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A NEW APPROACH TO THE LIFE CYCLE OF MEDITERRANEAN FRUIT FLY WITH A VIEW TO ESTIMATING SOME PARAMETERS

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Abstract. Ceratitis capitate, commonly known as the Mediterranean fruit fly, is one of the most destructive fruit pests in the world. The Mediterranean fruit fly is the most economically important fruit fly species because it causes severe destruction. In this paper, a dynamical model based on a delay differential equation is introduced that is described the life cycle of the insect. As follows, Hopf bifurcation is checked by changing the time delays.

In the next issue is estimated unknown parameters. It is proposed a method, based on the least square approach and a finite set of observation, to estimate the parameters which are identifiable. Finally, using numerical simulation, the stability at endemic equilibrium point is investigated.

Keywords: Delay differential system, Mediterranean fruit fly, Hopf bifurcation, Asymptotically stable, Parameter estimation.
2020 MSC: 92D30, 34D05, 34C23, 93E24.

1. Introduction

Ceratitis capitate, commonly known as the Mediterranean fruit fly (Medfly), is an exotic pest originating from sub-Saharan Africa. It is recognized as one of the most destructive agricultural pests, attacking a wide variety of plants. The Medfly can infest over 260 different plant species, including fruits, vegetables, nuts, and flowers [20]. Particularly susceptible hosts include apples, cherries, pears, peaches, pomegranates, figs, apricots, citrus fruits, tomatoes, and peppers—especially those with thin skins. The preferred hosts can vary by region [3].

Although native to African tropical regions, Medfly has spread to many parts of the world, including Western Australia, the Mediterranean region, the Middle East, South and Central America, Hawaii, and southern Europe. Despite having no close relatives in these areas, Medfly has occasionally infested regions such as California, Texas, and Florida over the last century [5]. Their

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adaptability has been a key factor in their global proliferation [6]. In Iran, Medfly was quarantined until 1975 when the first larva was detected in a peach garden in Mashhad. Although it was temporarily eradicated by cold weather and quarantine measures in 1978, Medfly reappeared in 1983, targeting not only peach gardens but also tangerine, orange, and persimmon orchards [21].

The Medfly life cycle consists of four stages: egg, larva, pupa, and adult. Under optimal conditions, this cycle can be completed in approximately 21 days, but in colder climates, it can extend to nearly 100 days. Each female Medfly lays about 22 eggs beneath the skin of a susceptible fruit. The eggs hatch in 2-4 days during summer or 19-20 days during winter, after which the larvae emerge and feed on the fruit pulp. The larval stage lasts about 5-14 days in summer and 25-45 days in winter. Fully grown larvae then move to the soil to pupate, with the pupal stage lasting 6-14 days in summer and 25-50 days in winter. Finally, adult Medflies emerge from the soil after 8-46 days and begin laying eggs within 2-3 days. The primary agricultural damage caused by Medflies includes oviposition in fruit and plant tissues, larval feeding that opens entry points for bacteria and fungi, and the subsequent decay of plant tissue due to secondary microbial invasions [15].

Several control methods for Medfly have been proposed, including Aerial and Ground Bait Spray Application, Sterile Insect Technique (SIT), Insecticide Application to Soil Under Host Trees and trapping. The economic impact of Medfly, due to quarantine and monitoring programs, is significant. Thus, not only have governmental measures been implemented to control Medfly, such as strict regulations to prevent its entry via travelers or imported goods, but scientific research is also crucial for understanding the Medfly's life cycle and spread. Accurate modeling of these phenomena is essential for developing effective control protocols.

Mathematical models have been widely used in biological studies to provide accurate predictions of phenomena crucial to fields such as agriculture, ecology, and medicine [1, 12, 22]. Many such models focus on analyzing a "target population," which has helped researchers effectively address biological problems. In the context of Medfly populations, numerous studies have applied mathematical modeling to yield valuable insights into mortality rates, aging processes, reproductive behavior, and population dynamics. For example, a 1995 study modeled the mortality rates of 600,000 Medflies across 167 cohorts [3], while subsequent research in 2000 highlighted differences in mortality trajectories between virgin and mixed-sex female Medflies, linking these patterns to physiological aging [16]. A 2006 review further discussed key aspects of modeling biological systems, particularly models relevant to limited lifespan, aging, and death in Medflies [17]. Additionally, studies have modeled egg-laying behavior [8], explored the relationship between fecundity and longevity across species [16], and applied statistical methods to understand Medfly dispersion and age structure in field populations [5, 14]. These studies have collectively advanced the understanding of Medfly biology, offering critical data for developing effective control strategies.

