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# GENERATORS AND JOINT SPECTRA FOR A SPECIAL CLASS OF TOPOLOGICAL ALGEBRAS

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Abstract. Let $u$ be a generator for a commutative Banach algebra with unit. It is well known that the spectrum of $u$ is homeomorphic to the carrier of this algebra. In this paper, we extend this result for a broader class of complete metrizable topological algebras, particularly those satisfying the properties of fundamental strongly sequential(FSS) and linearly complete algebras. Specifically, we establish that the homeomorphism between the spectrum Sp(u) and the carrier space holds for FSS-algebras and linearly complete regular algebras. Thus, we generalize the classical result known for Banach algebras. Furthermore, by assuming that the boundedness radius $\beta$ is subadditive, we prove that the spectrum $Sp(u)$ is polynomially convex. This assumption also enables us to derive a more general result on the polynomial convexity of joint spectra in finitely generated algebras. To demonstrate the significance and nontrivial nature of these extensions, we provide illustrative examples that highlight how the introduced conditions substantially broaden the applicability of existing results.

*Keywords*: Strongly sequential algebras, Fundamental algebras, Boundedness radius, Carrier space, Joint generators.
*2020 MSC*: 46H05, 46H20.

## 1. Introduction

The study of non-normed topological algebras was initiated in 1940 by Arens, Gel'fand, Kaplansky and others (see [14]). In 1952, Michael conducted a thorough investigation of locally multiplicatively convex (LMC) algebras and highlighted the issue of identifying which algebras possess continuous multiplicative linear functionals [18]. This is commonly known as Michael's problem. In the same paper, Michael presented algebras in which the property that every multiplicative linear functional is continuous holds. This problem plays a crucial role in the development of research in topological and even normed algebras. In 1979, Husain introduced the concept of strongly sequential topological algebras [16]. Moreover, in 2008, Honary and Najafi Tavani demonstrated that this idea holds in another class of topological algebras, known as Q-algebras [15].

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In 1991, Ansari introduced the concept of fundamental algebras as an expansion of Cohen's factorization theorem [3]. Ten years later, in 2001, he established a new class called fundamental locally multiplicative (FLM) algebras by imposing an additional condition on fundamental algebras [4]. Finally, in 2010, Ansari proved that every FLM-algebra with unit is also a locally bounded algebra [6]. For further exploration of locally bounded algebras, see the following books [10, 11, 21].

In 2016, Ansari and Sabet were the first to investigate algebras that are both fundamental and strongly sequential [8]. In this paper, we refer to such algebras as FSS-algebras, which strictly contain FLM-algebras. In the same paper, the authors demonstrated that FSS-algebras are Q-algebras. For further discussion on Q-algebras, see [10, p. 149], [17, p. 43], or [14, p. 69]. Many properties of Q-algebras closely resemble those of normed algebras, with the main difference lying in proof techniques. For instance, the well-known Gleason–Kahane–Zelazko theorem holds in FSS-algebras [9].

This naturally raises the question of which results from the theory of Banach algebras or locally bounded algebras can be extended to FSS-algebras. First, in a commutative Banach algebra $A$ with unit, and a generator $u$ for it, we establish that the spectrum of $u$, denoted Sp(u), is homeomorphic to the carrier space of the algebra $A$, denoted by $\Phi_A$. By imposing an additional condition on FSS-algebras, we demonstrate that this result remains valid. The details of this extension are presented in Subsection 3.1 of this paper.

Secondly, in a commutative Banach algebra $A$ with unit, and a generator $u$ for it, the spectrum of $u$ is a polynomially convex set. In Subsection 3.2, we prove that Sp(u) remains a polynomially convex set by imposing two additional conditions on FSS-algebras.

## 2. **Preliminaries**

First, we mention some of the basic concepts of topological algebras.
A complex algebra $A$ is called a *topological algebra* if it is equipped with a Hausdorff topology and satisfies the following conditions:

(1) The function $(x, y) \mapsto x + y$, from $A \times A$ to $A$, is continuous,
(2) The function $(\alpha, x) \mapsto \alpha x$, from $\mathbb{C} \times A$ to $A$, is continuous,
(3) The function $(x, y) \mapsto xy$, from $A \times A$ to $A$, is continuous.

