Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

OPTIMIZATION OF A TIME CONTINUOUS PORTFOLIO OF ASSETS AND DERIVATIVES

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Article type: Research Article
(Received: 20 October 2024, Received in revised form 05 June 2025)
(Accepted: 08 August 2025, Published Online: 08 August 2025)

Abstract. We consider an incomplete market and suggest a self-financing time continuous investment strategy consisting of a risk-free asset (bond), a risky asset, and a financial derivative whose value moves inversely to that of the risky asset. We optimize the wealth process by introducing a parametric convex utility function that simultaneously maximizes wealth and minimizes the mean square of it. Using the HJB equation, we compute precisely the optimal portfolio process, where notably, a range of processes can optimize the problem. This advantage enables investors to gain fringe benefits while maintaining their overall investment strategy by adjusting their portfolios accordingly. As an application of the results, we optimize a portfolio process with a European put as the derivative and compute the corresponding optimal wealth numerically. Additionally, we will outline a method to calculate the market return rate and the martingale parameter, which are necessary for optimization.

Keywords: Self-financing portfolio, Dynamic programming, Wealth optimization, HJB equation.
2020 MSC: 62P20, 49L25, 37H10.

1. Introduction

In financial mathematics, investment strategies and portfolio optimization are two critical areas of study that seek to maximize returns while minimizing risks. Investors are constantly faced with the challenge of allocating their assets wisely according to market dynamics oriented by interactions between various financial elements such as bonds, risky assets, derivative securities and etc. This task becomes even more complicated when one considers the competing interests of risk aversion and the desire for high returns.

Investors have always found simultaneous investment in risk-free and risky markets intriguing, leading to numerous studies on optimizing such investments using various mathematical models. These studies often explore the delicate balance between securing stable returns through risk-free assets and pursuing higher potential gains afforded by volatile stocks or alternative investments.

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https://doi.org/10.22103/jmmr.2025.24238.1711

Publisher: Shahid Bahonar University of Kerman
How to cite: O. Rabieimotlagh, *Optimization of a time continuous portfolio of assets and derivatives*, J. Mahani Math. Res. 2026; 15(1): 183-198.

Previous leading studies (see for example [1,7,8,13,14] and references therein), have been continued by recent research on simultaneous investment optimization across parallel markets to achieve an optimal stock portfolio. In two very fundamental researches [9] and [20] (see also [10]), the authors proposed a general time continuous portfolio selection model and carried out an analytical approach to the optimal utility. Later in [4], the author considered a time continuous portfolio consisting of one risk-less asset and a group of risky assets and used a mean variance minimization problem to a discretized model for optimizing the wealth process. The same purpose followed in [19], where the author considered a portfolio consisting of one risk-free asset (bond) and one risky asset (stock) and optimized the portfolio by maximizing a utility function of Von Neumann–Morgenstern type. A different study was completed in [17], where the author inserts the factor of investors age into the investment strategies. In another study [6], the authors demonstrated that by discretizing a portfolio model influenced by a Lévy process, optimization problems involving price changes with jumps can be effectively addressed using equivalent jump-free models. In [5], the authors combined a Markov decision process and a Reinforcement Learning method for dynamic portfolio optimization with risk assessment. The same idea was followed by [18], where the authors used a Reinforcement Learning method along with Hamilton-Jacobi-Bellman equations (in brief, HJB equation) to optimize a portfolio process. Artificial intelligence tools have recently attracted researchers' interest for portfolio optimization [11]. While AI-based methods remain largely theoretical, the potential for implementing and evaluating advanced algorithms using these tools is significant.

The optimization of the portfolio process is still among the researchers' concerns, and several methods have been used by researchers for this purpose. Many of these studies are focused on minimizing the investment risk by implementing well-known mathematical tools such as mean square minimization methods, HJB equations, optimal filtering of the over/under-reaction and etc.. The relevant papers are numerously available; and interested readers can refer to works such as [2, 3, 12, 21] and their citations.

In this paper, we will consider an incomplete market and propose a portfolio process consisting of a risk-free asset (bond), a risky asset, and a financial derivative based on the risky asset but its value moves inversely to that of the risky asset. Such a portfolio is called in brief a BAD-portfolio (abbreviated of Bond-Asset-Derivative portfolio). As we will demonstrate, the fact that the values of the asset and the derivative move in opposite directions allows investors to effectively manage financial risk by adjusting their holdings between the bond, the risky asset, and the derivative. The paper is organized as follows. In the next section, we will detail the BAD-portfolio process and calculate the corresponding discounted wealth process. Additionally, we will present the optimization problem as the minimization of a parametric convex utility function that can simultaneously maximize wealth and minimize mean square of it. In Section 3, we will compute the equivalent martingale probabilities and reformulate the discounted wealth process and the optimization problem with respect to these new probabilities. In Section 4, we will calculate the corresponding HJB equation and precisely compute the optimal portfolio process. Notably, a range of processes can optimize the problem (see Theorem 4.1 and (22)), which is advantageous because it enables investors to gain fringe benefits while maintaining their overall investment strategy by adjusting their portfolios accordingly in real-world applications. In Section 5, we will consider an asset value chart and construct a BAD-portfolio process with a European put option as the derivative. We will compute the optimal portfolio process and the corresponding wealth process numerically. Additionally, we will outline a method to calculate the market return rate and a martingale parameter necessary for determining the optimal portfolio process in Section 4.