Given the presence of time delays in the Medfly life cycle—specifically in the transitions from egg to larva, larva to pupa, and pupa to adult—a differential system with time delays is appropriate for modeling these processes. This paper introduces a dynamical model based on the Medfly life cycle and the quantities of susceptible and infected fruits. The model's parameters include both scientifically meaningful and estimable ones, with the estimation being critical for dynamical models. This paper employs the method introduced in [2] to estimate the parameters, which not only allows for estimation but also investigates their identifiability.

In 2008, a new Medfly control technique, the Spatial Decision Support System (SDSS), was introduced. Known as MedCila, this method consider various models—binary, linear, logarithmic, and biological-based—to include all recognized factors in its approach. MedCila's recommendations were nearly always convincing for coordinators, leading to a reduction in decision-making efforts regarding plot management or spraying when no Medfly threat was present [9].

In 2010, a dynamic model and two type models were compared and applied to the 1992 Medfly dataset [13], showing that these models were particularly suitable for analyzing Medfly mortality data [19]. A 2012 study introduced a statistical regression method based on time-to-death data, providing estimates of average age changes in Medfly populations. This technique complemented conventional approaches, especially those using mark-recapture survival analysis like the one in [8], and applied to age structure studies employing advanced strategies [6].

Further developments include a 2018 study that discussed the concept of identity and its applications to Medfly populations, yielding good estimates for age structure and mean age in wild populations [7]. In 2019, a life-table invasion model for Medflies was proposed, featuring a progression model and a partitioning model of invasions. This model helped formulate invasion dynamics and predict related spatiotemporal progression [23]. Recently, a game theory-based framework was introduced to analyze the issue of infertile cases selecting wild mates in a population, validated using Medfly data. This led to the creation of a formula for controlling wild Medflies [10].

The structure of this paper is as follows: Section 2 introduces a mathematical model as a differential system with time delays. In Sections 3 and 4, equilibrium points are obtained, and the dynamical behavior at the disease-free equilibrium point is investigated. Section 4 also covers the estimation of unknown parameters. Finally, using the estimated parameters, the stability of the endemic equilibrium point is examined, along with the effects of glue and plowing on this stability.

2. A time delay model for Medfly and its equilibrium

In the Medfly life cycle, the transitions from egg to larva, larva to pupa, and pupa to adult involve two time delays. The model assumes that the durations of the egg-to-larva and larva-to-pupa transitions are equal, denoted as the time delay τ_1. The second time delay, τ_2, represents the maturation period. Additionally, the quantity of garden fruits available to Medflies affects their reproduction rate, so the model also considers the amounts of healthy and infected fruits.

2.1. A time delay model for Medfly life span and fruit infection dynamics.

Consider a garden of fruits where a population of Medflies is established for the first time. Let $E(t)$, $L(t)$, $P(t)$, and $M(t)$ denote the number of eggs, larvae, pupae, and mature Medflies at time t, respectively. Assume that the total number of fruits, which are prime targets of the Medflies' attack, is K, which is constant. Let $S(t)$ and $I(t)$ represent the quantities of susceptible fruits and infected fruits at time t, respectively.

At a given time t, the number of mature Medflies positively affects the production of eggs at a rate of γ. Fertilized eggs develop into larvae at a rate of β. The larvae then transform into pupae at a rate of α, with this transformation influenced by the amount of susceptible and infected fruits. Pupae develop into mature Medflies at a rate of μ_P. The mature Medfly population decreases due to natural death and population control. The natural death rate is μ_M, and the population control rate by glue is λ_1. Plowing, which destroys pupae and disrupts the life cycle, is another control method with a rate of γ_1.

Overpopulation can lead to natural population reduction due to factors such as emigration, starvation, and reduced birth rates. There are time delays in each process: egg production by mature flies, larval development, pupal formation, and Medfly emergence. Since these delays are approximately equal, they are collectively denoted by τ_1. Additionally, τ_2 represents the puberty duration. Note that γ, β, α, μ_P, μ_M, γ_1 and λ_1 are all non-negative parameters.