A topological algebra $A$ is called *strongly sequential*, if there exists a neighborhood of zero $U$ such that for every $x \in U$, the sequence $(x^n)_n$ converges to zero in its topology [16].
The *boundedness radius* of an element $a$ in the topological algebra $A$, denoted

by $\beta(a)$, is defined as follows [1]

$$\beta(a) = \inf\{r > 0 : \frac{a^n}{r^n} \to 0\}.$$

Additionally, the above definition includes the special case where $\inf \phi = +\infty$. If $\beta(a) < +\infty$, $a$ is said to be a *bounded member.*

*Remark* 2.1. Let $A$ be a strongly sequential topological algebra, and let $a \in A$. Then there exists a neighborhood of zero $U_0$ such that for every $x \in U_0$, the sequence $(x^n)_n$ converges to zero in the topology of $A$. Since the sequence $(\frac{1}{n})_n \to 0$, it follows that the sequence $(\frac{1}{n}a)_n$ converges to zero in A. Hence, there exists a natural number $N$ such that for every $n \geq N$, we have $\frac{1}{n}a \in U_0$. In particular $\frac{1}{N}a \in U_0$. Now, considering the above relations and noting that $\frac{1}{N}a \in U_0$, we obtain $(\frac{1}{N}a)^n \to 0$. Therefore

$$\beta(\frac{1}{N}a) \leq 1,$$

and thus,

$$\beta(a) \leq N.$$

This shows that every element of a strongly sequential topological algebra is bounded.

*Proposition* 2.2. *[8] A topological algebra A is strongly sequential if and only if $\beta$ is continuous at zero.*

Topological algebra A is called *fundamental* if there exists $b > 1$ such that, for every sequence $(x_n)_n$ in $A$, the convergence of $b^n(x_{n+1} - x_n) \to 0$ in $A$ implies the sequence $(x_n)_n$ is *a* Cauchy sequence [3].

*Proposition* 2.3. *If $A$ is* a *metrizable fundamental algebra and $M$ is* a *closed two-sided ideal of it, then quotient algebra $\frac{A}{M}$ is also* a *metrizable fundamental algebra.*

*Proof.* Suppose that the algebra $A$ has an invariant metric $d$. It is known that the following metric is invariant metric for the quotient algebra $\frac{A}{M}$ (see [19, p. 30])

$$\rho(a + M, b + M) := \inf\{d(a - b, m) : m \in M\}.$$

Let $b > 1$ and $(a_n + M)_n$ be *a* sequence in $\frac{A}{M}$ such that

$$\rho(b^n((a_{n+1} + M) - (a_n + M)), 0) \to 0.$$

For any $\varepsilon > 0$, there exists $N$ such that for all $n \geqslant N$, there exists an element $c_n \in M$ satisfying the following relation

$$d(b^n(a_{n+1} - a_n + c_n), 0) < \varepsilon.$$

Now we define the sequence $(m_n)_n$ in $M$ by the following recurrence relation

$$m_1 = 0, \quad m_{n+1} = c_1 + c_2 + ... + c_n (n \geqslant 1).$$

By the given condition and the fact $A$ is a fundamental algebra, it follows that the sequence $(a_n+m_n)_n$ is Cauchy. Consequently, for given $\varepsilon>0$, there exists $N$ such that for every $s>r\geqslant N$, we have

$$d(a_s+m_s, a_r+m_r)<\varepsilon,$$

therefore, it follows that

$$\rho(a_s+M, a_r+M)=\inf\{d(a_s-a_r, m)\colon m\in M\}\leqslant d(a_s-a_r, m_r-m_s)<\varepsilon.$$

Thus, the proof is complete. □

*Remark* 2.4. Proposition 2.3 was stated in [3, 5], without proof.

In this paper, we will use the abbreviation FSS-algebra for the term fundamental strongly sequential topological algebra. Additionally, we denote the set of invertible elements in a topological algebra with unit $A$ by $Inv(A)$, and the set of non-invertible elements by $Sing(A)$. Finally, a topological algebra with unit $A$ is called a Q-algebra if $Inv(A)$ is an open set.

*Proposition* 2.5. *[10] In every Q-algebra,* a *maximal ideal is closed.*

*Proposition* 2.6. *Every maximal ideal in* a *complete metrizable FSS-algera is closed.*

*Proof.* Every complete metrizable *FSS*-algebra is a Q-algebra [8]. Therefore, by Prposition 2.5 the result follows. □

An algebra with unit, in which $Inv(A)=A-\{0\}$, is called a *division algebra.* The spectrum of $a\in A$, denoted by $Sp(A,a)$, is defined as follows

$$Sp(A,a):=\{\lambda:\lambda-a\in Sing(A)\}.$$

For brevity and without ambiguity, we use the notation $Sp(a)$ instead of $Sp(A,a)$.

*Proposition* 2.7. *If $D$ is a complex division algebra that $Sp(a)\neq\phi$ for every $a\in A$, then $D=1_D\mathbb{C}$.*

*Proof.* Suppose that $a\in D$. Since $Sp(a)\neq\phi$, there exists $\lambda\in Sp(a)$ such that $\lambda-a\in Sing(D)=\{0\}$ which implies that $a=\lambda 1_D$. □

In this paper, we use the symbol $A^*$ to denote the set of all continuous linear functionals on $A$.