2. Mathematical Model, Financial Assumptions and Concepts.

Throughout this paper, we assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $b_1(t), b_2(t)$ are two independent standard Brownian motions imposing a filtration $\mathcal{F}_t$, $t \geq 0$. For $0 < T \in \mathbb{R}$ and $n \in \mathbb{N}$, $\mathbb{L}^n[0,T]$ stands for the space of all adapted and progressively measurable processes X_t such that[1]

$$\mathbb{E}\Big(\int_0^T |X_t|^n dt\Big) < \infty.$$

We also assume that there exists an underlying risky asset in a frictionless market whose price at time $0 \leq t \leq T$ is S_t and is modeled by

$$\frac{dS_t}{S_t} = \alpha(t)dt + \langle \sigma(t), db(t) \rangle, \tag{1}$$

where $\sigma(t) = \big(\sigma_1(t), \sigma_2(t)\big)$, $b(t) = \big(b_1(t), b_2(t)\big)$ and $\langle ., . \rangle$ is the inner product of the vectors. In (1), $\alpha(t) \in \mathbb{L}^1[0,T]$ and $0 < \sigma_1(t), \sigma_2(t) \in \mathbb{L}^2[0,T]$ show respectively the expected return rate and the volatility rates of the risky asset. Also, $b_1(t)$ simulates the random movements caused by the asset, and $b_2(t)$ simulates the random movements imposed by unconsidered market parts. While the Black-Scholes model primarily uses b_1 for pricing, secondary factors like related market activity necessitate simulation with additional equations. When these secondary factors are not the primary focus, many authors simplify these effects by adding independent Brownian motions, as we do here with b_2.

We also consider a bond whose value at each time $0 \leq t \leq T$ is B_t and modeled by

$$\frac{dB_t}{B_t} = r(t)dt, \tag{2}$$

[1] $\mathbb{E}(.)$ stands for the mathematical expectation.

where $0 < r(t) \in \mathbb{L}^1[0,T]$ is an adapted process which shows the return rate of the bond.

In what follows, we assume that an agent invests in the market with a portfolio consisting of a_0 shares of the bond, a_1 shares of the risky asset, and a_2 shares of a European derivative with maturity T and linked to the underlying asset S_t with the payoff $\Phi(S_T) \geq 0$, where $\Phi(x)$ is a decreasing convex function. We also assume that he neglects the transaction costs; thus, his wealth at each time $0 \leq t \leq T$ is determined by

$$W_t = a_0(t)B_t + a_1(t)S_t + a_2(t)P_t,$$

where $P_t = P(t, S_t)$ shows the value of the derivative at time $0 \leq t \leq T$. As mentioned, such portfolios are called BAD-portfolios and are widely used to control investment risks in a risky market. Indeed, with an imprecise expression, the value of the derivative and risky asset move in the opposite directions to balance each other to control the investment risk.

Since the agent balances his portfolio at each time $0 \leq t \leq T$ based on the information up to time t, it is reasonable to assume that $a_i(t),\ i = 0, 1, 2$, are adapted processes; and therefore, the wealth process W_t is also an adapted process. For notational convenience, we let $A(t) = \big(a_1(t), a_2(t)\big)$ and $X_t = \big(S_t, P_t\big)$, thus the wealth process above is rewritten in the form:

$$W_t = a_0(t)B_t + \langle A(t), X_t\rangle. \tag{3}$$

Definition 2.1. With the above notations, the BAD-portfolio (or equivalently the wealth process W_t in (3) is called self-financing if the following assumptions hold.

(A1) $a_0(t)r(t)B_t \in \mathbb{L}^1[0,T]$ and $A(t) \in \mathcal{A}$, the set of all adapted processes with values in $\mathbb{R}^2$ such that

1) $a_1(t)\alpha(t),\quad a_1(t)r(t) \in \mathbb{L}^1[0,T],\quad a_1(t)\sigma_j(t) \in \mathbb{L}^2[0,T],\quad (j = 1, 2),$
2) $a_2(t) \in \mathbb{L}^2[0,T]$ with $\sup\limits_{0\leq t\leq T} |a_2(t)| < M,$ almost surely for $\mathbb{P}$.

(A2) The self-financing constraint is satisfied:

$$\begin{aligned} dW_t &= a_0(t)dB_t + a_1(t)dS_t + a_2(t)dP_t \\ &= a_0(t)dB_t + \langle A(t), dX_t\rangle. \end{aligned} \tag{4}$$

The assumption **(A1)** ensures that the integral form of the self-financing constraint **(A2)** is well-defined. The self-financing constraint **(A2)** indicates that the agent manages $W(t)$ by adjusting asset shares at each time $0 \leq t \leq T$, with no external funds required for investment on the time interval $0 \leq t \leq T$. This ability to dynamically allocate the shares allows him to efficiently manage wealth over time, maximizing returns and minimizing risks without the need for external capital injections (see [16] for financial conventions of the self-financing constraint).

In the rest of this paper, for convenience in calculations, we assume $B_0 = 1$ and $W_0 = 1$. This makes our wealth process a numeraire; and the corresponding discounted processes are

$$\widetilde{W}_t = \frac{W_t}{B_t}, \qquad \widetilde{S}_t = \frac{S_t}{B_t}, \qquad \widetilde{P}_t = \frac{P_t}{B_t}, \qquad \widetilde{X}_t = \frac{1}{B_t}X_t = (\widetilde{S}_t, \widetilde{P}_t);$$

and the wealth process (3) becomes

$$\widetilde{W}_t = a_0(t) + \langle A(t), \widetilde{X}_t\rangle. \tag{5}$$

Definition 2.2. With the above notations, a self-financing wealth process W_t is called admissible if
1) $\widetilde{W}_t \geq 0$, $\mathbb{P}$-a.s. for all $t \in [0,T]$,
2) $\mathbb{E}(\widetilde{W}_T) \geq W_0$.
The set of all admissible wealth processes is shown by $\mathcal{W}_+$. A self-financing portfolio process $A(t) = \big(a_1(t), a_2(t)\big) \in \mathcal{A}$ for which $\widetilde{W}_t \in \mathcal{W}_+$ is called admissible. The set of all admissible portfolio processes is shown by $\mathcal{A}_+$.

