $s(x) \odot s(x) = x$,
(S2) For all $x, y \in H$, if $y \odot y \leq x$, then $y \leq s(x)$.

Example 2.14. *a) Let $I = [0, 1]$ be the unit closed interval of real numbers. Define the operations $\odot$ and $\to$ on I as follows: For any $x, y \in I$,*

$$x \odot y = \min(x, y) \quad \textit{and} \quad x \to y = \begin{cases} 1 & \textit{if } x \leq y \\ y & \textit{if } x > y \end{cases}.$$

*The structure **Goa** $:= (I; \odot, \to, 1)$ defined with these operations is a hoop algebra. This algebraic structure is known as the Gödel algebra (see [20]). In **Goa**, define the unary operation $s : I \to I$ as $s(x) = x$ for every $x \in I$. Then, s is a square root on **Goa**.*
b) Let $I = [0, 1]$ be the unit closed interval of real numbers. Define the operations $\odot$ and $\to$ on I as follows. For any $x, y \in I$,

$$x \odot y = x \cdot y \quad \textit{and} \quad x \to y = \begin{cases} 1 & \textit{if } x \leq y \\ \frac{y}{x} & \textit{if } x > y \end{cases}.$$

*The structure **Pra** $:= (I; \odot, \to, 1)$ is a hoop algebra (see [20]). This algebraic structure is known as the product algebra. In the algebra **Pra**, define the unary operation $s : I \to I$ as $s(x) = \sqrt{x}$, $\forall x \in I$. Then, it is clear s is a square root on **Pra**.*

Notice that: 1) Not all hoop algebras have a square root (see Example 3.2(iv) below) and 2) those that have a square root, then this square root is uniquely determined (see [17, Theorem 3.5]).

Theorem 2.15. [17, Theorem 3.3] *Let H be a hoop algebra and s a square root on H. Then,*
(i) *$x \leq s(x)$, for every $x \in H$. In particular, if H is bounded and $s(x) = 0$, then $x = 0$,*
(ii) *$s(1) = 1$.*

Theorem 2.16. [17] *Let H be a hoop algebra and s a square root on H. Then:*
(i) *s is one to one,*
(ii) *$x^2 = x$ if and only if $s(x) = x$, for every $x \in H$,*
(iii) *$x = 1$ if and only if $s(x) = 1$, for every $x \in H$,*
(iv) *$x \leq y$ if and only if $s(x) \leq s(y)$, for every $x, y \in H$,*

(v) $s(x) \odot s(y) \leq s(x \odot y)$, *for every* $x, y \in H$,
(vi) $s(x) \to s(y) = s(x \to y)$, *for every* $x, y \in H$.

Proposition 2.17. [17, Proposition 3.12] *Let* H *be a* $\vee$*-hoop algebra and* s *a square root on* H. *For every* $x, y \in H$ *one has*
(i) *If* $s(x) \odot s(y) = s(x \odot y)$, *then* $s(x \wedge y) = s(x) \wedge s(y)$. *So* $s(x \vee y) = s(x) \vee s(y)$,
(ii) *If* $s(x) \odot s(y) = s(x \odot y)$, *then* $x = s(x^2)$. *In particular, if* H *is bounded, then* $s(0) = 0$,
(iii) *If* H *is bounded and* $s(0) \neq 0$, *then* $s(x) \odot s(y) \neq s(x \odot y)$.

Definition 2.18. [17, Definition 4.13] Let H be a bounded hoop algebra and s be a square root on H. If $s(0) = 0$, then H is called *good*.

As shown in Proposition 2.17, if for all $x, y \in H$, we have $s(x \odot y) = s(x) \odot s(y)$, or $s(x^2) = x$, then H is good.

Example 2.19. *Let* $I = [0, 1]$ *be the unit closed interval of reals. In the hoop Łukasiewicz algebra,* $\mathbf{L}_1 = (I; \max, \min, \odot, \to, 0, 1)$, *we have* $a \odot b = \max(0, a + b - 1)$ *and* $a \to b = \min(1, 1 - a + b)$, *for all* $a, b \in I$. *Define the unitary operation* $s : I \to I$ *by* $s(x) = \frac{x+1}{2}$, *for every* $x \in I$, *it is clear that* $s(x) \in I$. *It is easy to check that* s *is a square root on* $\mathbf{L}_1$. *However, we have* $s(0) = 0.5$. *Therefore, the Łukasiewicz algebra is not good.*

3. Derivation on hoop algebras

In this section, while defining the multiplicative derivation on hoop algebras, we examine some of its properties as well as its relationship with filters. Throughout this section, we assume that H is a $\vee$-hoop algebra.

Definition 3.1. (See also [18]) A mapping $d : H \to H$ is called a *multiplicative derivation* or simply a *derivation* if it satisfies the following condition:

$$d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)).$$

In addition, a derivation d on a $\vee$-hoop H is called:

- *Isotone* if $x \leq y$, then $d(x) \leq d(y)$.
- *Contractive* if $d(x) \leq x$,
- An *ideal derivation* if d is both isotone and contractive.
- *Idempotent* if $d^2 = d$, where $d^2(x) = d(d(x))$, for all $x \in H$.

We denote the set of all derivations on H by $\mathrm{Der}(H)$.

Example 3.2.
(i) *For a bounded hoop* H *if we define* $d_0 : H \to H$ *by* $d_0(x) = 0$, *for all* $x \in H$, *then* d_0 *is a derivation on* H, *which is called a zero derivation.*

(ii) *If we define* $d_1 : H \to H$ *by* $d_1(x) = x$, *for all* $x \in H$, *then* d_1 *is a derivation on* H, *which is called an identity derivation.*

(iii) *Let* $(H = \{0, a, b, 1\}, \leq)$ *be a chain. Define the operations* $\odot$ *and* $\rightarrow$ *on* H *as follows:*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(H; \odot, \rightarrow, 0, 1)$ *is a bounded hoop algebra. We now define some derivations on* H *as follows:*

$$d_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } x = a, b, 1, \end{cases} \quad \text{and} \quad d_2(x) = \begin{cases} 0 & \text{if } x = 0, b, 1, \\ a & \text{if } x = a. \end{cases}$$

Then d_1 *and* d_2 *are derivations on* H*. Also,* d_1 *is an idempotent and ideal derivation. Additionally,* d_2 *is contractive and idempotent, but fails to be isotone, and hence, it is not an ideal derivation. It is also clearly seen that* $d(x) \odot y = x \odot d(y)$ *does not necessarily hold. Because we have* $d_2(a) \odot b = a \odot b = a$ *and* $a \odot d_2(b) = a \odot 0 = 0$.

iv) Let $(H = \{0, a, b, 1\}, \leq)$ *be a chain. Define the operations* $\odot$ *and* $\rightarrow$ *on* H *as follows:*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then, $(H; \odot, \rightarrow, 0, 1)$ *is a bounded hoop algebra. We now define map* d *on* H *as follows:*

$$d(x) = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } x = a, b, 1, \end{cases}$$

Then, d *is an ideal derivation on* H*.*

v) Consider the free monoid generated by a single element a *such that* $0 < a < 1$*. Put* $C_a = \{1 = a^0, a, a^2, \cdots\}$ *and define the operations* $\odot$ *and* $\rightarrow$ *on* C_a *as follows:*

$$a^n \cdot a^m = a^{n+m}, \quad a^n \rightarrow a^m = a^{\max(m-n,0)}$$

for any $n, m \in \mathbb{N} \cup \{0\}$*. It is evident that* C_a *satisfies Definition 2.1. Therefore, it is a hoop algebra. We define some mappings as follows:*

a) $d_m(x) = a^m \odot x$ *for all* $x \in C_a$ *and* $m \in \mathbb{N} \cup \{0\}$*. It is easy to check that* d_m *is an ideal derivation.*

b)

$$d_a(x) = \begin{cases} a \odot x & \text{if } x = a, a^2, a^3, \cdots \\ a^2 & \text{if } x = 1. \end{cases}$$

It is easy to check that d_a is an ideal derivation.
vi) Let $(G, +, -, 0, \vee, \wedge)$ is an Abelian l-group. Then, the positive cone $P(G)$ of G is the set $\{x \in G : x \vee 0 = x\}$. In this case $(P(G), \odot, \rightarrow, 1_{P(G)})$ is a hoop algebra with the following operations:

$$x \odot y = x + y; \quad x \rightarrow y = (y - x) \vee 0; \quad 1_{P(G)} := 0$$

Note that the partial order of $P(G)$ is the converse of the partial order inherited from G. For every $a \in P(G)$, define the mapping d_a as $d_a(x) = a \odot x$ for all $x \in P(G)$. It is easy to check that d is an ideal derivation on $P(G)$.

Proposition 3.3. [18, Proposition 3.1] *Let H be a $\vee$-hoop algebra and d a derivation on H. Then the following properties hold for all $x, y \in H$:*
(i) $x \odot d(1) \leq d(x)$,
(ii) $d(x^n) = x^{n-1} \odot d(x)$, *for all* $n \in \mathbb{N}$,
(iii) $d(1) = 1$ *if and only if* $x \leq d(x)$, *for all* $x \in H$,
(iv) $d(x \odot y) \leq d(x) \vee d(y)$.