The growth of the susceptible fruit population is modeled by a logistic function with a growth rate λ_2. At time t, the rate of harvesting susceptible fruits is μ_S, and the rate of infection of susceptible fruits by Medflies is ν. Infected fruits are removed at a rate of w due to damage caused by Medflies. The rate of harvesting damaged fruits is μ_I. The parameters λ_2, ν, μ_S, w and μ_I are also non-negative.

Figure 1 illustrates the schematic diagram of both the Medfly life cycle (upper boxes) and the time evolution of fruits (lower boxes). In this diagram, black arrows represent population transfers between boxes, blue arrows indicate the effects of susceptible and infected fruits on Medfly attacks, and brown arrows depict the negative impact of the larval population on susceptible and infected fruits. The purple arrow represents the effect of the mature Medfly

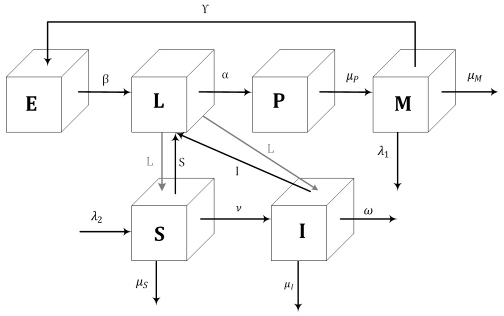

Figure 1. Schematic diagram illustrating the life span of Medflies (upper boxes) and the time evolution of fruits (lower boxes).

population on the number of eggs. The mathematical model is:

$$
(1) \qquad \begin{cases}
\dfrac{dE}{dt} = \gamma M(t) - \beta E(t-\tau_1) \\
\dfrac{dL}{dt} = \beta E(t-\tau_1) - \alpha L(t-\tau_1)(S(t)+I(t)) \\
\dfrac{dP}{dt} = \alpha L(t-\tau_1)(S(t)+I(t)) - \mu_P P(t-\tau_1) - \gamma_1 P(t) \\
\dfrac{dM}{dt} = \mu_P P(t-\tau_1) - \mu_M M(t-\tau_2) - \lambda_1 M(t) \\
\dfrac{dS}{dt} = \lambda_2 S(t)\left(1 - \dfrac{S(t)}{K}\right) - \nu S(t)L(t) - \mu_S S(t) \\
\dfrac{dI}{dt} = \nu S(t)L(t) - wL(t)I(t) - \mu_I I(t)
\end{cases},
$$

2.2. **Equilibrium states.** To determine the equilibrium states, consider the system at equilibrium where

$$\left(\frac{dE}{dt}, \frac{dL}{dt}, \frac{dP}{dt}, \frac{dM}{dt}, \frac{dS}{dt}, \frac{dI}{dt}\right) = \mathbf{0}.$$

It is evident that $G_0 = (0,0,0,0,0,0)$ represents a trivial equilibrium state with no significant dynamics. Similarly, $S = 0$ leads to a trivial equilibrium state.

Assume $S \neq 0$. At equilibrium, we have:

$$(2) \qquad M = \frac{\beta}{\gamma} E.$$

$$E = \frac{\alpha}{\beta} L(S + I). \tag{3}$$

$$\alpha L(S + I) = (\mu_P + \gamma_1)P. \tag{4}$$

If $P = 0$, from Eqs. (2) and (4) we find $M = 0$ and $L = 0$, respectively. Substituting $L = 0$ into Eqs. (2) and (3) lead to $E = 0$ and $I = 0$. Therefore, the equilibrium state is given by:

$$G_1 = \left(0, 0, 0, 0, \frac{K(\lambda_2 - \mu_S)}{\lambda_2}, 0\right), \text{ where } \mu_S < \lambda_2.$$

If $P \neq 0$, substituting Eqs. (3) and (4) into Eq. (2) yields

$$M = \frac{\mu_P}{\mu_M + \lambda_1} P. \tag{5}$$

Using Eq. (5) and $P > 0$, the condition $\frac{dM}{dt} = 0$ is satisfied if:

$$\frac{\gamma \mu_P}{(\mu_P + \gamma_1)(\mu_M + \lambda_1)} = 1, \tag{6}$$

which implies that the rate of egg laying exceeds the combined rate of death of adult insects and glue application. If $\lambda_2 > \mu_S$ and $L < \frac{\lambda_2 - \mu_S}{\nu}$, an endemic equilibrium point exists. Thus, from $\dfrac{dS}{dt} = 0$, we obtain:

$$S_* = \frac{K(\lambda_2 - \nu L_* - \mu_S)}{\lambda_2}. \tag{7}$$

Finally, $\dfrac{dI}{dt} = 0$ results in:

$$I_* = \frac{\nu L_* S_*}{\omega L_* + \mu_I}. \tag{8}$$

2.3. **Boundedness, existence and uniqueness solutions of system.** The boundedness property of the system is examined.