It must be clear that, for a wealth process W_t and a portfolio process $A(t)$,

$$a_0(t) = \widetilde{W}_t - \langle A(t), \widetilde{X}_t\rangle.$$

As we will see soon, we can somehow omit $a_0(t)$ in the calculations, thus our main concern will be on the portfolio process $A(t) = \big(a_1(t), a_2(t)\big) \in \mathcal{A}_+$. If an agent begins with an initial wealth $W_0 = 1$ and invests just in the bond, i.e. $A(t) = (0,0)$ and $a_0(t) = 1$, then $\widetilde{W}_t = 1$ is admissible; thus $(0,0) \in \mathcal{A}_+$. On the other hand, it is easy to see that, if $A_1, A_2 \in \mathcal{A}_+$ then $\lambda A_1 + (1-\lambda)A_2 \in \mathcal{A}_+$ for all $0 \leq \lambda \leq 1$; thus $\mathcal{A}_+$ is a convex subset of $\mathcal{A}$. Thus we have the following lemma:

Lemma 2.3. *With the above notations, $\mathcal{A}_+$ is a non-empty convex set containing the origin.*

The challenge of finding an optimal portfolio process is a complicated issue that can be formulated in various ways depending on the portfolio structure and the agent's objective. In this paper, based on the wealth process (3), the problem is to find a portfolio process that maximizes the expected wealth at the maturity and minimizes the divergence from this expected wealth at the same time, i.e.

$$\max_{A(t)\in\mathcal{A}_+} \mathbb{E}\big(\widetilde{W}_T\big), \qquad \min_{A(t)\in\mathcal{A}_+} \underbrace{\mathbb{E}(\widetilde{W}_T^2) - \mathbb{E}^2(\widetilde{W}_T)}_{=\mathbb{V}(\widetilde{W}_T)}.$$

Since, in general, there might not exist a portfolio process $A(t)$ satisfying simultaneously both conditions above, we can compromise the optimization problem to the following form:

$$\mathcal{V}(\delta) = \min_{A(t)\in\mathcal{A}_+} \mathbb{E}\Big(\delta\widetilde{W}_T^2 + \frac{1-\delta}{\widetilde{W}_T}\Big), \qquad 0 \leq \delta \leq 1. \tag{6}$$

In the case of (6), for $\delta = 1$, the optimal problem is a mean square minimization problem. Such problems appear in many mathematical finance problems (see for example [4, 7], and references therein). If $\delta = 0$, then Holder‘s inequality implies that

$$\mathbb{E}(\frac{1}{\widetilde{W_T}}) \geq \frac{1}{\mathbb{E}(\widetilde{W_T})}.$$

Thus, the process $A \in \mathcal{A}_+$ which minimizes the left-hand side, relatively maximizes the expected wealth; or roughly speaking, (6) relatively maximizes $\mathbb{E}(\widetilde{W_T})$, for $\delta = 0$. By selecting $0 \leq \delta \leq 1$, we can balance the strategy between mean-variance minimization and wealth maximization. It's important to note that minimizing $\mathbb{E}(\widetilde{W_t^2})$ and maximizing $\mathbb{E}(\widetilde{W_t})$ directly influence variance minimization, which can also be seen as risk minimization. In the next section, we will replace (6) with its alternative form with respect to the martingale probabilities.

3. **Equivalent martingale probabilities**

When we talk about an incomplete market, the equivalent martingale probability measure might not be unique. As we will see in this section, there exists a family of equivalent martingale probability measures which affect the optimal problem (6).

Lemma 3.1. *The discounted wealth process $\widetilde{W_t}$ follows the self-financing constraint*

$$d\widetilde{W_t} = \langle A(t), d\widetilde{X_t} \rangle. \tag{7}$$

Proof. Combining the self-financing constraint **(A2)**, the wealth process (3) and the Itö product rule, we obtain:

$$B_t \, da_0(t) + \langle dA(t), X_t \rangle + \langle dA(t) \, , \, dX_t \rangle = 0,$$

or equivalently

$$da_0(t) + \langle dA(t), \widetilde{X_t} \rangle + \langle dA(t) \, , \, \frac{1}{B} dX_t \rangle = 0. \tag{8}$$

On the other hand,

$$d\widetilde{X}_t = \frac{1}{B_t} dX_t - r(t)\widetilde{X}_t dt \quad \Rightarrow \quad \frac{1}{B_t} dX_t = d\widetilde{X}_t + r(t)\widetilde{X}_t dt.$$

Thus, considering $\langle dA(t) \, , \, r(t)\widetilde{X}(t)dt \rangle = 0$, we obtain:

$$da_0(t) + \langle dA(t), \widetilde{X_t} \rangle + \langle dA(t) \, , \, d\widetilde{X}_t \rangle = 0.$$

Using the last equation, the self-financing constraint (7) is obtained by differentiating both sides of (5). □

Remark 3.2. Considering (5), a consequence of Lemma 3.1 is

$$a_0(t) = 1 + \int_0^t \langle A(u), d\widetilde{X_u}\rangle - \langle A(t)\,,\,\widetilde{X}_t\rangle. \tag{9}$$

That is, as soon as we obtain the portfolio process $A(t)$ in (7), we can obtain $a_0(t)$ by (9).