Corollary 3.4. *Let H be a $\vee$-hoop algebra with a derivation d. Then for all $x, y \in H$:*
(i) $x \leq d(1) \rightarrow d(x)$,
(ii) $d(x \wedge y) \leq d(x) \vee d(x \rightarrow y)$,
(iii) $x \rightarrow y \leq d(x) \rightarrow d(x \wedge y)$ *and* $d(x \rightarrow y) \leq x \rightarrow d(x \wedge y)$,
(iv) *If H is totally ordered and $x \leq y$, then $y \rightarrow x \leq d(y) \rightarrow d(x)$ and $d(y \rightarrow x) \leq y \rightarrow d(x)$.*

Proof. (i) According to Proposition 3.3(i) and Theorem 2.3(ii), it is straightforward.
(ii) According to Theorem 2.3(i), we have $x \wedge y = x \odot (x \rightarrow y)$. Now the inequality is a consequence of Proposition 3.3(iv).
(iii) We know $x \wedge y = x \odot (x \rightarrow y)$. So we have $d(x \wedge y) = d(x \odot (x \rightarrow y))$. By Definition 3.1, we have $x \odot d(x \rightarrow y) \leq d(x \wedge y)$ and $d(x) \odot (x \rightarrow y) \leq d(x \wedge y)$. Therefore, by Theorem 2.3(ii), the result is obtained.
(iv) Suppose $x \leq y$. Using part (iii), we directly obtain $y \rightarrow x \leq d(y) \rightarrow d(y \wedge x) = d(y) \rightarrow d(x)$ and $d(y \rightarrow x) \leq y \rightarrow d(y \wedge x) = y \rightarrow d(x)$. □

Proposition 3.5. *Let H be a bounded $\vee$-hoop algebra and d a derivation on H. Then the following properties hold for all $x, y \in H$:*
(i) $d(0) = 0$,
(ii) $d(x) \odot x' = x \odot d(x') = 0$,
(iii) *If* $x \leq y$, *then* $d(x) \leq y''$,
(iv) $d(x) \leq x''$. *If H has (DNP) or $x^2 = x$, then $d(x) \leq x$,*
(v) $d(x) = d(x) \vee (x \odot d(1))$ *and* $x \odot d(1) \odot d(x)' = 0$,
(vi) *If $d(x) = 1$, then $x'' = 1$ and $x' = 0$. Moreover, if H has (DNP) or $x^2 = x$, then $x = 1$,*
(vii) *If $x \leq y'$, then $d(y) \leq x'$ and $d(x) \leq y'$,*
(viii) $d(x') \leq d(x)'$,

(ix) *If* $d(x \odot y) = 0$ *or* $x \wedge y = 0$*, then* $d(x) \odot y = 0$ *and* $d(y) \odot x = 0$*.*

Proof. To prove parts (i) through (viii), see [18, Proposition 3.1]. Note that the moreover part of (vi) is also true for $x^2 = x$. Indeed, if $x^2 = x$, then $d(x) = d(x^2) = x \odot d(x)$. Since $d(x) = 1, 1 = d(x) = x \odot d(x) = x \odot 1 = x$.
(ix) Suppose $d(x \odot y) = 0$. Using the definition, we have $(d(x) \odot y) \vee (x \odot d(y)) = 0$. So the result is obtained. Also, if $x \wedge y = 0$, By Theorem 2.3(iii), we have $x \odot y = 0$ and then $d(x \odot y) = 0$. So the result is obtained. □

Remark 3.6. According to Proposition 3.5(iv), in the following cases, d is a contraction on H.

1. H is a bounded Wajsberg hoop,
2. H is a bounded hoop algebra with (DNP),
3. H is idempotent.

Corollary 3.7. *Let* H *be an idempotent* ∨*-hoop algebra with a derivation* d *on* H*. We define* $d^0(x) = x$ *and for each* $n \in \mathbb{N}$, $d^{n+1}(x) = d(d^n(x))$*, for each* $x \in H$*. Then* $d^n(x) = d(x)$*, for all* $x \in H$ *and* $n \in \mathbb{N}$*.*

Proof. We show the result by induction on n. If $n = 1$, then it is clear. Suppose $n = 2$. According to the assumption, for each $x \in H$, we have $x \odot x = x$ and $d(x) \odot d(x) = d(x)$. We have, $d^2(x) = d(d(x)) = d(d(x) \odot d(x)) = (d(d(x)) \odot d(x)) \vee (d(x) \odot d(d(x))) = d(d(x)) \odot d(x)$ and so, $d^2(x) \leq d(x)$. On the other hand, one has

$$\begin{aligned}
d^2(x) &= d(d(x)) \\
&= d(d(x \odot x)) \\
&= d((d(x) \odot x) \vee (x \odot d(x))) \\
&= d(d(x) \odot x) \\
&= (d(d(x)) \odot x) \vee (d(x) \odot d(x)) \\
&= (d^2(x) \odot x) \vee d(x).
\end{aligned}$$

and then $d^2(x) \geq d(x)$. Therefore, the equality holds. Now, suppose that the result is true for $n = k$. Then we have $d^{k+1}(x) = d(d^k(x)) = d(d(x)) = d^2(x) = d(x)$, which shows that the result is true for $n = k + 1$. □

Theorem 3.8. *For* $a \in H$*, define the map* $d_a : H \to H$ *by* $d_a(x) = a \odot x$*, for all* $x \in H$*. Then* d_a *is an ideal derivation on* H*, which is called a principal ideal derivation.*

Proof. It is similar to the proof of [27, Theorem 3.8]. □

Remark 3.9. According to Theorem 3.8, for all $a \in H$, d_a is an ideal derivation on H. Now we have $d_a(x) \odot y = a \odot x \odot y = x \odot a \odot y = x \odot d_a(y)$. So we have, $d_a(x \odot y) = d_a(x) \odot y = x \odot d_a(y)$, for all $x, y \in H$.

Example 3.10. *In Example 3.2, items (v)(a) and (vi), the ideal derivations are principal. Also, d and d_a in Example 3.2, items (iv) and (v)(b), respectively, are ideal derivations that are not principal.*

Definition 3.11. Let $d : H \to H$ be a derivation on H. Then d is called *regular* if $d(1) \in \mathrm{Id}(H)$.

Example 3.12. *(i) In every hoop algebra H, for $a = 1$, the principal ideal derivation d_1, as defined in Theorem 3.8 is a regular derivation.*
(ii) Suppose H is a hoop algebra and $a \notin \mathrm{Id}(H)$. Then, the derivation d_a is not a regular derivation.

Proposition 3.13. *Let $d : H \to H$ be an isotone derivation on H. Then for all $x, y, z \in H$:*
(i) *If $z \leq x \to y$, then $z \leq d(x) \to d(y)$ and $x \leq d(z) \to d(y)$,*
(ii) $x \to y \leq d(x) \to d(y)$ *and* $d(x \to y) \leq x \to d(y)$,
(iii) $x \to y \leq d(x \vee y) \to d(y)$,
(iv) $d(x) \vee d(y) \leq d(x \vee y)$. *If H is totally ordered, then equality holds,*
(v) $d(x \wedge y) \leq d(x) \wedge d(y)$. *If H is totally ordered, then equality holds,*
(vi) $d(x) \leq d(1)$,
(vii) $x \odot d(y) \leq d(x)$. *Therefore,* $x \leq d(y) \to d(x)$,
(viii) $d^n(x) \geq d(x)^n$ *for all* $n \in \mathbb{N}$.

Proof. Parts (i) and (ii) are similar to the proof of [27, Proposition 3.14].
(iii) We know that for every $x, y \in H$, we have $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. Since d is isotone, it follows $d(x \vee y) \leq d((x \to y) \to y)$. Now, by part (ii), $d(x \vee y) \leq (x \to y) \to d(y)$. Therefore, using Theorem 2.3(ii), $x \to y \leq d(x \vee y) \to d(y)$.
(iv) and (v) are clear.
(vi) For all $x \in H$, we have $x \leq 1$. Since d is isotone, $d(x) \leq d(1)$.
(vii) We have $x \odot y \leq x, y$. Since d is isotone, $d(x \odot y) \leq d(x)$, so $(d(x) \odot y) \vee (x \odot d(y)) \leq d(x)$. Therefore, $x \odot d(y) \leq d(x)$.
(viii) We show by induction on n that $d^n(x) \geq d(x)^n$. For $n = 1$, it is clear. Let $n = 2$. Then, $d^2(x) = d(d(x) \odot 1) = d(d(x)) \vee (d(x) \odot d(1))$. Since d is isotone, $d(x) \leq d(1)$ and we have $d^2(x) \geq d(x) \odot d(1) \geq d(x) \odot d(x) = d(x)^2$.