*Proposition* 2.8. *[2] Suppose $A$ is* a *fundamental complete metrizable algebra, in which every element is bounded. If $A^*$ separates the elements of $A$, then the spectrum of every element in $A$ is nonempty.*

A linear functional $\varphi$ on $A$ is called a *multiplicative linear functional* if it satisfies the following condition: For any two arbitrary elements $x,y\in A$

$$\varphi(xy)=\varphi(x)\varphi(y).$$

The set of nonzero multiplicative linear functionals on $A$, equipped with the A-topology, is called the *carrier space.* In this paper, we denote this set by $\Phi_A$.

*Proposition* 2.9. *[7] Let $A$ be* a *complete metrizable $FSS$-algebra. Then, the following results hold:*

(1) *If $f \in \Phi_A$, then $|f(x)| \leqslant \beta(x)$, for all $x \in A$.*
(2) *The carrier space of $A$ is weak$^*$ compact.*

*Proposition* 2.10. *[15] If $A$ is* a *complete metrizable Q-algebra, then every multiplicative linear functional on $A$ is continuous.*

## 3. Main results

We present and prove the main theorems in two separate subsections.

### 3.1. On the spectrum of the generators in monothetic algebras.

*Definition* 3.1. For every topological algebra $A$, we say that $A$ is a linearly complete regular algebra if, for every closed subspace $M$ of $A$ and every element $a \in A$ such that $a \notin M$, there exists an element $\Lambda \in A^*$ satisfying $\Lambda a = 1$ and $\Lambda b = 0$ for all $b \in M$. For brevity, we use the abbreviation LCR instead of linearly complete regular.

*Remark* 3.2. According to the definition of an LCR-algebra, if an algebra $A$ is an LCR-algebra, then $A^*$ separates the points of $A$. This can be seen in the fact that every locally convex algebra satisfies the $LCR$-algebra property (see [19, p. 59]).

*Lemma* 3.3. *Suppose that $A$ is* a *complete metrizable $FSS$ and $LCR$-algebra. Moreover, let $M$ be* a *closed ideal of $A$. Then the quotient algebra $\frac{A}{M}$ is* a *complete metrizable algebra, and the spectrum of every element in $\frac{A}{M}$ is nonempty.*

*Proof.* It is clear that $\frac{A}{M}$ is *a* complete metrizable algebra. On the other hand, for every $a \in A$, the following inequality holds

$$\beta_{\frac{A}{M}}(a+M) \leqslant \beta(a).$$

From Proposition 2.3, the algebra $\frac{A}{M}$ is fundamental. Moreover, as noted in Remark 2.1, every element of this algebra is bounded. According to Proposition 2.8, to establish the result, it suffices to show that $(\frac{A}{M})^*$ separates the points of $\frac{A}{M}$. Let $a+M$ be an arbitrary element of $\frac{A}{M}$ such that $a \notin M$. Since the algebra $A$ is $LCR$-algebra, there exists $\Lambda \in A^*$ such that $\Lambda a = 1$ and $\Lambda x = 0$ for all $x \in M$. Based on this, we define the function $f : \frac{A}{M} \to \mathbb{C}$ as follows

$$f(x+M) = \Lambda(x).$$

To further support this claim, we will prove that $f \in (\frac{A}{M})^*$. To prove that $f \in (\frac{A}{M})^*$, we proceed as follows.
Suppose that $x_1, x_2 \in A$ and $\alpha \in \mathbb{C}$, and assume that $x_1 + M = x_2 + M$. Then we have

$$\begin{aligned} x_1 - x_2 \in M &\Longrightarrow \Lambda(x_1 - x_2) = 0 \\ &\Longrightarrow \Lambda(x_1) = \Lambda(x_2). \end{aligned}$$

Thus, we have

$$\begin{aligned} f(\alpha(x_1+M)+(x_2+M)) &= f(\alpha x_1+x_2+M) \\ &= \Lambda(\alpha x_1+x_2) \\ &= \alpha\Lambda(x_1)+\Lambda(x_2) \\ &= \alpha f(x_1+M)+f(x_2+M). \end{aligned}$$