We next turn to derive equivalent martingale probabilities. For beginning, the market price of risk for the risky asset is given by (see [16, chapter 5]) :

$$\theta_\lambda(t) = \big(\alpha(t) - r(t)\big)\Big(\frac{\lambda}{\sigma_1(t)}\,,\,\frac{1-\lambda}{\sigma_2(t)}\Big), \qquad 0 \le \lambda \le 1.$$

For $0 \le \lambda \le 1$, let $\mathbb{P}_\lambda$ be the equivalent probability measure with $d\mathbb{P}_\lambda/d\mathbb{P} = Z_\lambda(T)$, where

$$Z_\lambda(t) = \exp\Big[-\int_0^t \langle\theta_\lambda(u)\,,\,db(u)\rangle - \frac{1}{2}\int_0^t \|\theta_\lambda(u)\|^2 du\Big]. \tag{10}$$

If we further consider the assumptions:

(A3) $\|\theta_\lambda(t)\|^2 Z(t) \in \mathbb{L}^2[0,T], \qquad 0 \le \lambda \le 1,$

(A4) $\sup_{\lambda\in[0,1]} \mathbb{E}\Big(\exp\Big[\int_0^T \|\theta_\lambda(u)\|^2 du\Big]\Big) < \infty,$

then the Girsanov theorem (see [16, Theorem 5.4.1]) implies that, for $0 \le \lambda \le 1$, $Z_\lambda(t)$ is a martingale with respect to $\mathbb{P}$ and

$$\begin{aligned}\widetilde{b}_{1\lambda}(t) &= \lambda\int_0^t \frac{\alpha(u) - r(u)}{\sigma_1(u)}du + b_1(t),\\ \widetilde{b}_{2\lambda}(t) &= (1-\lambda)\int_0^t \frac{\alpha(u) - r(u)}{\sigma_2(u)}du + b_2(t),\end{aligned}$$

are independent standard Brownian motions with respect to $\mathbb{P}_\lambda$. Furthermore,

$$\frac{d\widetilde{S}_t}{\widetilde{S}_t} = \langle\sigma(t), d\tilde{b}_\lambda(t)\rangle, \qquad \widetilde{b}_\lambda(t) = \big(\widetilde{b}_{1\lambda}(t), \widetilde{b}_{2\lambda}(t)\big) = \int_0^t \theta_\lambda(u)du + b(t), \tag{11}$$

which implies that $\widetilde{S}_t$ is a martingale with respect to $\mathbb{P}_\lambda$, $(0 \le \lambda \le 1)$, and from the no arbitrage option pricing formula, we have

$$P_t = B_t\mathbb{E}_\lambda\Big(\frac{\Phi(S_T)}{B_T}\Big|\mathcal{F}_t\Big),$$

where $\mathbb{E}_\lambda$ denotes the mathematical expectation with respect to $\mathbb{P}_\lambda$.[2]

Lemma 3.3. *With the above notations, the discounted wealth process* $\widetilde{W}_t$ *is a martingale with respect to* $\mathbb{P}_\lambda$.

[2] While some studies minimize the derivative prices by finding the appropriate λ, this is not the focus here for two reasons. First, our aim is portfolio optimization, not derivative pricing. Second, and more importantly, here as we will see later in Eq. (16), the derivative price is independent of λ.

Proof. Let $0 \le s \le t \le T$. We first show that $\widetilde{P}_t$ is a martingale:

$$\begin{aligned} \mathbb{E}_\lambda\Big(\widetilde{P}_t\Big|\mathcal{F}_s\Big) &= \mathbb{E}_\lambda\Big[\mathbb{E}_\lambda\Big(\frac{\Phi(S_T)}{B_T}\Big|\mathcal{F}_t\Big)\Big|\mathcal{F}_s\Big] \\ \text{(using the iterated conditioning rule)} \quad &= \mathbb{E}_\lambda\Big[\frac{\Phi(S_T)}{B_T}\Big|\mathcal{F}_s\Big] = \frac{P_s}{B_s} \\ &= \widetilde{P}_s. \end{aligned}$$

Now, from the martingale representation theorem (see [16, Theorem 5.4.2]), there exists an adapted process $\Gamma(t)$ such that

$$\widetilde{P}_t = \widetilde{P}_0 + \int_0^t \langle \Gamma(u), d\tilde{b}_\lambda(u)\rangle.$$

Substituting the above equality in the self-financing constraint (7), we obtain

$$d\widetilde{W}_t = a_1(t)\widetilde{S}_t\langle \sigma(t), d\tilde{b}_\lambda(t)\rangle + a_2\langle \Gamma(t), d\tilde{b}_\lambda(t)\rangle.$$

This shows that $\widetilde{W}_t$ is an Itö integral with respect to $\widetilde{b}_\lambda(t)$ and therefore is a martingale with respect to $\mathbb{P}_\lambda$. □

Lemma 3.4. *Let $0 \le s \le t \le T$. With the above notations, we have*

$$\mathbb{E}\big(\widetilde{W}_t\big|\mathcal{F}_s\big) = \mathbb{E}_\lambda\big(\frac{\widetilde{W}_t}{Z_\lambda(t)}\Big|\mathcal{F}_s\big), \qquad \mathbb{P} \text{ almost surely.} \tag{12}$$

$$\mathbb{E}\big(\widetilde{W}_t Z_\lambda(t)\big|\mathcal{F}_s\big) = \mathbb{E}_\lambda\big(\widetilde{W}_t\big|\mathcal{F}_s\big), \qquad \mathbb{P} \text{ almost surely.} \tag{13}$$