Now, suppose the result is true for $n = k$. In the light of Definition 3.1, we have

$$\begin{aligned} d^{k+1}(x) &= d(d^k(x)) \\ &= d(d^k(x) \odot 1) \\ &= (d(d^k(x)) \odot 1) \vee (d^k(x) \odot d(1)) \\ &\geq d^k(x) \odot d(1) \\ &\geq d(x)^k \odot d(x) \\ &= d(x)^{k+1}. \end{aligned}$$

Therefore, $d^n(x) \geq d(x)^n$, for $n = k + 1$. □

Proposition 3.14. *Let $d : H \to H$ be a contractive derivation on H. Then for all $x, y \in H$:*
(i) $d(x) \odot d(y) \leq d(x \odot y)$,
(ii) *If d is isotone, then* $d(x \to y) \leq d(x) \to d(y) \leq d(x) \to y$,
(iii) *If $d(1) = 1$, then d is the identity derivation,*
(iv) *By assumption $d^0(x) = x$, we have $d^n(x) \leq d^{n-1}(x)$, for all $n \in \mathbb{N}$,*
(v) *If $d(x) \leq d(1)$, for any $x \in H$, then $d^n(x) \geq d(x)^n$, for all $n \in \mathbb{N}$.*

Proof. Parts (i), (ii), and (iii) are similar to the proof of [27, Proposition 3.15] and [18, Proposition 4.6].
(iv) Based on the definition of contraction derivation, it is clear.
(v) It is similar to the proof of Proposition 3.13(viii). □

Proposition 3.15. *Let $d : H \to H$ be a derivation on H. Then the following conditions are equivalent:*
(i) *d is an idempotent ideal derivation;*
(ii) *d satisfies $d(x) \to d(y) = d(x) \to y$, for all $x, y \in H$.*

Proof. It is similar to the proof of [27, Proposition 3.16] and [18, Theorem 4.1]. □

Proposition 3.16. *Let $d : H \to H$ be a contractive derivation on bounded ∨-hoop algebra H. If $d(1) \in B(H)$, then the following conditions are equivalent:*
(i) *d is an ideal derivation on H,*
(ii) *$d(x) \leq d(1)$, for all $x \in H$,*
(iii) *$d(x) = d(1) \odot x$, for all $x \in H$,*
(iv) *$d(x \wedge y) = d(x) \wedge d(y)$, for all $x, y \in H$,*
(v) *$d(x \vee y) = d(x) \vee d(y)$, for all $x, y \in H$,*
(vi) *$d(x \odot y) = d(x) \odot d(y)$, for all $x, y \in H$,*
(vii) *If H is cancellative, then $d(x \to y) = d(x) \to d(y)$, for all $x, y \in H$.*

Proof. (i) $\Longrightarrow$ (ii). It is clear, as d is isotone.
(ii) $\Longrightarrow$ (iii). Suppose $d(x) \leq d(1)$, for every $x \in H$. Because of $d(1) \in B(H)$, using Lemma 2.9(ii) and Theorem 2.3(viii), we have

$$d(x) = d(1) \wedge d(x) = d(1) \odot d(x) \leq d(1) \odot x.$$

On the other hand, according to Proposition 3.3(i), we know $x \odot d(1) \leq d(x)$. Thus the result holds.
(iii) $\Longrightarrow$ (iv). Suppose $d(x) = d(1) \odot x$, for all $x \in H$. By Lemma 2.9(ii) we have

$$\begin{aligned} d(x \wedge y) &= d(1) \odot (x \wedge y) \\ &= d(1) \wedge (x \wedge y) \\ &= (d(1) \wedge x) \wedge (d(1) \wedge y) \\ &= (d(1) \odot x) \wedge (d(1) \odot y) \\ &= d(x) \wedge d(y), \end{aligned}$$

for all $x, y \in H$.
(iv) $\Longrightarrow$ (i). Suppose $x, y \in H$ so that $x \leq y$. Based on the assumption $d(x) = d(x \wedge y) = d(x) \wedge d(y)$. Hence $d(x) \leq d(y)$. Therefore, d is isotone. Since d is contractive on H, we have d is an ideal derivation on H.
(iii) $\Longrightarrow$ (v). Suppose $d(x) = d(1) \odot x$, for any $x \in H$. Now, by the help of Proposition 2.5(i), for all $x, y \in H$, we have: $d(x \vee y) = d(1) \odot (x \vee y) = (d(1) \odot x) \vee (d(1) \odot y) = d(x) \vee d(y)$.
(v) $\Longrightarrow$ (i). This is similar to (iv) $\Longrightarrow$ (i).
(iii) $\Longrightarrow$ (vi). Suppose that for every $x \in H$, $d(x) = d(1) \odot x$. According to Lemma 2.10(i), for all $x, y \in H$, we have $d(x \odot y) = d(1) \odot (x \odot y) = (d(1) \odot x) \odot (d(1) \odot y) = d(x) \odot d(y)$.
(vi) $\Longrightarrow$ (ii). By the assumption, $d(x) = d(x \odot 1) = d(x) \odot d(1) = d(x) \wedge d(1)$. Therefore $d(x) \leq d(1)$, for all $x \in H$.
(iii) $\Longrightarrow$ (vii). Suppose H is cancellative. It suffices to show $d(1) \odot (x \to y) = d(1) \odot x \to d(1) \odot y$, for all $x, y \in H$. This is by (iii) equivalent to $d(x \to y) = d(x) \to d(y)$. We know $x \odot (x \to y) \leq y$, so $d(1) \odot x \odot (x \to y) \leq d(1) \odot y$. Hence $d(1) \odot (x \to y) \leq d(1) \odot x \to d(1) \odot y$. On the other hand, we have $d(1) \odot (x \to y) \leq d(1) \odot (x \to y)$. Then, according to Proposition 2.11(i), $d(1) \odot (d(1) \odot x \to d(1) \odot y) \leq d(1) \odot (x \to y)$. As H is cancellative and $d(1)$ is idempotent, by Proposition 2.11(ii), we get $d(1) \odot x \to d(1) \odot y \leq d(1) \odot (x \to y)$. Therefore, we get the result.
(vii) $\Longrightarrow$ (i). Suppose $x, y \in H$ so that $x \leq y$. By the assumption $d(1) = d(x \to y) = d(x) \to d(y)$. Then, $d(x) \odot d(1) \leq d(y)$. Since $d(1) \in B(H)$, we have $d(x) \odot d(1) \leq d(y) \odot d(1)$. Using Proposition 2.11(ii), we obtain $d(x) \leq d(y)$. Therefore, d is isotone. Since d is contractive, d is an ideal derivation on H. □

Corollary 3.17. *Assume that $d : H \to H$ is an ideal derivation on bounded $\vee$-hoop algebra H. If $d(1) \in B(H)$, then:*
(i) $d(x \odot y) = d(x) \odot y = x \odot d(y)$, *for all* $x, y \in H$,
(ii) $d^n(x) = d(x)$, *for all* $n \in \mathbb{N}$.

Proof. (i) By Proposition 3.16(iii), $d(x \odot y) = d(1) \odot x \odot y = d(y) \odot x = d(x) \odot y$, for all $x, y \in H$.
(ii) By Proposition 3.16(iii), $d(d(x)) = d(x) \odot d(1) = x \odot d(1) \odot d(1) = d(x)$, for all $x \in H$. Now the result follows by induction on n. □

Definition 3.18. Let $d : H \to H$ be a derivation on H. We denote the set of all fixed points of d in H by $\mathrm{Fix}_d(H)$. That is, $\mathrm{Fix}_d(H) = \{x \in H : d(x) = x\}$.

Example 3.19. *In Example 3.2(iii),* $\mathrm{Fix}_{d_1}(H) = \{0, a\}$. *Also, in Example 3.2(v)(a), if* $m \neq 0$*, then* $\mathrm{Fix}_{d_m}(H) = \emptyset$.

Proposition 3.20. *Let* $d : H \to H$ *be a derivation on* H*. Then*
(i) $\mathrm{Fix}_d(H)$ *is closed under* $\odot$,
(ii) *If* d *is an ideal derivation, then* $\mathrm{Fix}_d(H)$ *is closed under* $\vee$,
(iii) *If* d *is an ideal derivation and* $d(1) \in B(H)$*, then* $\mathrm{Fix}_d(H) = d(H)$.

Proof. (i) Let $x, y \in \mathrm{Fix}_d(H)$. Then, $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)) = (x \odot y) \vee (x \odot y) = x \odot y$. Thus, $x \odot y \in \mathrm{Fix}_d(H)$.
(ii) Let $x, y \in \mathrm{Fix}_d$(H). Then, $d(x) = x$ and $d(y) = y$. Now, according to Proposition 3.13(iv), we have: $x \vee y = d(x) \vee d(y) \leq d(x \vee y) \leq x \vee y$. Hence $d(x) \vee d(y) = d(x \vee y) = x \vee y$ and $x \vee y \in \mathrm{Fix}_d(H)$.
(iii) Let $a \in d(H)$. There exists $b \in H$ such that $a = d(b)$. Now, according to Corollary 3.17(ii), we have $d(a) = d(d(b)) = d(b) = a$. Hence $a \in \mathrm{Fix}_d(H)$. Also, if $a \in \mathrm{Fix}_d(H)$, then it is clear that $a \in d(H)$. Therefore, the assertion is concluded. □

Theorem 3.21. *Let* H *be a bounded* $\vee$*-hoop algebra,* $d : H \to H$ *be an ideal idempotent derivation on* H *and* $d(1) \in B(H)$*. Then* $(\mathrm{Fix}_d(H); \odot, \to_d, 0, \bar{1})$ *is a bounded hoop algebra, where* $\bar{1} = d(1)$ *and* $x \to_d y = d(x \to y)$*, for all* $x, y \in \mathrm{Fix}_d(H)$.