Now, suppose that $x_n+M \to 0_{\frac{A}{M}} = M$. Then we have

$$\rho(x_n+M, M) \to 0.$$

For any $\varepsilon > 0$, there exists a natural number $N$ such that for all $n \geqslant N$ we have

$$\rho(x_n+M, M) < \frac{\varepsilon}{2}.$$

By the definition of $\rho(x_n+M) = \inf\{d(x_n, m) : m \in M\}$, there exists $m_n \in M$ such that

$$d(x_n, m_n) < \rho(x_n+M, M) + \frac{\varepsilon}{2}.$$

Therefore, for every $n \geqslant N$ we obtain

$$d(x_n - m_n, 0) < \varepsilon.$$

Thus, the sequence $(x_n - m_n)_n$ converges to 0. Additionally, since $\Lambda$ is both linear and continuous, we can conclude for all $n, m > N$, the following holds

$$\begin{aligned} f(x_n - m_n + M) &= \Lambda x_n - \Lambda m_n \\ &= \Lambda(x_n - m_n) \\ &\to 0. \end{aligned}$$

Thus, $f$ is continuous, which implies that $f \in (\frac{A}{M})^*$. Consequently, the proof is complete. □

In the remainder of this paper, we shall assume that $A$ is a commutative algebra with unit.

*Lemma* 3.4. *If $A$ is* a *complete metrizable $FSS$ and $LCR$-algebra, then the maximal modular ideals of $A$ are the kernels of multiplicative linear functionals.*

*Proof.* We know that the two-sided maximal modular ideals of $A$ with codimension 1 are the kernels of multiplicative linear functionals of $A$ (see Theorem 4 in Section 11 of [12]). To prove this lemma, it is sufficient to show that every maximal modular ideal $A$ in an $FSS$-algebra has co-dimension 1. Let $M$ be *a* maximal ideal of $A$ with modular unit $j$. By Proposition 2.6, the ideal $M$ is closed, therefore, $B = \frac{A}{M}$ is *a* topological algebra [10]. In the continuation of the proof, we show that the only ideals of $B$ are $B$ and $\{0\}$. For this purpose,

suppose $J$ is an ideal of $B$ other than $B$ and $\{0\}$.
Let $K = \{k \in A \colon k + M \in J\}$. Now, if $k, k' \in K$, $\alpha \in \mathbb{C}$ and $a \in A$, we have

$$\begin{aligned} k, k' \in K &\Longleftrightarrow (k + k') + M = (k + M) + (k' + M) \in J, \\ \alpha k \in K &\Longleftrightarrow (\alpha k) + M = \alpha(k + M) \in J, \\ ak \in K &\Longleftrightarrow (ak) + M = (a + M)(k + M) \in J. \end{aligned}$$

Therefore, $K$ is an ideal of $M$. Moreover, the following relations hold

$$\begin{aligned} J \neq B &\Longrightarrow \exists a + M \in B \colon a + M \notin J \Longrightarrow a \notin K, \\ m \in M &\Longrightarrow m + M = M = 0_B \in J, \\ J \neq \{0_B\} = \{M\} &\Longrightarrow \exists a + M \in B \colon a + M \notin J \Longrightarrow (a \notin M, a \in K). \end{aligned}$$

Thus, we conclude that

$$M \subsetneq K \subsetneq A.$$

Since $M$ is a maximal ideal of $A$, the latter result leads to a contradiction. Moreover, $j + M$ is the unit element of $B$ because

$$(a + M)(j + M) = (j + M)(a + M) = a + M \quad (since \quad aj - a = ja - a \in M).$$

Therefore, $B$ is a division algebra. By Lemma 3.3, the spectrum of every element in $B$ is nonempty. Consequently, by Proposition 2.7, we have

$$B = C\,(j + M).$$

Now, for an arbitrary element $a \in A$, there exists a unique scalar $\alpha$ such that

$$a - \alpha j = m \rightarrow a = m + \alpha j.$$

Thus, we can write $A = M \oplus < j >$, meaning the co-dimension of $M$ is 1. Therefore, the proof is complete. □

*Lemma* 3.5. *Assume that $A$ is a complete metrizable FSS and LCR-algebra, and let $a \in A$. Then we have*

$$sp(a) = \{\varphi(a) : \varphi \in \Phi_A\}.$$

*Proof.* To prove this, we need to show that

$$\lambda \in sp(a) \Leftrightarrow \lambda \in \{\varphi(a) : \varphi \in \Phi_A\}.$$

If $\lambda \neq 0$, the proof follows a similar argument to the one presented in Bonsall's book (see Proposition 9 in Section 16 of [12]). Thus, it suffices to demonstrate that

$$0 \notin sp(a) \Leftrightarrow 0 \notin \{\varphi(a) : \varphi \in \Phi_A\}.$$

First, assume that $0 \notin sp(a)$. This means that there exists an inverse element $a^{-1}$ such that $aa^{-1} = 1$. To prove this, we use proof by contradiction. Suppose that $0 \in \{\varphi(a) : \varphi \in \Phi_A\}$. Then, there exists a $\varphi \in \Phi_A$ such that $0 = \varphi(a)$. Consequently, we have

$$1 = \varphi(1) = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1}) = 0.$$