Theorem 2.1. *The solutions of the system are uniformly bounded.*

Proof. Consider the new variable

$$X(t) = E(t) + L(t) + P(t) + M(t) + S(t) + I(t).$$

We have

$$\frac{dX}{dt} = \frac{dE}{dt} + \frac{dL}{dt} + \frac{dP}{dt} + \frac{dM}{dt} + \frac{dS}{dt} + \frac{dI}{dt}.$$

The carrying capicity of the population of fruits is K. Hence, $S(t) \leq K$ for all t. also, there is $K' > 0$, $M(t) \leq K'$. Because of lack of facilities and food, the adult population does not grow uncontrollably. Rather, according to biological data, they begin to migrate when they reach a number close to K'. Put $\Lambda = min\{\gamma, \mu_M, \lambda_1, \mu_S, \mu_I, \omega\}$ and $K'' = max\{K, K'\}$. Thus, $\gamma M + \Lambda E + \lambda_2 S \leq \Lambda' K''$, where, $\Lambda' = \gamma + \Lambda + \lambda_2$ and

$$\frac{dX}{dt} \leq \gamma M + \lambda E - \Lambda X \leq \Lambda' K" - \Lambda X$$

. By solving the above linear differential equation in terms of variable X one gets,

$$X(t) \leq \frac{\Lambda' K''}{\Lambda} + (X_0 - \frac{\Lambda' K''}{\Lambda}) e^{-\Lambda t}.$$

Now, by tending $t \longrightarrow \infty$, we obtain $0 \leq X \leq \frac{\Lambda' K''}{\Lambda}$. Hence, all solutions of the system are in the region

$$\Omega := \{X \in \mathbb{R}^6 \quad : \quad 0 \leq X \leq \frac{\Lambda' K''}{\Lambda} + \epsilon\}$$

for any $\epsilon > 0$ as $t \longrightarrow \infty$. □

Theorem 2.2. *Consider the model 1 subjected to the following initial conditions*

$$E(0) = E_0, L(0) = L_0, P(0) = P_0, M(0) = M_0, S(0) = S_0, I(0) = I_0.$$

The model solutions exist and are unique.

Proof. Let
$f(E, L, P, M, S, I) = (\gamma M(t) - \beta E(t - \tau_1), \beta E(t - \tau_1) - \alpha L(t - \tau_1)(S(t) + I(t)), \alpha L(t-\tau_1)(S(t)+I(t)) - \mu_P P(t-\tau_1) - \gamma_1 P(t), \mu_P P(t-\tau_1) - \mu_M M(t-\tau_2) - \lambda_1 M(t), \lambda_2 S(t) \left(1 - \dfrac{S(t)}{K}\right) - \nu S(t)L(t) - \mu_S S(t), \nu S(t)L(t) - wL(t)I(t) - \mu_I I(t),$
f is a continuous function on $\mathbb{R}^6$. Using [1], for any initial value, $\sigma = (E - 0, L_0, P_0, M_0, S_0, I_0)$, there is a solution of the model passing through σ. To prove the uniqueness of the solution with initial condition σ, we should prove, the model is satisfied in Lipschitz condition.
Consider σ in a compact set Γ of $\mathbb{R}^6$ and $\sigma_1 = (E_1, L_1, P_1, M_1, S_1, I_1)$ and $\sigma_2 = (E_2, L_2, P_2, M_2, S_2, I_2)$ are in Γ.
Put $Y = (\gamma(M_1 - M_2) - \beta(E_1 - E_2))^2 + (\beta(E_1 - E_2) - \alpha(L_1(S_1 + I_1) - L_2(S_2 + I_2))^2 + (\alpha(L_1(S_1 + I_1) - L_2(S_2 + I_2)) - (\mu_P + \gamma_1)(P_1 - P_2))^2$
$+(\mu_P(P_1 - P_2) - (\mu_M + \lambda_1)(M_1 - M_2))^2 + (\lambda_2(S_1(1 - \frac{S_1}{K}) - S_2(1 - \frac{S_2}{K})) - \nu(S_1 L_1 - S_2 L_2) - \mu_S(S_1 - S_2))^2 + (\nu(S_1 L_1 - S_2 L_2) - \omega(L_1 I_1 - L_2 I_2) - \mu_I(I_1 - I_2))^2$.
Since, K is carrying capaicity of fruits, we have
$\parallel f(E_1, L_1, P_1, M_1, S_1, I_1) - f(E_2, L_2, P_2, M_2, S_2, I_2) \parallel$
$= \sqrt{Y} \leqslant \sqrt{Y_1}$
$\leqslant \sqrt{K'}((E_1 - E_2)^2 + (L_1 - L_2)^2 + (P_1 - P_2)^2 + (M_1 - M_2)^2 + (S_1 - S_2)^2 + (I_1 - I_2)^2)^{\frac{1}{2}}$
$\leqslant \sqrt{K'} \parallel \sigma_1 - \sigma_2 \parallel$.
where,
$(\gamma^2 + (\mu_M + \lambda_1)^2)(M_1 - M_2)^2 + 2\beta^2(E_1 - E_2)^2 + (8\alpha^2 + 2\nu^2 + \omega^2)K^2(L_1 - L_2)^2$
$+ (\mu_p^2 + (\mu_p + \gamma_1)^2)(P_1 - P_2)^2 + (\lambda_2^2 + \mu_S^2)(S_1 - S_2)^2 + \mu_I^2(I_1 - I_2)^2$,
and