Proof. We will check the conditions of Definition 2.1. As d is idempotent, we have $d(d(1)) = d(1)$. Therefore, $d(1) \in \mathrm{Fix}_d(H)$. Now, by Proposition 3.16(iii), we have $x \odot d(1) = d(x) = x$ and $x \to_d d(1) = d(d(x) \to d(1)) = d(1)$, for all $x \in \mathrm{Fix}_d(H)$. Therefore, $(\mathrm{Fix}_d(H); \odot, d(1))$ is a commutative monoid and the condition $H1$ is satisfied. Also, $x \to_d x = d(x \to x) = d(1)$, for every $x \in \mathrm{Fix}_d(H)$. Therefore, the condition $H2$ is satisfied. Suppose $x, y \in \mathrm{Fix}_d(H)$. By Proposition 3.16(vi), we have

$$\begin{aligned} x \odot (x \to_d y) &= d(x) \odot d(x \to y) \\ &= d(x \odot (x \to y)) \\ &= d(y \odot (y \to x)) \\ &= d(y) \odot d(y \to x) \\ &= y \odot (y \to_d x). \end{aligned}$$

Hence, the condition $H3$ is satisfied. Similar to the proof of [27, Theorem 3.18(3)], the condition $H4$ is also satisfied. So $(\mathrm{Fix}_d(H); \odot, \to_d, d(1))$ is a hoop algebra. By Proposition 3.5(i), $0 \in \mathrm{Fix}_d(H)$. Also, for all $x \in \mathrm{Fix}_d(H)$, we have

$$d(0) \to_d x = d(0) \to_d d(x) = d(d(0) \to d(x)) = d(1),$$

i.e., $d(0) \leq x$. Therefore $(\mathrm{Fix}_d(H); \odot, \to_d, 0, \bar{1})$ is a bounded hoop algebra. □

Proposition 3.22. *Let $d : H \to H$ be a contractive derivation on H. Then the following conditions are equivalent:*
(i) $d(1) = 1$,
(ii) $\mathrm{Fix}_d(H) = H$,
(iii) $\mathrm{Fix}_d(H)$ *is a filter of* H.

Proof. (i) $\Longrightarrow$ (ii). Let $d(1) = 1$. Proposition 3.14(iii) implies $d(x) = x$, for all $x \in H$. Thus, $\mathrm{Fix}_d(H) = H$. The other implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) are clear. □

Proposition 3.23. *Let d and d' be two derivations on H. Then, $d \vee d'$ given by $(d \vee d')(x) = d(x) \vee d'(x)$, for all $x \in H$, is a derivation on H.*

Proof. Suppose $d, d' \in \mathrm{Der}(H)$. We will show that $d \vee d' \in \mathrm{Der}(H)$. By Proposition 2.5(i), we have

$$\begin{aligned}(d \vee d')(x \odot y) &= d(x \odot y) \vee d'(x \odot y)\\ &= ((d(x) \odot y) \vee (x \odot d(y))) \vee ((d'(x) \odot y) \vee (x \odot d'(y)))\\ &= ((d(x) \vee d'(x)) \odot y) \vee (x \odot (d(y) \vee d'(y)))\\ &= ((d \vee d')(x) \odot y) \vee (x \odot (d \vee d')(y)),\end{aligned}$$

for all $x, y \in H$. Therefore, the assertion is concluded. □

Corollary 3.24. *Suppose H is a $\vee$-hoop algebra and $\mathrm{Der}_c(H)$ is the set of all contractive derivations on H. Then $(\mathrm{Der}_c(H); \vee, d_1)$ is a $\vee$-semilattice with the following definitions. For $d, d' \in \mathrm{Der}_c(H)$*

$$(d \vee d')(x) = d(x) \vee d'(x), \quad d_1(x) = x \text{ for all } x \in H.$$

If H is a bounded, then $(\mathrm{Der}_c(H); \vee, d_0, d_1)$ is a bounded $\vee$-semilattice such that $d_0(x) = 0$.

Proof. For all $d, d' \in \mathrm{Der}_c(H)$ define

$$d \leq d' \quad \text{if and only if} \quad d(x) \leq d'(x) \quad \text{for all} \quad x \in H.$$

Then, it can be easily checked that $(\mathrm{Der}_c(H); \leq)$ is a partially ordered set. Also, according to Proposition 3.23, $\mathrm{Der}_c(H)$ is closed under $\vee$. Because, if $d, d' \in \mathrm{Der}_c(H)$, then $d(x) \leq x$ and $d'(x) \leq x$, for all $x \in H$. So $(d \vee d')(x) = d(x) \vee d'(x) \leq x \vee x = x$. Therefore $(d \vee d') \in \mathrm{Der}_c(H)$. Also, if $d \in \mathrm{Der}_c(H)$ then, we have $(d \vee d_1)(x) = d(x) \vee d_1(x) = d(x) \vee x = x = d_1(x)$, for all $x \in H$. Also if H is bounded, then we have $(d \vee d_0)(x) = d(x) \vee d_0(x) = d(x) \vee 0 = d(x)$. So $d_0 \leq d \leq d_1$ and thus $(\mathrm{Der}_c(H); \vee, d_0, d_1)$ is a bounded $\vee$-semilattice. □

Corollary 3.25. *Suppose H is a $\vee$-hoop algebra such that for all $x, y \in H$, $x \odot y = x \wedge y$ and $\mathrm{Der}_{id}(H)$ is the set of all ideal derivations on H. Then $\mathrm{Der}_{\mathbf{id}}(\mathbf{H}) = (\mathrm{Der}_{id}(H); \vee, \wedge, d_1)$ is a lattice with the largest element d_1 with the following definitions:*

$$(d \vee d')(x) = d(x) \vee d'(x), \ (d \wedge d')(x) = d(x) \wedge d'(x), \ d_1(x) = x$$

for all $x \in H$ *and* $d, d' \in \mathrm{Der}_{id}(H)$. *If* H *is bounded, then* $\mathrm{Der}_{\mathbf{id}}(\mathbf{H}) = (\mathrm{Der}_{id}(H); \vee, \wedge, d_0, d_1)$ *is a distributive bounded lattice such that* $d_0(x) = 0$, *for all* $x \in H$.

Proof. First, we show that the operations $\vee$ and $\wedge$ on $\mathrm{Der}_{id}(H)$ are well-defined. Suppose $d, d' \in Der_{id}(H)$ and $x \leq y$. Hence $d(x) \leq d(y)$ and $d'(x) \leq d'(y)$, for all $x \in H$. So $(d \vee d')(x) = d(x) \vee d'(x) \leq d(y) \vee d'(y) = (d \vee d')(y)$. Therefore, $d \vee d'$ is an isotone derivation. Now, using Corollary 3.24, it is easy to show that $\mathrm{Der}_{id}(H)$ is closed under $\vee$. Now, suppose $d, d' \in \mathrm{Der}_{id}(H)$. According to Proposition 2.5(ii), we have:

$$\begin{aligned}(d \wedge d')(x \odot y) &= d(x \odot y) \wedge d'(x \odot y)\\ &= ((d(x) \odot y) \vee (x \odot d(y))) \wedge ((d'(x) \odot y) \vee (x \odot d'(y)))\\ &= ((d(x) \odot y) \wedge (d'(x) \odot y)) \vee ((d(x) \odot y) \wedge (x \odot d'(y))) \vee\\ &\quad ((x \odot d(y)) \wedge (d'(x) \odot y)) \vee ((x \odot d(y)) \wedge (x \odot d'(y)))\\ &= (d(x) \odot d'(x) \odot y) \vee (d(x) \odot y \odot x \odot d'(y)) \vee\\ &\quad (x \odot d(y) \odot d'(x) \odot y) \vee (x \odot d(y) \odot d'(y))\end{aligned}$$

for all $x, y \in H$. Note that the last equality is true by the assumption that $x \odot y = x \wedge y$, for all $x, y \in H$. On the other hand, Proposition 3.13(vii) implies that $d(x) \odot y \odot x \odot d'(y) \leq d(y) \odot x \odot d'(y)$ and $x \odot d(y) \odot d'(x) \odot y \leq d(x) \odot d'(x) \odot y$. Then, we obtain

$$\begin{aligned}(d \wedge d')(x \odot y) &= (d(x) \odot d'(x) \odot y) \vee (x \odot d(y) \odot d'(y))\\ &= ((d \odot d')(x) \odot y) \vee (x \odot (d \odot d')(y))\\ &= ((d \wedge d')(x) \odot y) \vee (x \odot (d \wedge d')(y))\end{aligned}$$

for all $x, y \in H$. Therefore, $d \wedge d'$ is derivation. It is also easy to show that $d \wedge d'$ is an isotone, contractive, and thus an ideal derivation. Hence $d \wedge d' \in \mathrm{Der}_{id}(H)$. Also, Since the derivations are contractive, we have $(d \wedge d_1)(x) = d(x) \wedge d_1(x) = d(x) \wedge x = d(x)$. Therefore, $\mathbf{Der_{id}(H)} = (\mathrm{Der}_{id}(H); \vee, \wedge, d_1)$ is a lattice with the largest element d_1. Also H is bounded, then we have $(d \wedge d_0)(x) = d(x) \wedge d_0(x) = d(x) \wedge 0 = 0 = d_0(x)$. So $\mathbf{Der_{id}(H)} = (\mathrm{Der}_{id}(H); \vee, \wedge, d_0, d_1)$ is a bounded lattice. Regarding distributivity of $\mathrm{Der}_{id}(H)$, since H is a distributive lattice, it is easily seen that $\mathrm{Der}_{id}(H)$ is a distributive lattice as well. □