Proof. We prove (12), the proof of (13) is quite similar. Let $0 \le s \le t \le T$, $U \in \mathcal{F}_s \subseteq \mathcal{F}_t$ and $\mathbb{I}_U$ be the characteristic function of U. As $Z_\lambda(t)$ is a martingale with respect to $\mathbb{P}$, by the iterated conditioning property of the conditional expectation, we have:

$$\begin{aligned} \mathbb{E}_\lambda\big(\frac{\mathbb{I}_U\widetilde{W}_t}{Z_\lambda(t)}\big) &= \mathbb{E}\big(\frac{\mathbb{I}_U\widetilde{W}_t}{Z_\lambda(t)}Z_\lambda(T)\big) = \mathbb{E}\Big(\mathbb{E}\big(\frac{\mathbb{I}_U\widetilde{W}_t}{Z_\lambda(t)}Z_\lambda(T)\Big|\mathcal{F}_t\big)\Big) \\ &= \mathbb{E}\Big(\frac{\mathbb{I}_U\widetilde{W}_t}{Z_\lambda(t)}\mathbb{E}\big(Z_\lambda(T)\Big|\mathcal{F}_t\big)\Big) = \mathbb{E}\big(\mathbb{I}_U\widetilde{W}_t\big). \end{aligned} \tag{14}$$

On the other hand, from the partial averaging property of the conditional expectation we have

$$\begin{aligned} \int_U \mathbb{E}\big(\widetilde{W}_t\big|\mathcal{F}_s\big)d\mathbb{P} &= \int_U \widetilde{W}_t d\mathbb{P} = \mathbb{E}\big(\mathbb{I}_U\widetilde{W}_t\big) \\ (\text{ from (14) }) \quad &= \mathbb{E}_\lambda\big(\frac{\mathbb{I}_U\widetilde{W}_t}{Z_\lambda(t)}\big) = \int_U \frac{\widetilde{W}_t}{Z_\lambda(t)}d\mathbb{P}_\lambda \\ &= \int_U \mathbb{E}_\lambda\big(\frac{\widetilde{W}_t}{Z_\lambda(t)}\Big|\mathcal{F}_s\big)d\mathbb{P}_\lambda. \end{aligned}$$

Since the above equality holds for all $U \in \mathcal{F}_s$, so (12) holds. □

Given the simplicity of calculations provided by martingale property, we rewrite the optimization problem (6) with respect to the martingale probability measures $\mathbb{P}_\lambda$; thus we have:

$$\mathcal{V}(\delta) = \min_{A\in\mathcal{A}_+} \mathbb{E}_\lambda\Big(\frac{\delta \widetilde{W}_T}{Z_\lambda(T)} + \frac{1-\delta}{Z_\lambda(T)\widetilde{W}_T^2}\Big), \qquad 0\le\delta\le 1. \tag{15}$$

In the next section, we will derive a family of optimal portfolio processes by constructing the corresponding HJB equation for the above optimal problem.

4. **Optimum portfolio process**

In this section, we will consider (7) and present enough conditions for the portfolio processes $A(t)\in\mathcal{A}_+$ satisfying (6) (or respectively (15)). To this end, we first consider the optimal problem $\mathcal{V}(\delta)$ in (6) (or equivalently (15)). The Markov property of $\widetilde{S}_t$ and the risk neutral pricing formula

$$\widetilde{P}_t = \frac{1}{B_T}\mathbb{E}_\lambda\Big(\Phi(S_T)|\mathcal{F}_t\Big) = \mathbb{E}_\lambda\Big(\frac{\Phi(B_T\widetilde{S}_T)}{B_T}|\mathcal{F}_t\Big)$$

imply that $\widetilde{P}_t$ is a function of $(t,\widetilde{S}_t)$; hence, the Itö formula implies:

$$d\widetilde{P}_t = \Big(\frac{\partial\widetilde{P}_t}{\partial t} + \frac{1}{2}\|\sigma(t)\|^2\widetilde{S}_t^2\frac{\partial^2\widetilde{P}_t}{\partial\widetilde{S}_t^2}\Big)dt + \frac{\partial\widetilde{P}_t}{\partial\widetilde{S}_t}d\widetilde{S}_t.$$

Substituting this equation into (7), we obtain

$$d\widetilde{W}_t = \Big(a_1(t) + a_2(t)\frac{\partial\widetilde{P}_t}{\partial\widetilde{S}_t}\Big)d\widetilde{S}_t + a_2(t)\Big(\frac{\partial\widetilde{P}_t}{\partial t} + \frac{1}{2}\|\sigma(t)\|^2\widetilde{S}_t^2\frac{\partial^2\widetilde{P}_t}{\partial\widetilde{S}_t^2}\Big)dt.$$

Finally, the Feynman-Kac theorem (see [16, Theorem 6.4.1]) implies that $\widetilde{P}_t = \widetilde{P}(t,x)$ is the solution of the boundary value problem:

$$\frac{\partial\widetilde{P}_t}{\partial t} + \frac{1}{2}\|\sigma(t)\|^2x^2\frac{\partial^2\widetilde{P}_t}{\partial x^2} = 0, \quad \widetilde{P}(T,x) = \frac{\Phi(B(T)x)}{B(T)}. \tag{16}$$

Thus, we have

$$\begin{aligned} d\widetilde{W}_t &= \Big(a_1(t) + a_2(t)\frac{\partial\widetilde{P}}{\partial x}(t,\widetilde{S}_t)\Big)d\widetilde{S}_t \\ &= \Big(a_1(t) + a_2(t)\frac{\partial\widetilde{P}}{\partial x}(t,\widetilde{S}_t)\Big)\widetilde{S}_t\langle\sigma(t), d\tilde{b}_\lambda(t)\rangle. \end{aligned} \tag{17}$$