This is a contradiction. Therefore, $0 \notin \{\varphi(a) : \varphi \in \Phi_A\}$. Assume $0 \in sp(a)$. Since $a \in aA$ and $aA$ is an ideal of $A$, there exists a maximal ideal of $A$ containing $aA$. By Lemma 3.4, there exists a $\varphi \in \Phi_A$ such that $aA \subseteq ker\varphi$, implying $\varphi(a) = 0$, which contradicts our assumption. Therefore, the proof is complete. □

Let $\{a_1, \cdots, a_n\}$ be a finite subset of $A$. The smallest closed subalgebra containing $\{a_1, \cdots, a_n\}$ is called the *generated subalgebra* by $\{a_1, \cdots, a_n\}$ and is denoted by $A(a_1, \cdots, a_n)$. The set $\{a_1, \cdots, a_n\}$ is called a set of joint generators for $A$ if $A(a_1, \cdots, a_n) = A$. In particular, an element $a$ is called a generator of $A$ if $A(a) = A$. An algebra that has a generator is called a *monothetic algebra*. An algebra $A$ is called *finitely generated* if there exists a finite set of joint generators for it.

*Theorem* 3.6. *Let $A$ be* a *complete metrizable $FSS$ and $LCR$-algebra, and let $u$ be* a *generator for it. Then the mapping*

$$\varphi \mapsto \varphi(u) \quad (\varphi \in \Phi_A)$$

*is* a *homeomorphism from $\Phi_A$ to $Sp(u)$.*

*Proof.* The mapping $G : \Phi_A \to Sp(u)$, given by

$$G(\varphi) = \varphi(u),$$

is continuous by definition. Moreover, as stated in Lemma 3.5, this function is surjective. To establish that $G$ is a homeomorphism, it suffices to show that it is injective. Indeed, according to Proposition 2.9, since $\Phi_A$ is equipped with the weak-star compact topology, the continuity of its inverse function follows immediately. Consequently $G$ is a homeomorphism.
Now, suppose $\varphi, \psi \in \Phi_A$ satisfy $G(\varphi) = G(\psi)$, meaning that $\varphi(u) = \psi(u)$. Let $x$ be an arbitary element of $A$. If $A(u) = A$, then there exists a sequence of polynomials $(p_n)_n$ with complex coefficients, such that

$$x = \lim_{n\to\infty} p_n(u).$$

By Proposition 2.10, the functions $\varphi$ and $\psi$ are continuous, so we obtain

$$\begin{aligned}\varphi(x) &= \varphi(\lim_{n\to\infty} p_n(u))\\ &= \lim_{n\to\infty} \varphi(p_n(u))\\ &= \lim_{n\to\infty} \psi(p_n(u))\\ &= \psi(\lim_{n\to\infty} p_n(u))\\ &= \psi(x).\end{aligned}$$

Thus, we conclude that $\varphi = \psi$, which proves that $G$ is injective. Therefore, the proof is complete. □

The following examples provide evidence for the existence of a complete metrizable FSS and LCR-algebra. For more details, refer to references [2] and [8].

*Example* 3.7. *Let $A$ be a commutative locally bounded but non-locally convex algebra equipped with the metric $d_1$ defined by*

$$d_1(x,y) = \|x-y\|_p, \text{ where } \|.\|_p \text{ is the } p-norm.$$

*Also, let $X$ be* a *complete metrizable locally convex but non-locally bounded topological vector space with the metric $d_2$.*
*Denote by $e$ the unit element of $A$. Suppose the mapping $(a,x) \mapsto xa$ is* a *bilinear and continuous mapping from $A \times X$ into $X$, satisfying the following conditions :*
⋇ $x(ab) = (xa)b$,
⋇ $xe = x$,
*for all $a, b \in A$ and $x \in X$.*
*Then, $X$ is* a *topological unit-linked right $A$-module, with the module multiplication defined by $(a,x) \mapsto xa$. Moreover, the space $Y = X \times A$ is a non-locally bounded, non-locally convex, fundamental topological vector space with pointwise operations and metric $d$, where*

$$d((x_1,a_1),(x_2,a_2)) = d_1(a_1,a_2) + d_2(x_1,x_2).$$

*Define the multiplication on $Y$ by*

$$(x_1,a_1)(x_2,a_2) = (x_1a_2 + x_2a_1, a_1a_2)$$

*for all $x_1, x_2 \in X$ and $a_1, a_2 \in A$. Now, $Y$ is an algebra, and since the module multiplication is continuous, $Y$ is a topological algebra [2].*
*Ansari, Sabet and Sharifi have shown that the algebra $Y$ is a complete metrizable $FSS$-algebra with unit [8]. Now, suppose that the algebra $A$ is an $LCR$-algebra. We will demonstrate that the algebra $Y$ is also an $LCR$-algebra.*
*Let $M = M_X \times M_A$ be* a *closed subspace of $Y$, and let $(x_0,a_0)$ be an element of $Y$ that does not belong to $M$, meaning that at least one of the following conditions fails to hold*
⋇ $x_0 \in M_X$,
⋇ $a_0 \in M_A$.
*Assume $a_0 \notin M_A$. Since $M_A$ is* a *closed subspace of $A$ and $A$ is an $LCR$-algebra, there exists a continuous linear functional $\Lambda_A$ on $A$ such that*