Proposition 3.26. *Let* $d : H \to H$ *be a derivation and* F *a filter of* H. *Then,*

(i) *If* $d(1) \in F$, *then* $d(x)^n \in F$ *and* $d^n(x) \in F$, *for all* $x \in F$ *and any* $n \in \mathbb{N}$,

(ii) *If* d *is isotone and* $d(x) \in F$, *then* $d(y)^n, d^n(y), d^n(x) \in F$, *for all* $n \in \mathbb{N}$ *and* $y \in H$ *such that* $x \leq y$,

(iii) *If* d *is contractive, then* $\langle x \rangle \subseteq \langle d(x) \rangle$, *for all* $x \in H$. *If, moreover* $d(1) = 1$, *then the equality holds.*

Proof. (i) Suppose $d(1), x \in F$. Then, $x \odot d(1) \in F$. But by Proposition 3.3(i), $x \odot d(1) \leq d(x)$. Therefore, $d(x) \in F$. Now, by mathematical induction, we show that $d^n(x) \in F$ for all $n \in \mathbb{N}$. If $n = 1$, this is clear by the above. Suppose $n = 2$. Now we have $d^2(x) = d(d(x)) \geq d(x) \odot d(1)$. Hence $d^2(x) \in F$. Suppose $d^k(x) \in F$. Now we have $d^{k+1}(x) \geq d^k(x) \odot d(1)$. Therefore, $d^{k+1}(x) \in F$, as $d(1) \in F$ and F is a filter of H, we are done.

(ii) According to conditions $F2$ and $F3$ of Definition 2.12 and Proposition 3.13(viii), it is clear.

(iii) As d is contractive, $\langle x \rangle \subseteq \langle d(x) \rangle$, for all $x \in H$. The second statement is true by Proposition 3.14(iii). □

Proposition 3.27. *Let $d : H \to H$ be a contractive derivation on bounded $\vee$-hoop algebra H such that $d(1) \in B(H)$ and also F is a filter of H such that $d(1) \in F$. Then F is prime if and only if, $d(x) \vee d(y) \in F$ implies $d(x) \in F$ or $d(y) \in F$, for all $x, y \in H$.*

Proof. ($\Rightarrow$) If F is prime, then the statement is clear.
($\Leftarrow$) Let $x \vee y \in F$. According to Proposition 3.3(i), we have $(x \vee y) \odot d(1) \leq d(x \vee y)$. Given the assumptions of the proposition, we obtain $d(x \vee y) \in F$. Now according to Proposition 3.16(v), $d(x \vee y) = d(x) \vee d(y) \in F$. By the assumption, we have $d(x) \in F$ or $d(y) \in F$. Since d is contractive, as F is a filter, we have $x \in F$ or $y \in F$. Therefore, F is prime. □

Proposition 3.28. *Let $d : H \to H$ be an ideal derivation on H such that $d(1) \in B(H)$ and put $F = \{x \in H : d(x) = d(1)\}$. Then, F is a filter of H.*

Proof. We check the conditions of Definition 2.12. It is clear that $1 \in F$. Thus $F1$ holds. Suppose $x, y \in F$ then we have $d(x) = d(y) = d(1)$. By Proposition 3.16(vi), $d(x \odot y) = d(x) \odot d(y) = d(1) \odot d(1) = d(1)$. Therefore $x \odot y \in F$. Hence $F2$ also holds. Now let $x \in F$ and $x \leq y$. Since d is an ideal derivation, we have $d(1) = d(x) \leq d(y) \leq d(1)$. Then, $d(y) = d(1)$. Therefore, $y \in F$ and $F3$ also is satisfied. Thus, F is a filter of H. □

Suppose H is a bounded $\vee$-hoop algebra. For a derivation d on H, define $\ker_0(d) = \{x \in H : d(x) = 0\}$. It is clear that $\ker_0(d) \neq \emptyset$, because $d(0) = 0$.

Proposition 3.29. *Suppose H is a bounded $\vee$-hoop algebra and d a derivation on H. Then:*

(i) *If $x, y \in \ker_0(d)$, then $x \odot y \in \ker_0(d)$.*
Moreover, if d is isotone, then:
(ii) *If $x \leq y$ and $y \in \ker_0(d)$, then $x \in \ker_0(d)$,*
(iii) *If $x \in \ker_0(d)$, then $x \odot y \in \ker_0(d)$ and $x \wedge y \in \ker_0(d)$, for all $y \in H$.*

Proof. Suppose $x, y \in \ker_0(d)$. We have $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)) = 0 \vee 0 = 0$. Items (ii) and (iii) are direct. □

For a lattice L, recall that a non-empty subset I of L is an ideal of L if $x, y \in I$ implies $x \vee y \in I$ and $x \in L$, $y \in I$ and $x \leq y$ imply $x \in I$. Now, we have

Proposition 3.30. *Suppose H is a bounded $\vee$-hoop algebra and $d : H \to H$ an ideal derivation on H and $d(1) \in B(H)$. Then $\ker_0(d)$ is an ideal of the lattice H.*

Proof. According to Remark 2.6, H is a lattice. Then, by Propositions 3.29(ii) and 3.16(v) and the paragraph before Proposition 3.29, it is easily seen $\ker_0(d)$ is an ideal of H. □

Remark 3.31. Let H and G be two $\vee$-hoop algebras, $d : H \to H$ a derivation on H and $f : H \to G$ be a hoop monomorphism. We define the map $\rho : \mathrm{Im(f)} \to \mathrm{Im(f)}$ by $\rho(f(x)) = f(d(x))$. We will show that ρ is a derivation on Im(f). It is clear that ρ is well-defined because f is a monomorphism. Suppose that $a, b \in \mathrm{Im(f)}$. There exist unique elements $x, y \in H$ such that $a = f(x)$ and $b = f(y)$. Now, we have

$$\begin{aligned}
\rho(a \odot b) &= \rho(f(x) \odot f(y)) \\
&= \rho(f(x \odot y)) \\
&= f(d(x \odot y)) \\
&= f((d(x) \odot y) \vee (x \odot d(y)) \\
&= f(d(x) \odot y)) \vee f(x \odot d(y)) \\
&= (f(d(x)) \odot f(y)) \vee (f(x) \odot f(d(y))) \\
&= (\rho(f(x)) \odot f(y)) \vee (f(x) \odot \rho(f(y))) \\
&= (\rho(a) \odot b) \vee (a \odot \rho(b)).
\end{aligned}$$

Consequently, ρ is a derivation on Im(f).

Corollary 3.32. *Let H and G be two $\vee$-hoop algebras, $d : H \to H$ a derivation on H and $f : H \to G$ be a hoop isomorphism. Then $\rho : G \to G$ by $\rho(f(x)) = f(d(x))$ is a derivation on G.*

Proof. It is clear according to Remark 3.31. □

Proposition 3.33. *Let $(H; \odot_H, \to_H, 1_H)$ and $(G; \odot_G, \to_G, 1_G)$ be two $\vee$-hoop algebras and d_H and d_G be two derivations on H and G, respectively. Then $d_{H\times G}$ given by $d_{H\times G}(x_1, x_2) = (d_H(x_1), d_G(x_2))$, for all $x_1 \in H$ and $x_2 \in G$, is a derivation on $H \times G$.*

Proof. We show that $d_{H\times G}$ holds in Definition 3.1. Suppose $x_1, y_1 \in H$ and $x_2, y_2 \in G$. By Proposition 2.2,

$$\begin{aligned}
&d_{H\times G}((x_1,x_2)\odot(y_1,y_2))\\
&= d_{H\times G}((x_1\odot_H y_1),(x_2\odot_G y_2))\\
&= (d_H(x_1\odot_H y_1), d_G(x_2\odot_G y_2))\\
&= ((d_H(x_1)\odot_H y_1)\vee(x_1\odot_H d_H(y_1)),((d_G(x_2)\odot_G y_2)\vee(x_2\odot_G (d_G(y_2)))\\
&= ((d_H(x_1)\odot_H y_1),((d_G(x_2)\odot_G y_2))\vee((x_1\odot_H d_H(y_1)),(x_2\odot_G (d_G(y_2)))\\
&= ((d_H(x_1),d_G(x_2))\odot_{H\times G}(y_1,y_2))\vee((x_1,x_2)\odot_{H\times G}(d_H(y_1),d_G(y_2)))\\
&= (d_{H\times G}(x_1,x_2)\odot_{H\times G}(y_1,y_2))\vee((x_1,x_2)\odot_{H\times G} d_{H\times G}(y_1,y_2)).
\end{aligned}$$

Therefore, $d_{H\times G}$ is a derivation on $H\times G$. □

Proposition 3.34. *With the assumptions of Proposition 3.33, so that H and G are bounded, we have:*

(i) $d_{H\times G}$ *is contractive (isotone, ideal, idempotent) if and only if* d_H *and* d_G *is contractive (isotone, ideal, idempotent),*
(ii) $(0,0)\in Fix_{d_{H\times G}}(H\times G)$,
(iii) $(x,y)\in \mathrm{Fix}_{d_{H\times G}}(H\times G)$ *if and only if* $x\in \mathrm{Fix}_{d_H}(H)$ *and* $y\in \mathrm{Fix}_{d_G}(G)$,
(iv) $(x,y)\in \ker_0(d_{H\times G})$ *if and only if* $x\in\ker_0(d_H)$ *and* $y\in\ker_0(d_G)$.