Considering (10) and (11), we can replace $b(t)$ with $\tilde{b}_\lambda(t)$, and find

$$Z_\lambda(t) = \exp\Big(\frac{1}{2}\int_0^t \|\theta_\lambda(u)\|^2du - \int_0^t\langle\theta_\lambda(u)\,,\, d\tilde{b}_\lambda(u)\rangle\Big).$$

Thus, by Ito's formula,

$$\begin{aligned} dZ_\lambda(t) &= \frac{1}{2}\|\theta_\lambda(t)\|^2 Z_\lambda(t)dt - \langle\theta_\lambda(t), \tilde{b}_\lambda(t)\rangle Z_\lambda(t) \\ &+ \frac{1}{2}\langle\theta_\lambda(t), \tilde{b}_\lambda(t)\rangle\langle\theta_\lambda(t), \tilde{b}_\lambda(t)\rangle Z_\lambda(t) \\ &= Z_\lambda(t)\Big(\|\theta_\lambda(t)\|^2 dt - \langle\theta_\lambda(t), d\widetilde{b}_\lambda(t)\rangle\Big). \end{aligned}$$

Now, we consider the system of equations

$$(18) \qquad \begin{cases} d\widetilde{W}_t = \Big(a_1(t) + a_2(t)\frac{\partial\widetilde{P}}{\partial x}(t, \widetilde{S}_t)\Big)\widetilde{S}_t\langle\sigma(t), d\widetilde{b}_\lambda(t)\rangle, \\ dZ_\lambda(t) = Z_\lambda(t)\Big(\|\theta_\lambda(t)\|^2 dt - \langle\theta_\lambda(t), d\widetilde{b}_\lambda(t)\rangle\Big). \end{cases}$$

By using the verification theorem (see [15, Theorem 3.5.2]) for the optimal problem $V(\delta)$ in (15), the corresponding HJB-equation is obtained as

$$(19) \qquad \begin{aligned} \frac{\partial V}{\partial t} &+ \min_{A\in\mathcal{A}_+}\Big\{\frac{1}{2}\Big(a_1(t) + a_2(t)\frac{\partial\widetilde{P}}{\partial x}(t, \widetilde{S}_t)\Big)^2\|\sigma(t)\|^2\widetilde{S}_t^2\frac{\partial^2 V}{\partial x^2} \\ &- \Big(a_1(t) + a_2(t)\frac{\partial\widetilde{P}}{\partial x}(t, \widetilde{S}_t)\Big)\big(\alpha(t) - r(t)\big)\widetilde{S}_t\, z\frac{\partial^2 V}{\partial x\,\partial z} \\ &+ \frac{1}{2}\|\theta_\lambda(t)\|^2 z^2\frac{\partial^2 V}{\partial z^2} + \|\theta_\lambda(t)\|^2 z\frac{\partial V}{\partial z}\Big\} = 0 \end{aligned}$$

subject to the boundary condition

$$(20) \qquad V(T, x, z) = \frac{\delta x^2}{z} + \frac{1-\delta}{xz}, \qquad (x > 0,\ z > 0).$$

For $t \in [0, T]$ and $x, z > 0$, the minimization in (19) involves a quadratic polynomial in $\Big(a_1(t) + a_2(t)\frac{\partial\widetilde{P}}{\partial x}(t, \widetilde{S}_t)\Big)^2$. Therefore, if $\partial^2 V(t, x, z)/\partial x^2 > 0$, the minimizer is

$$\hat{a}_1(t) + \hat{a}_2(t)\frac{\partial\widetilde{P}}{\partial x}(t, \widetilde{S}_t) = \left(\frac{z\big(\alpha(t) - r(t)\big)}{|\sigma(t)|^2\widetilde{S}_t}\right)\left(\frac{(\partial^2 V)/(\partial x, \partial z)}{(\partial^2 V)/(\partial x^2)}\right).$$

Conversely, if $\partial^2 V(t, x, z)/\partial x^2 < 0$, the minimization problem has no solution, as the minimum is attained when $a_1(t) + a_2(t)(\partial\widetilde{P}/\partial x)(t, \widetilde{S}_t) \to \pm\infty$. On the other hand, from (19), we have

$$\partial^2 V(T, x, z)/\partial x^2 = 2\delta/z + (1-\delta)/(x^3 z) > 0, \qquad (x > 0,\ z > 0),$$

and therefore the minimization problem has a solution at least in the final phase before the expiration time T. If $\big(\partial^2 V(t, x, z)\big)/\big(\partial x^2\big) \le 0$ we put $a_1(t) = a_2(t) = 0$. This means that the agent invests her wealth on the bond to keep $A(t) = \big(a_1(t), a_2(t)\big) \in \mathcal{A}_+$. Substituting the above processes in (19), finally we see that if $V(t,x,z)$ is the solution of the equation

$$
\begin{aligned}
\frac{\partial V}{\partial t} &+ \frac{1}{2}\|\theta_\lambda(t)\|^2 z^2 \frac{\partial^2 V}{\partial z^2} + \|\theta_\lambda(t)\|^2 z \frac{\partial V}{\partial z} \\
&- \frac{1}{2}\left(\frac{z\big(\alpha(t)-r(t)\big)}{\|\sigma(t)\|}\right)^2 \frac{\big(\partial^2 V/\partial x\, z\big)^2}{(\partial^2 V/\partial x^2)} = 0, \\
& V(T,x,z) = \frac{\delta x^2}{z} + \frac{1-\delta}{xz}, \qquad x>0, z>0;
\end{aligned}
\tag{21}
$$

then the portfolio process $\hat{A}_\lambda(t) = \big(\hat{a}_1(t), \hat{a}_2(t)\big),\ \ 0 \le \lambda \le 1$, with

$$
\hat{a}_1(t) + \hat{a}_2(t)\frac{\partial \widetilde{P}}{\partial x}(t, \widetilde{S}_t) = \mathcal{L}(t), \tag{22}
$$

where

$$
\mathcal{L}(t) = \left(\frac{\big(\alpha(t)-r(t)\big)Z_\lambda(t)}{\|\sigma(t)\|^2 \widetilde{S}_t}\right)\left(\frac{(\partial^2 V/\partial x\,\partial z)}{(\partial^2 V/\partial x^2)}(t, \widetilde{W}_t, Z_\lambda(t))\right)
$$

is an optimum portfolio process for (6) (or equivalently (15)). We summarize the above results in the below.