$$\Lambda_A(m_A) = 0 \quad for\, all\, m_A \in M_A \, and \, \Lambda_A(a_0) = 1.$$

*Define $\Lambda(x,a) = \Lambda_A(a)$. Clearly $\Lambda$ is* a *continuous linear functional on $Y$. Moreover, for every element $b \in M$, we have*

$$\Lambda(b) = 0, \quad while \quad \Lambda(x_0,a_0) = 1.$$

*Similarly, if $x_0 \notin M_X$, then since every lovally convex space is an $LCR$-algebra the result follows. Hence, the algebra $Y$ is an $LCR$-algebra.*

*Example* 3.8. *Let* $0 < p < 1$,*and let* $T = \{t_1, t_2, \cdots\}$ *be* a *set of symbols. Define* $S$ *as the commutative semigroup generated by* $T$ *with the operation*

$$t_i t_j = t_{\min\{i,j\}} \quad if \quad i \neq j, \quad and \quad t_i^n t_i = t_i^{n+1}.$$

*Thus, the semigroup* $S$ *consists of elements of the form*

$$S = \{t_i^j : i, j \in \mathbb{N}\}.$$

*Now, define the algebra* $A$ *as follows*

$$A = \{ \sum_{i,j=1}^{\infty} \alpha_{ij} t_i^j : \ \sum_{i,j=1}^{\infty} |\alpha_{ij}|^p (j+1)^p < \infty, \quad \alpha_{ij} \in \mathbb{C}\}.$$

*Let* $A$ *be* a *locally bounded non-locally convex algebra, generated by* $S$*, with* $p$*-norm defined as*

$$\| \sum_{i,j=1}^{\infty} \alpha_{ij} t_i^j \| = \sum_{i,j=1}^{\infty} |\alpha_{ij}|^p (j+1)^p.$$

*Let* $\tilde{A} = A \bigoplus \mathbb{C}$ *be the unitization of* $A$*. Then* $\tilde{A}$ *is unital, locally bounded, and non-locally convex algebra.*
*Now, define the space* $X$ *by*

$$X = \{ \sum_{i,j=1}^{\infty} \alpha_{ij} t_i^j : \sum_{i,j=1}^{\infty} |\alpha_{ij}|^p p_m(t_i^j) < \infty \, for \, all \, m \in \mathbb{N}\},$$

*where the function* $p_m$ *is defined as*

$$p_m(t_i^j) = \begin{cases} (j+1)^p, & i \leqslant m \\ 1, & i > m \end{cases},$$

*and the corresponding seminorm is*

$$p_m(\sum_{i,j=1}^{\infty} \alpha_{ij} t_i^j) = \sum_{i,j=1}^{\infty} |\alpha_{ij}|^p p_m(t_i^j).$$

*Thus,* $p_m$ *is* a *seminorm on* $X$*, making* $X$ a *locally convex and non-locally bounded algebra.*
*Moreover,* $A$ *is* a *subalgebra of* $X$*, so* $X$ *is* a *locally convex right* $\tilde{A}$*-module with module multiplication defined as*

$$x(a, \lambda) = xa + \lambda x, \quad for \quad a \in A, x \in X, \, and \, \lambda \in \mathbb{C}.$$

*Therefore, the space* $Y = X \times \tilde{A}$*, equipped with the algebraic operation defined in Example 3.7, is an* $FSS$*-algebra, which is neither locally bounded nor locally convex.*
*Next, we show that* $A$ *is an* $LCR$*-algebra. Let* $M$ *be* a *closed subspace of* $A$*, and assume*

$$a = \sum_{i,j=1}^{\infty} \alpha_{ij}(a) t_i^j \notin M.$$

*Suppose that,* $t_{i_0}^{j_0} \notin M$ *and* $\alpha_{i_0 j_0}(a) \neq 0$. *It has been shown that the linear functionals* $\Lambda_{k,l}$ *(for natural numbers* $k, l$*) defined by*

$$\Lambda_{k,l}(\sum_{i,j=1}^{\infty} \alpha_{ij} t_i^j) = \alpha_{kl}$$

*are continuous on* $A$ *[2]. Now, define*

$$\Lambda = \frac{1}{\Lambda_{i_0,j_0}(a)} \Lambda_{i_0,j_0}.$$

*Then,* $\Lambda$ *is* a *continuous functional on* $A$ *such that*

$$\Lambda(a) = 1, \quad for\ all\ b \in M,\ \Lambda(b) = 0.$$

*Thus,* $A$ *is an* $LCR$*-algebra.*

3.2. **On the polynomial convexity of the spectrum of generators and its extensions.**