Proof. It is straightforward. □

4. Square root and derivations in hoop algebras

In this section, while examining the relationship and effect of the square root on the derivation of hoop algebras, we introduce the critical point and examine some of its properties. Throughout this section, we assume that H is a $\vee$-hoop algebra with a square root $s: H\to H$ and $d: H\to H$ is a multiplicative derivation on H.

Proposition 4.1. *Let H, s and d be defined as above. Then, for each $x\in H$:*

(i) $d(x)=ds(x)\odot s(x)$. *Therefore,* $d(x)\le s(x)$, $d(x)\le ds(x)$ *and* $d(x)\odot s(x)\le x$,
(ii) $ds(x)\le s(x)\to d(x)$ *and* $s(x)\le ds(x)\to d(x)$,
(iii) $d(x)^2\le x$ *and* $d(x)\odot d(y)\le s(x\odot y)$, *for all* $y\in H$,
(iv) $d(x)\le ds(x)\wedge sd(x)\le ds(x)\vee sd(x)$. *If d is contractive, then* $ds(x)\vee sd(x)\le s(x)$,
(v) $x\le ds(x)\to sd(x)$,
(vi) *If* $s(x)\le y$, *then* $d(x)\odot d(y)\le d(y^2)$, *for all* $y\in H$,
(vii) *If* $sd(x)=s(x)$, *then* $d(x)=x$,
(viii) $d(x^n)=ds(x)\odot s(x)^{2n-1}$, *for all* $n\in\mathbb{N}$,
(ix) *If d is contractive, then* $d(x)\le s(x\odot s(x))$,
(x) $d(x\odot s(x))=x\odot ds(x)$,

Proof. (i) Based on Definitions 2.13 and 3.1, we have:

$$d(x) = d(s(x) \odot s(x)) = (ds(x) \odot s(x)) \vee (ds(x) \odot s(x)) = ds(x) \odot s(x).$$

By Theorem 2.3(iii), we have $d(x) \leq s(x)$ and $d(x) \leq ds(x)$.
(ii) Using part (i) and Theorem 2.3(ii), the result is obtained.
(iii) According to part (i), $d(x)^2 = d(x) \odot d(x) \leq s(x) \odot s(x) = x$. Also, by Theorem 2.16(v), we have $d(x) \odot d(y) \leq s(x) \odot s(y) \leq s(x \odot y)$.
(iv) Based on part (i), we know $d(x) \leq ds(x)$. On the other hand, by Theorem 2.15, $d(x) \leq sd(x)$. Therefore, we will have $d(x) \leq sd(x) \wedge ds(x)$. If d is contractive, we have $d(x) \leq x$ and $ds(x) \leq s(x)$, $sd(x) \leq s(x)$ from monotonicity of s (Proposition 2.16(iv)). Therefore, $sd(x) \vee ds(x) \leq s(x) \vee s(x) = s(x)$.
(v) By part (i) we have, $sd(x) = s(ds(x) \odot s(x)) \geq ds(x) \odot s(x)$. Therefore, $x \leq s(x) \leq ds(x) \to sd(x)$, as $x \leq s(x)$.
(vi) Suppose that $s(x) \leq y$. Then, by part (i), $d(x) \odot d(y) \leq s(x) \odot d(y) \leq y \odot d(y) = d(y^2)$.
(vii) According to Theorem 2.16(i), it is clear.
(viii) Suppose $x \in H$ and $n \in \mathbb{N}$. Based on Proposition 3.3(ii), Definition 2.13 and part (i), we have $d(x^n) = x^{n-1} \odot d(x) = (s(x)^2)^{n-1} \odot s(x) \odot ds(x) = ds(x) \odot s(x)^{2n-1}$.
(ix) Suppose that the derivative d is contractive. In this case, we have $d(x) \leq x \leq s(x)$. As a result $d(x) \odot d(x) \leq x \odot s(x)$. Therefore $d(x) \leq s(x \odot s(x))$.
(x) According to Definition 3.1 and part (i), we have:

$$\begin{aligned} d(x \odot s(x)) &= (d(x) \odot s(x)) \vee (x \odot ds(x)) \\ &= (ds(x) \odot s(x) \odot s(x)) \vee (x \odot ds(x)) \\ &= (ds(x) \odot x) \vee (x \odot ds(x)) = x \odot ds(x). \end{aligned}$$

□

Proposition 4.2. *Let H be a bounded ∨-hoop algebra and s and d be defined as above. Then, for all $x, y \in H$:*
(i) $sd(x) \odot sd(x') \leq s(0)$,
(ii) *If $sd(x) = 1$, then $x'' = 1$ and $x' = 0$. Therefore, if H has (DNP) or $x^2 = x$, then $x = 1$,*
(iii) *If H is good and $d(x \odot y) = 0$, then $sd(x) \leq s(y)'$,*
(iv) $ds(0) \odot s(0) = 0$.

Proof. (i) Using Proposition 3.5(viii), $d(x') \leq d(x)'$. Now according to Theorem 2.16(iv,vi), $sd(x') \leq s(d(x)') = s(d(x) \to 0) = sd(x) \to s(0)$. Then, $sd(x') \odot sd(x) \leq s(0)$.
(ii) Using Proposition 3.5(vi) and Corollary 2.16(iii), we get the result.
(iii) We know $d(x) \odot y = 0$. Then, $sd(x) \odot s(y) = 0$. Thus, $sd(x) \leq s(y)'$.
(iv) Take $x = 0$, according to Propositions 4.1(i) and 3.5(i), the result is obtained. □

Proposition 4.3. *Let H and s be defined as above, and $d : H \to H$ be an isotone derivation on H. Then, for all $x, y \in H$:*
(i) $d(x \odot y) \leq s(d(x) \odot d(y))$,
(ii) $d(x \odot y) \leq d(x \wedge y) \leq sd(x \wedge y) \leq sd(x) \wedge sd(y)$,
(iii) $s(x) \to s(y) \leq sd(x) \to sd(y)$. *If d is contractive, then* $sd(x \to y) \leq x \to sd(y)$ *and* $s(x) \to s(y) \leq sd(x) \to s(y)$,
(iv) $s(x) \leq sd(y) \to sd(x)$.

Proof. (i) Since d is isotone, we have $d(x \odot y) \leq d(x)$ and $d(x \odot y) \leq d(y)$. Thus $d(x \odot y) \odot d(x \odot y) \leq d(x) \odot d(y)$. Then, by the definition of square root, we get $d(x \odot y) \leq s(d(x) \odot d(y))$.
(ii) According to Theorem 2.3(iii), $x \odot y \leq x \wedge y$. Hence $d(x \odot y) \leq d(x \wedge y)$. Also, we know $d(x \wedge y) \leq d(x)$ and $d(x \wedge y) \leq d(y)$. Now, according to Theorem 2.16(iv), we have $sd(x \wedge y) \leq sd(x)$ and $sd(x \wedge y) \leq sd(y)$. Therefore, $sd(x \wedge y) \leq sd(x) \wedge sd(y)$.
(iii) According to Proposition 3.13(ii), we have $x \to y \leq d(x) \to d(y)$. Therefore, by Theorem 2.16(iv,vi), we conclude that $s(x) \to s(y) \leq sd(x) \to sd(y)$. Suppose now that d is contractive. Then, according to Proposition 3.13(ii) and Theorem 2.16(iv,vi), $sd(x \to y) \leq s(x) \to sd(y) \leq x \to sd(y)$ and $s(x) \to s(y) \leq sd(x) \to sd(y) \leq sd(x) \to s(y)$.
(iv) By applying Proposition 3.13(vii), for all $x, y \in H$, we have $x \odot d(y) \leq d(x)$ and according to Theorem 2.16(iv,v), $s(x \odot d(y)) \leq sd(x)$, $s(x) \odot sd(y) \leq sd(x)$. Hence, $s(x) \leq sd(y) \to sd(x)$. □

Proposition 4.4. *Let H and s be defined as above, and $d : H \to H$ be an isotone derivation on H. Then $d^n(x) \geq d^{n-k}s(x) \odot d^k s(x)$, for all $x \in H$, $n \geq 2$ and $k \in \mathbb{N}$ that $1 \leq k < n$.*