Theorem 4.1. *With the above notations; the optimization problem* (6) *has a family of solutions given by optimum processes* $\hat{A}_\lambda(t),\ \ 0 \le \lambda \le 1$, *given by* (22).

Remark 4.2. It must be noted that, the equation (21) and the optimal portfolio process (22) indicate that to compute $V(t,x,z)$ and its derivatives, we require $\alpha(t)$ and $\sigma(t)$ over the entire interval $[0,T]$. However, in a real application, our market information is limited to the current time $t \in [0,T]$. To address this, we note that in application, we have to approximate these values appropriately (as we will see in Section 5), thus in numerical computation, we can assume that they are constant. This enables us to solve (21) numerically for all $t \in [0,T]$.

Remark 4.3. As shown in (22), the optimal investment processes form a family of random processes. Specifically, if the investment ratio in the risky assets and the derivative follows the (22), then the optimization problem (6) holds. This is advantageous as, in real-world applications, minor adjustments to the main investment strategy can yield incidental benefits for investors.

Remark 4.4. The family of optimum portfolios (22) includes two subsets: one focused exclusively on risky assets (i.e., $a_2(t) \equiv 0$) and the other solely on the derivative (i.e., $a_1(t) = 0$). Consequently, the investor can optimally choose a portfolio without investing in either type of assets. The first subset aligns with earlier studies while the second is less often addressed in recent research.

Section 5 presents application of the above result and its efficiency/deficiency for managing BAD-portfolio processes.

5. Application, numerical optimization and simulation

The existence of a family of optimal portfolio processes means that different investors can achieve the optimal result with different strategies. However, probably the most important deficiency of the family of optimal processes (22) is the presence of $Z_\lambda(t)$ in the structure of these processes. In fact, while $\widetilde{S}_t$ is calculated directly from observing the price of the underlying asset at the moment, $Z_\lambda(t)$ has to be calculated and inserted manually from the second equation of (18). The parameters $\alpha(t)$, $r(t)$ and $\sigma_{1,2}(t)$ also should be approximated appropriately, therefore we assume that they are positive constants, then in the case $\lambda = \sigma_1^2/\|\sigma\|^2$, we have:

$$\theta_\lambda = \frac{\alpha - r}{\|\sigma\|^2}\sigma.$$

In this case, since

$$\begin{aligned} \widetilde{S}_t &= S_0 \exp\left(\sigma.\widetilde{b}_t - \frac{1}{2}\|\sigma\|^2 t\right) \Rightarrow \exp\left(\sigma.\widetilde{b}_t\right) = \exp\left(\frac{1}{2}\|\sigma\|^2 t\right)\frac{\widetilde{S}_t}{S_0}, \\ Z_\lambda(t) &= \exp\left(-\theta_\lambda.\widetilde{b}_t + \frac{1}{2}\|\theta_\lambda\|^2 t\right) = \left(\exp\left(\sigma.\widetilde{b}_t\right)\right)^{\frac{r-\alpha}{\|\sigma\|^2}} \exp\left(\frac{1}{2}\frac{(\alpha - r)^2}{\|\sigma\|^2}t\right), \end{aligned}$$

so we have

$$Z_\lambda(t) = \left(\frac{S_0}{\widetilde{S}_t}\right)^{\frac{\alpha-r}{\|\sigma\|^2}} \exp\left(\frac{t(\alpha - r)^2}{2\|\sigma\|^2} - \frac{t(\alpha - r)}{2}\right). \tag{23}$$

This enables us to compute $Z_\lambda(t)$, and therefore the optimal process $\hat{A}(t)$ in (22), directly by observing $\widetilde{S}_t$ from the market moments.

We consider a bond with a return rate of 20% per year, giving us a daily rate of approximately $r \approx 0.00055$. Additionally, we examine a risky asset, illustrated by V_t its price variation over 45 days in Fig. 1(a). Thus, the corresponding normalized discounted price can be computed as below (see Fig. 1(b)):

$$S_t = V_t/V_0, \qquad \widetilde{S}_t = \frac{S_t}{B_t} = \frac{V_t}{V_0}e^{-0.00055t}.$$

We also assume that the derivative of the portfolio is a European put with maturity $T = 45$ and the strike price $3V_0/2 = 198$; thus the normalized strike price is given by $K = 3/2$; and we have $\Phi(S_T) = (3/2 - S_T)^+$. We also assume the agent invests equally in the risky asset and derivative, that is

$$\hat{a}_1(t) = \hat{a}_2(t) = \frac{\mathcal{L}(t)}{\left(1 + \frac{\partial \widetilde{P}}{\partial x}(t, \widetilde{S}_t)\right)}. \tag{24}$$

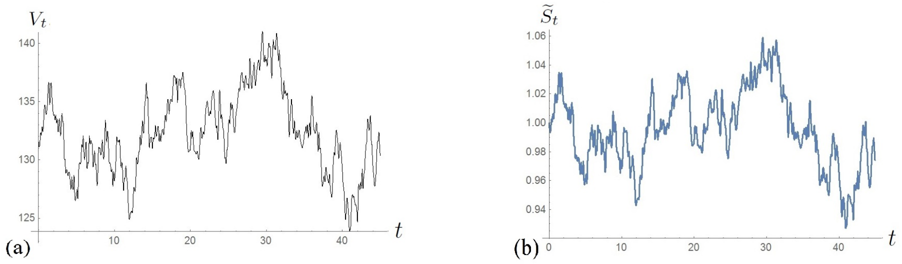

Figure 1. (a): risky asset price V_t, (b): corresponding discounted price $\widetilde{S}_t$.