*Proposition* 3.9. *Suppose* $A$ *is* a *strongly sequential algebra, and let* $a$ *be* a *generator for* $A$. *If* $\lambda \in \mathbb{C} - Sp(a)$ *and* $P(\mathbb{C})$ *denotes the set of all polynomials with complex coefficients, then there exists* a *polynomial* $p \in P(\mathbb{C})$ *such that*

$$|p(\lambda)| \geqslant \beta(p(a)).$$

*Proof.* According to the assumptions, there exists an element $b \in A$ such that $(\lambda - a)b = 1$. Since $a$ is a generator for $A$, there exists a sequence of polynomials $p_k \in P(\mathbb{C})$ such that $p_k(a) - b \to 0$. By Proposition 2.2, the function $\beta$ is continuous at zero. Hence, there exists a natural number $N$ such that

$$\beta(b - p_N(a)) < 1/(\beta(\lambda - a)).$$

On the other hand, for all $x, y \in A$ the relation $\beta(xy) \leqslant \beta(x)\beta(y)$ holds [13]. Thus, it follows that

$$\begin{aligned}\beta(1 - (\lambda - a)p_N(a)) &= \beta((\lambda - a)b - (\lambda - a)p_N(a)) \\ &= \beta((\lambda - a)(b - p_N(a))) \leqslant \beta(\lambda - a)\beta(b - p_N(a)) < 1.\end{aligned}$$

Now, if we put $p(z) = 1 - (\lambda - z)p_N(z)$, then it is clear that, $p \in P(\mathbb{C})$, $p(\lambda) = 1$, and $\beta(p(a)) < 1$. Thus, the proof is complete. □

In the following, we extend Proposition 3.9 to finitely generated algebra. Let $\mathbb{C}^n = \overbrace{\mathbb{C} \times \cdots \times \mathbb{C}}^{n\,times}$, $A^n = \overbrace{A \times \cdots \times A}^{n\,times}$ and let $P(\mathbb{C}^n)$ denote the set of all polynomials in $n$ complex variables. For $p \in P(\mathbb{C}^n)$ and $a = (a_1, a_2, \ldots, a_n) \in A^n$, the element $P(a)$ is defined by evaluating the polynomial $P$ at $a$, replacing $a = (a_1, a_2, \ldots, a_n)$ by $z = (z_1, z_2, \ldots, z_n)$. Given $a = (a_1, a_2, \ldots, a_n) \in A^n$, the joint spectrum of $a$ denoted by $Sp(A, a)$, is the subset of $\mathbb{C}^n$ given by

$$Sp(A, a) = \{\varphi(a) : \varphi \in \Phi_A\},$$

where $\varphi(a) = (\varphi(a_1), \ldots, \varphi(a_n))$. As in the case $n = 1$, we write $Sp(a)$ instead of $Sp(A, a)$ when there is no risk of confusion.

*Proposition* 3.10. *Let $A$ be* a *complete merizable $FSS$ and $LCR$-algebra. Define the set*

$$J = \{\sum_{k=1}^{n}(\lambda_k - a_k)b_k : b = (b_1, b_2 \dots, b_n) \in A^n\}$$

*where $a = (a_1, a_2, \dots, a_n) \in A^n$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n$.*
*Then the following statements are equivalent:*

(1) $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in Sp(a)$.
(2) *$J$ is* a *proper ideal of $A$.*
(3) $J \subset SingA$.
(4) $1 \notin J$.

*Proof.* By applying Lemma 3.4, the proof proceeds along lines similar to those presented in Bonsall's book (see [12, p. 100]). □

Suppose $A$ is a topological algebra in which every element is bounded. A real-valued function $\varphi$ is defined on $A$ is called a subadditive function if, for all $x, y \in A$, it satisfies the inequality

$$\varphi(x + y) \leqslant \varphi(x) + \varphi(y).$$

Kinani, Oubbi and Oudadess established that the boundedness radius function $\beta$ is subadditive in commutative locally convex algebras [13].