Proof. We prove the statement by induction on n. For $n = 2$, by Proposition 4.1(i), we have

$$\begin{aligned} d^2(x) &= d(d(x)) \\ &= d(ds(x) \odot s(x)) \\ &= (d^2 s(x) \odot s(x)) \vee (ds(x) \odot ds(x)) \\ &\geq ds(x) \odot ds(x). \end{aligned}$$

suppose, for $n = m \geq 2$ and every $k \in \mathbb{N}$ such that $1 \leq k < m, d^m(x) \geq d^{m-k}s(x) \odot d^k s(x)$. Let $n = m + 1$. Based on Definition 3.1 and Proposition 4.1(i), for every $1 \leq k < m$, we have,

$$\begin{aligned} d^{m+1}(x) &= d(d^m(x)) \\ &\geq d(d^{m-k}s(x) \odot d^k s(x)) \\ &= (d(d^{m-k}s(x)) \odot d^k s(x)) \vee (d^{m-k}s(x) \odot d(d^k s(x))) \\ &\geq d^{m+1-k}s(x) \odot d^k s(x) \end{aligned}$$

and

$$d^{m+1}(x) \geq d^{m-k}s(x) \odot d(d^k s(x)) = d^{m-k}s(x) \odot d^{k+1}s(x).$$

Consequently, for all $n \in \mathbb{N}, x \in H, n \geq 2$ and all $k \in \mathbb{N}$ such that $1 \leq k < n$, $d^n(x) \geq d^{n-k}s(x) \odot d^k s(x)$. □

Proposition 4.5. *Let H be a cancellative ∨-hoop algebra and s and d be defined as above. Let $ds(x) = sd(x)$, for all $x \in H$. Then d is one to one.*

Proof. Suppose that $x, y \in H$ such that $d(x) = d(y)$. Then, $ds(x) = sd(x) = sd(y) = ds(y)$. Now by Proposition 4.1(i), we have $ds(x) \odot s(x) = ds(y) \odot s(y)$. Since H is cancellative, we have $s(x) = s(y)$. Hence, $x = y$ by Theorem 2.16(i). □

Remark 4.6. Let H, s and d be as above. If H is idempotent, then $ds(x) = sd(x)$. Because, by Proposition 2.16(ii), we have $s(x) = x$ which implies that $ds(x) = d(x) = sd(x)$. Consequently, if H is idempotent cancellative ∨-hoop algebra, then by Proposition 4.5, d is one to one.

Proposition 4.7. *Let H be a bounded ∨-hoop algebra and s and d be defined as above. Then in the following cases, the equality $ds(x) = sd(x)$ holds.*
(i) *d is contractive and $d(1) = 1$,*
(ii) *$x \in Fix_d(H) \cap \mathrm{Id}(H)$,*
(iii) *$x \in Fix_d(H)$ and H is cancellative,*

Proof. (i) According to Proposition 3.14(iii), d is the identity function.
(ii) If $x \in \mathrm{Fix}_d(H) \cap \mathrm{Id}(H)$, then $s(x) = d(x) = x$ and the equality is clear.
(iii) Suppose $x \in Fix_d(H)$ and H is cancellative. We have $d(x) = x$ and know that $d(x) = ds(x) \odot s(x)$. By multiplying both sides by $s(x)$ we get $x \odot s(x) = ds(x) \odot x$. Then, $ds(x) = s(x)$ as H is cancellative. □

Example 4.8. *(a) Let $I = [0, 1]$ be the unit closed interval of real numbers. In the product algebra **Pra**, the operations $\odot$ and $\rightarrow$ on I are defined as follows.*

$$x \odot y = xy \quad \text{and} \quad x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{if } x > y \end{cases}.$$

*for any $x, y \in I$. We know **Pra** $= (I; \odot, \longrightarrow, 1)$ is a hoop algebra. In **Pra**, we define the unary operation $s : I \rightarrow I$ as $s(x) = \sqrt{x}$, $\forall x \in I$. Now suppose $a \in I$ and $d_a : I \longrightarrow I$ is a principal ideal derivation on I. Now for all $x \in I$, we have*

$$d_a s(x) = as(x) = a\sqrt{x} \quad \text{and} \quad sd_a(x) = s(a \odot x) = \sqrt{ax} = \sqrt{a}\sqrt{x}.$$

We know $a \leq \sqrt{a}$. Therefore we have $a\sqrt{x} \leq \sqrt{a}\sqrt{x}$ and as a result $d_a s(x) \leq sd_a(x)$.

*(b) Let $I = [0, 1]$ be the unit closed interval of real numbers. In the Gödel algebra **Goa**, the operations $\odot$ and $\rightarrow$ on I are as follows*

$$x \odot y = \min(x, y) \quad \text{and} \quad x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y \end{cases}$$

*for any $x, y \in I$. We know **Goa** $= (I; \odot, \longrightarrow, 1)$ is a hoop algebra. In **Goa**, we define the unary operation $s : I \rightarrow I$ as $s(x) = x$, $\forall x \in I$. Now, suppose $a \in I$ and $d_a : I \longrightarrow I$ is a principal ideal derivation on I. For all $x \in I$, we have*

$$d_a s(x) = d_a(x) = a \odot x \quad \text{and} \quad sd_a(x) = d_a(x) = a \odot x.$$

In this case, $d_a s(x) = sd_a(x)$.

Proposition 4.9. *Let H and s be defined as above, and $d : H \rightarrow H$ be a contractive derivation on H. Then $d(x) \leq ds(x) \leq sd(x) \leq s(x)$, for all $x \in H$.*

Proof. Since d is contractive, we have $d(x) \leq x$, for all $x \in H$. Now, by Proposition 4.1(i), we have $d(x) = ds(x) \odot s(x) \geq ds(x) \odot ds(x)$. Therefore, by definition of s, it follows $ds(x) \leq sd(x)$. Hence, we have $d(x) \leq ds(x) \leq sd(x) \leq s(x)$, for all $x \in H$. □

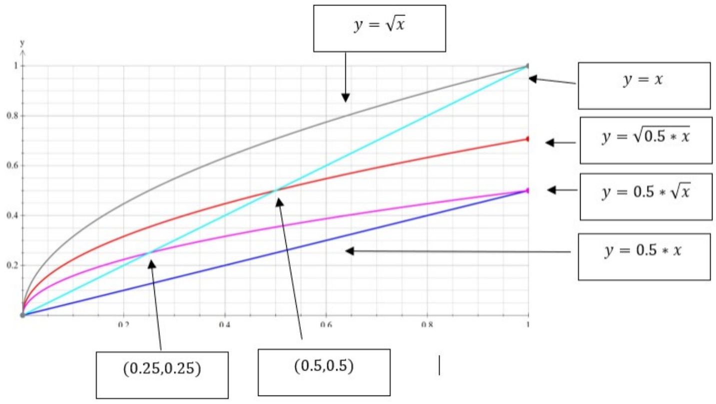
Definition 4.10. Let H, s and d be defined as above. We denote the set of all fixed points of sd in H by $\text{Fix}_{sd}(H)$. That is, $\text{Fix}_{sd}(H) = \{x \in H : sd(x) = x\}$. We call each element of $\text{Fix}_{sd}(H)$ a *critical point*.

Proposition 4.11. *Let H, s and d be defined as above. Then:*
(i) *If $x \in \text{Fix}_{sd}(H)$, then $d(x) = x^2$,*
(ii) *If $s(x \odot y) = s(x) \odot s(y)$, for all $x, y \in H$, then the converse of* (i) *holds.*

Proof. (i) Let $x \in \text{Fix}_{sd}(H)$. By Definition 2.13, we have $d(x) = sd(x) \odot sd(x) = x \odot x = x^2$.

(ii) Suppose that $x \in H$ and $d(x) = x^2$ so we have $sd(x) = s(x^2) = s(x) \odot s(x) = x$. Hence, $x \in \text{Fix}_{sd}(H)$. □

Example 4.12. *(a) Let H, s and d be defined as above, and $s(x \odot y) = s(x) \odot s(y)$, for all $x, y \in H$. Assume that H is bounded. Then, by Proposition 3.5(i) and Proposition 4.11(ii), we have $d(0) = 0 = 0 \odot 0$. Therefore, $0 \in \text{Fix}_{sd}(H)$. Also, if H is a good hoop, then $0 \in \text{Fix}_{sd}(H)$.*
(b) In the product algebra, Example 4.8(a), we have $s(x \odot y) = s(x) \odot s(y)$, for each $x, y \in H$. According to Proposition 4.11(ii), it is easily seen that $a \in \text{Fix}_{sd_a}(H)$. Because, if $sd_a(x) = x$ and $x \neq 0$, then $\sqrt{a}\sqrt{x} = x$. Hence $x = a$. Therefore, $\text{Fix}_{sd_a}(H) = \{0, a\}$. We have the following diagram:



Assuming $a = 0.5$, *the curves* $s(x) = \sqrt{x}$, $sd_{0.5}(x) = \sqrt{0.5 * x}$, $d_{0.5}s(x) = 0.5 * \sqrt{x}$ *and* $d_{0.5}(x) = 0.5 * x$ *have been drawn in the above diagram, where* $*$ *represents multiplication in real numbers. According to Theorem 3.8,* $d_{0.5}$ *is an ideal derivation. As it can be seen, only at points* $x = 0$ *and* $x = 0.5$, *we have* $sd_{0.5}(x) = x$ *and so* $\mathrm{Fix}_{sd_a}(H) = \{0, 0.5\}$. *It is also easy to see that in the interval* $[0, 1]$, *we have* $d_{0.5}(x) \le d_{0.5}s(x) \le sd_{0.5}(x) \le s(x)$. *Finally, suppose* $a \in [0, 1]$ *such that* $d_{0.5}s(a) = a$. *As a result, we have* $0.5 * \sqrt{a} = a$. *Hence* $a = 0.25$. *Therefore,* $\mathrm{Fix}_{d_{0.5}s}(H) = \{0, 0.25\}$.