Considering the price action V_t, we can approximate the volatility rate by (see [16, Sec. 3.4.3] for methods on approximation of the volatility rate)

$$\|\sigma\|^2 \quad \approx \quad \frac{1}{45}\sum_{j=0}^{44} \ln^2\left(\frac{V_{(j+1)}}{V_j}\right) \quad \Rightarrow \quad \|\sigma\| \approx 0.0228114.$$

Computation of the return rate of the market, i.e. α, is a more challenging problem. Let, $n > 0$ be an integer, $h = T/n$ and define

$$Y_n = \frac{1}{T}\sum_{j=0}^{n} \ln\left(\frac{V_{(j+1)h}}{V_{jh}}\right).$$

The central limit theorem, implies that as $n \to +\infty$, the distribution of

$$\frac{Y_n - (\alpha - \|\sigma\|^2/2)}{\|\sigma\|/\sqrt{T}}$$

tends to a standard normal distribution. Thus

$$\lim_{n\to+\infty} \mathbb{P}\left(|Y_n - \alpha + \|\sigma\|^2/2| < \epsilon\right) = \sqrt{\frac{2}{\pi}}\int_0^{\epsilon\sqrt{T}/\sigma} e^{-y^2/2}dy.$$

Here, for $n = 4500$, we have

$$\frac{\|\sigma\|^2}{2} + Y_n \approx 0.00026018 + \underbrace{\frac{1}{45}\sum_{j=0}^{4499} \ln\left(\frac{V_{(j+1)/100}}{V_{j/100}}\right)}_{\approx 0.00225325} \approx 0.00251343.$$

Thus, for $\epsilon = 0.01$

$$\mathbb{P}\left(|\alpha - 0.00251343| < 0.01\right) \approx \sqrt{\frac{2}{\pi}}\int_0^{\frac{0.01\sqrt{45}}{0.023}} e^{-y^2/2}dy \approx 0.996726.$$

This means that, we can appropriately assume that $0.0075 \leq \alpha \leq 0.0125$. In what follows, we will assume that α takes almost the mean value, i.e. $\alpha = 0.01$. This allows us, for a given $0 \leq \delta \leq 1$, approximate numerically $Z_\lambda(t)$ in (23) and also $V(t,x,z)$ in (21), and the optimal portfolio process $\hat{A}(t)$ in (24). Fig.2(a), ..., Fig.2(f) shows the discounted wealth process for respectively $\delta = 0$, ..., $\delta = 1$. Increasing the value of δ leads to a rise in the discounted wealth process,

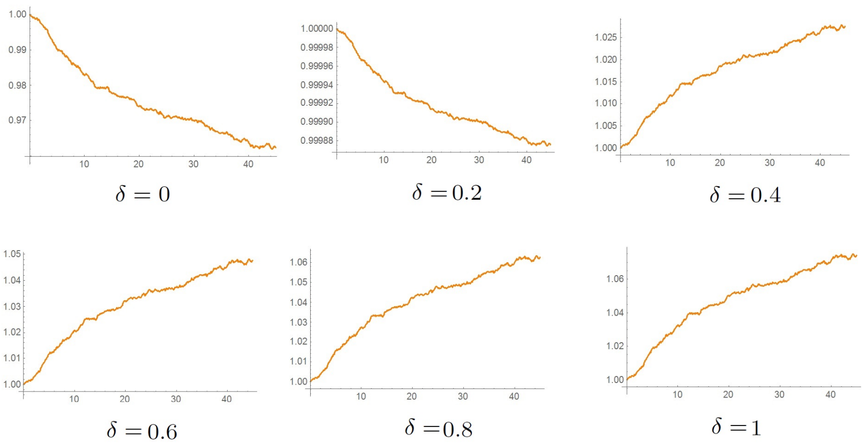

FIGURE 2. The discounted wealth processes for $\delta = 0, 0.2, 0.4, 0.6, 0.8, 1$.

which aligns with our expectation. From equation (6), we observe that raising δ from 0 to 1 shifts the optimization problem from mean-variance minimization to expectation maximization.

6. Conclusion

We analyzed a wealth process based on a BAD-portfolio and formulated the optimal problem (6), which seeks to maximize expected wealth at time $T > 0$ while minimizing its mean square depending on the parameter $0 \leq \delta \leq 1$. We calculated risk-neutral probabilities and demonstrated that the discounted wealth process is devoid of a drift term (see (17)) with respect to these probabilities. This allowed us to compute the minimizing portfolio processes (22) corresponding to the HJB-equation (19). As we mentioned before, these portfolios provide a family of optimal process allowing the agent the advantage of choosing and balancing between the asset and the derivatives at each time before T. Another advantage of the calculation is that the optimal portfolio processes can be precisely linked to the asset price by appropriately choosing the risk-neutral probability (Section 5, where we selected $\lambda = \sigma_1^2/\|\sigma\|^2$ and computed Z_λ with respect to $\widetilde{S}_t$). This is crucial; without it, the results were merely theoretical. Now, we can apply them in real-world scenarios, as demonstrated in Section 5.

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