*Proposition* 3.11. *Suppose that $A$ is* a *complete metrizable $FSS$ and $LCR$-algebra, and that $\beta$ is* a *subadditive function on $A$. If the set $\{a_1, a_2, \dots, a_n\}$ is* a *generator for $A$ and $\lambda \in \mathbb{C}^n - Sp(a)$, then there exists* a *polynomial $p \in P(\mathbb{C}^n)$ such that*

$$|p(\lambda)| \geqslant \beta(p(a)).$$

*Proof.* See Proposition 9 in Section 19 of Bonsall's book for a detailed proof [12]. □

Note, under the assumptions of the above Proposition, the results of Proposition 3.10 remain valid [12].
Let $K$ be a compact subset of a topological space $E$. The seminorm $|\cdot|_K$ on $C(E)$, is defined by

$$|f|_K = \sup\{|f(x)| : x \in K\}.$$

For a compact subset $K$ of $\mathbb{C}^n$, the polynomially convex hull of $K$ is defined as

$$hull(K) = \{z \in \mathbb{C}^n : \forall p \in P(\mathbb{C}^n), |p(z)| \leqslant |p|_K\}.$$

Clearly, we always have $K \subseteq hull(K)$. Furthermore, if $hull(K) \subseteq K$, then $K$ is called a polynomially convex set.

*Theorem* 3.12. *Assume that $A$ is* a *complete metrizable $FSS$ and $LCR$-algebra, and that $\beta$ is subadditive. If $\{a_1, a_2, \dots, a_n\}$ is* a *set of joint generators for $A$, then the joint spectrum $Sp(a)$, where $a = (a_1, \dots, a_n)$, is* a *nonempty, compact, and polynomially convex subset of $\mathbb{C}^n$.*

*Proof.* We consider the function $G : \Phi_A \to \mathbb{C}^n$ defined by

$$G(\varphi) = \varphi(a) = (\varphi(a_1), \varphi(a_2), ..., \varphi(a_n)).$$

For each i, let

$$e_i = (0, 0, \dots, 1, \dots, 0).$$

Then we can express $G(\varphi)$ as

$$G(\varphi) = \widehat{a_1}(\varphi)e_1 + \widehat{a_2}(\varphi)e_2 + \cdots + \widehat{a_n}(\varphi)e_n$$

where each function $\widehat{a_i} : \Phi_A \to \mathbb{C}$ is defined by

$$\widehat{a_i}(\varphi) = \varphi(a_i), \quad i = 1, 2, \dots, n.$$

Since each $\widehat{a_i}$ is continuous, it follows that $G$ is also continuous. By Proposition 2.9, $\Phi_A$ is a compact set, and by Lemma 3.4, it is nonempty. Therefore, $G(\Phi_A)$ is a nonempty, compact subset of $\mathbb{C}^n$. To complete the proof, it remains to show that $Sp(a)$ is polynomially convex, i.e., that

$$hull(Sp(a)) \subseteq Sp(a).$$

To this end, we first prove that the inequality $|p|_{Sp(a)} \leqslant \beta(p(a))$ holds. For any $z \in Sp(a)$ and any polynomial $p \in P(\mathbb{C}^n)$, by the definition of $Sp(a)$, there exists $\varphi \in \Phi_A$ such that $z = \varphi(a)$. Then by Proposition 2.9, it follows

$$|p(z)| = |p(\varphi(a))| = |\varphi(p(a)| \leqslant \beta(p(a)).$$

Thus,

$$|p|_{Sp(a)} \leqslant \beta(p(a)).$$

Now, assume $\lambda \in hull(Sp(a))$. By the definition of $hull(Sp(a))$, for any polynomial $p \in P(\mathbb{C}^n)$, we have

$$|p(\lambda)| \leqslant |p|_{Sp(a)}.$$

Since $|p|_{Sp(a)} \leqslant \beta(p(a))$, it follows that

$$|p(\lambda)| \leqslant \beta(p(a)).$$

Finally, by Proposition 3.11, this implies that $\lambda \in Sp(a)$, completing the proof. □

*Remark* 3.13. Theorem 2.3 presents a class of algebras that can be complete metrizable $FSS$ and $LCR$, where the function $\beta$ is subadditive, while at the same time, these algebras are not Banach algebras [20].

## 4. Conclusion

In this paper, we extend the classical correspondence between the space of multiplicative functions and the spectrum of a generator in Banach algebras to a broader class of topological algebras called complete metrizable FSS-algebras (i.e., algebras that are both fundamental and strongly sequential). We also show that, under suitable conditions, the spectrum of a generator in these algebras is polynomially convex. These findings may be useful for understanding the spectral structure and uniqueness of topology in non-normable algebras. As a continuation of this research, future investigations may focus on extending spectral results in the framework of commutative FSS-algebras, particularly examining the spectra of generators in relation to topological properties such as convexity and compactness.

## 5. Author Contributions

All authors have actively and substantially participated in every stage of the work related to this manuscript, from inception to completion.

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## 8. Conflict of interest

The authors declare no conflict of interest.

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