Proposition 4.13. *Let* H, s *and* d *be defined as above and* $x \in \mathrm{Id}(H)$. *Then* $x \in \mathrm{Fix}_{sd}(H)$ *if and only if* $d(x) = x$.

Proof. If $x \in \mathrm{Fix}_{sd}(H)$, then $x = x \odot x = sd(x) \odot sd(x) = d(x)$. Now suppose $x \in H$ such that $d(x) = x$. Then, by Theorem 2.16(ii), we have $sd(x) = s(x) = x$. So $x \in \mathrm{Fix}_{sd}(H)$. □

Proposition 4.14. *Let* H *and* s *be defined as above and* $d : H \to H$ *be an isotone derivation and* $x \in H$ *a critical point. Then for all* $a, y \in H$:
(i) *If* $a \le x \le y$, *then* $sd(a) \le x \le sd(y)$,
(ii) *If* d *is a monomorphism, then the converse of clause* (i) *holds,*
(iii) *If* d *is ideal and* $y \le x$, *then* $ds(y) \le x$,
(iv) *If* $ds(a) = a$, *then* $d(a) = s(a)^3$.

Proof. (i) Suppose $x \le y$. Since d is isotone, $d(x) \le d(y)$. Now, in view of Theorem 2.16(iv), $x = sd(x) \le sd(y)$. If $a \le x$, then $sd(a) \le sd(x) = x$.
(ii) Suppose for $x, y \in H$ and $x = sd(x) \le sd(y)$. According to Theorem 2.16(iv), we have $d(x) \le d(y)$. Since d is a monomorphism, $d(x) \to d(y) = 1$ which implies that $x \le y$.
(iii) Suppose $y \le x$. By Theorem 2.16(iv) s is isotone and so, $s(y) \le s(x)$. Since d is ideal derivation, i.e., d is isotone too, and according to Proposition 4.9, we have $ds(y) \le ds(x) \le sd(x) = x$.
(iv) Suppose $ds(a) = a$. Then, $d(a) = ds(a) \odot s(a) = a \odot s(a) = s(a)^3$. □

Proposition 4.15. *Let H be a bounded $\vee$-hoop algebra, s be defined as above, and $d : H \to H$ be an ideal derivation on H. Let $x, y \in \mathrm{Fix}_{sd}(H)$ and $s(a \odot b) = s(a) \odot s(b)$, for all $a, b \in H$. If $d(1) \in B(H)$, then $x \odot y \in \mathrm{Fix}_{sd}(H)$. Also, $(\mathrm{Fix}_{sd}(H); \vee, \wedge)$ is a distributive lattice.*

Proof. Suppose $x, y \in \mathrm{Fix}_{sd}(H)$. According to Proposition 3.16, we have

$$sd(x \odot y) = s(d(x) \odot d(y)) = sd(x) \odot sd(y) = x \odot y.$$

Also according to Propositions 2.17 and 3.16, we have

$$sd(x \wedge y) = s(d(x) \wedge d(y)) = sd(x) \wedge sd(y) = x \wedge y$$

and

$$sd(x \vee y) = s(d(x) \vee d(y)) = sd(x) \vee sd(y) = x \vee y.$$

Therefore $\mathrm{Fix}_{sd}(H)$ is closed under $\wedge$ and $\vee$. Since H is a distributive lattice, it is easily seen that $\mathrm{Fix}_{sd}(H)$ is also a distributive lattice. □

Example 4.16. *(a) In Example 2.19, suppose $a \in I$ and $d_a : I \to I$ a principal ideal derivation. Then for all $x \in I$,*

$$d_a s(x) = a \odot s(x) = a \odot \left(\frac{x+1}{2}\right) \quad \text{and} \quad sd_a(x) = s(a \odot x) = \frac{(a \odot x) + 1}{2}.$$

Suppose, $x \in \mathrm{Fix}_{sd}(\mathbf{L}_1)$. We have $sd_a(x) = \frac{(a \odot x)+1}{2} = x$. We consider the following situations:
Case 1. $a \odot x = a + x - 1$. Therefore, $sd_a(x) = \frac{a+x-1+1}{2} = x$. As a result $a + x = 2x$. Hence $x = a$. So $a \in \mathrm{Fix}_{sd}(\mathbf{L}_1)$. For example, assuming $a = 0.8$, we have $sd_{0.8}(0.8) = s(0.8 \odot 0.8) = s(0.6) = \frac{0.6+1}{2} = 0.8$.
Case 2. $a \odot x = 0$. Therefore $sd_a(x) = \frac{0+1}{2} = x$. As a result $x = \frac{1}{2}$. Hence $\frac{1}{2} \in \mathrm{Fix}_{sd}(\mathbf{L}_1)$.
Because assuming $a = 0.5$, we also have $sd_{0.5}(0.5) = s(0.5 \odot 0.5) = s(0) = \frac{1}{2} = 0.5$. Also, assuming $a+x \geq 1$, if $a \leq x$, then, $sd_a(x) = s(a \odot x) = s(a+x-1) = \frac{(a+x-1)+1}{2} = \frac{a+x}{2} \leq x$ and if $x \leq a$, then $sd_a(x) = s(a \odot x) = s(a + x - 1) = \frac{(a+x-1)+1}{2} = \frac{a+x}{2} \geq x$.
Assuming $a + x \leq 1$, we have $sd_a(x) = s(a \odot x) = s(0) = \frac{1}{2} = 0.5 \geq x$. Also according to Proposition 4.9, if $a \leq x$, then we have $d_a s(x) \leq sd_a(x) \leq x$.
(b) Let H and s be defined as above, and $d : H \to H$ be the identity derivative on H. If $x^2 = x$, for all $x \in H$, then $\mathrm{Fix}_{sd}(H) = H$. Because, we have $sd(x) = s(x) = x$, for all $x \in H$.

Proposition 4.17. *Let H and s be defined as above, and d_1 and d_2 be contractive derivations on H. Let $s(x \odot y) = s(x) \odot s(y)$, for all $x, y \in H$. Then:*
(i) *If $x \in \mathrm{Fix}_{sd_1}(H)$ and $x \in \mathrm{Fix}_{sd_2}(H)$, then $x \in \mathrm{Fix}_{s(d_1 \vee d_2)}(H)$,*
(ii) *If $x \in \mathrm{Fix}_{s(d_1 \vee d_2)}(H)$ and at least one of d_i, $i = 1, 2$, is the identity derivation, then s is the identity function.*

Proof. (i) Using Proposition 3.23, the result is obtained.

(ii) Suppose $x \in \mathrm{Fix}_{s(d_1 \vee d_2)}(H)$. Thus we have $s(d_1 \vee d_2)(x) = x$. Therefore $sd_1(x) \vee sd_2(x) = x$. Let d_1 be an identity derivation. Therefore we have $s(x) \vee sd_2(x) = x \geq s(x)$. On the other hand, by Theorem 2.15(i), $x \leq s(x)$. Thus $s(x) = x$. The case that d_2 is the identity derivation is proved similarly. □

Proposition 4.18. *Let H be a ∨-hoop algebra with a square root s and a contractive derivation d. Let F be the filter of H. Then for all $x \in H$:*
(i) *If $sd(x) \in F$, Then $d(x)^n, ds(x)^n, x^n, s(x)^n \in F$, for all $n \in \mathbb{N}$,*
(ii) $\langle s(x) \rangle = \langle x \rangle \subseteq \langle d(x) \rangle = \langle ds(x) \rangle = \langle sd(x) \rangle$,
(iii) *If x is a critical point, then in* (ii) *the equality holds.*

Proof. (i) Suppose $sd(x) \in F$. In this case, based on the condition $S1$ of Definition 2.13, the assumption of the proposition, and the condition $F2$ of Definition 2.12, we have $sd(x) \odot sd(x) = d(x) \in F$. Therefore, by the condition $F3$ of Definition 2.12 and Proposition 4.9, $d(x)^n, ds(x)^n, x^n, s(x)^n \in F$.
(ii) Using Proposition 4.9, the result is obtained.
(iii) Since $sd(x) = x$, therefore $\langle x \rangle = \langle sd(x) \rangle$. As a result, $\langle x \rangle = \langle d(x) \rangle$. □

5. Conclusion and Future Research

In this paper, by developing the concept of multiplicative derivation on ∨-hoop algebras and presenting some properties, we showed that the set of multiplicative derivations on a hoop algebra forms a bounded semilattice and, under certain conditions, forms a distributive bounded lattice. Also, by combining the square root with the derivation, other properties were presented on these algebras. In the following, by introducing the critical points in view of the combination of square root with derivation, we presented some properties of these points and showed that the set of critical points forms a distributive lattice. Some of our future plans are to investigate the types of multiplicative derivation on hoop algebras, such as f-multiplicative and fg-multiplicative and the effect of the square root on the aforementioned derivation. Also, checking the properties of the created lattices will be one of our other plans.

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7. Conflict of interest

The authors declare no conflict of interest.

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