$K' = max\{\gamma^2 + (\mu_M + \lambda_1)^2, 2\beta^2, (8\alpha^2 + 2\nu^2 + \omega^2)K^2, \mu_p^2 + (\mu_p + \gamma_1)^2, \lambda_2^2 + \mu_S^2, \mu_I^2\}$. □

3. Stability analysis of the steady state

In this section, stability at the disease free equilibrium point is investigated. Linearising the model near disease free equilibrium point

$$G_1 = (E^*, L^*, P^*, M^*, S^*, I^*) = \left(0, 0, 0, 0, \frac{K(\lambda_2 - \mu_S)}{\lambda_2}, 0\right),$$

The characteristic equation of system is determinant of the following matrix, J.

$$J = \begin{bmatrix} a_1 & 0 & 0 & \gamma & 0 & 0 \\ a_2 & a_3 & 0 & 0 & a_4 & a_4 \\ 0 & a_5 & a_6 & 0 & -a_4 & -a_4 \\ 0 & 0 & a_7 & a_8 & 0 & 0 \\ 0 & a_9 & 0 & 0 & a_{10} & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & a_{13} & 0 \end{bmatrix}. \tag{9}$$

Where,
$a_1 = -\lambda - \beta e^{-\tau_1\lambda}$, $a_2 = \beta e^{-\tau_1\lambda}$, $a_3 = -\lambda - \alpha(S + I)e^{-\tau_1\lambda}$, $a_4 = -\alpha L$, $a_5 = \alpha(S + I)e^{-\tau_1\lambda}$, $a_6 = -\gamma_1 - \lambda_1 - \mu_P e^{-\tau_1\lambda}$, $a_7 = \mu_P e^{-\tau_1\lambda}$, $a_8 = -\lambda_1 - \lambda - \mu_M e^{-\tau_2\lambda}$, $a_9 = -\nu S$, $a_{10} = H - \lambda$, $a_{11} = \nu S - \omega I$, $a_{12} = \nu L$, $a_{13} = -\omega L - \mu_I - \lambda$.
The characteristic equation at the equilibrium point G_1 is

$$(H - \lambda)(\mu_I + \lambda)P(\lambda) = 0, \tag{10}$$

where

$$\begin{aligned} P(\lambda) =& (\lambda + \beta e^{-\tau_1\lambda})(\lambda + \lambda_1 + \mu_M e^{-\tau_2\lambda})(\lambda + \alpha S^* e^{-\tau_1\lambda})(\gamma_1 + \lambda_1 + \mu_P e^{-\tau_1\lambda}) \\ & - \alpha\beta\gamma\mu_P S^* e^{-3\tau_1\lambda}. \end{aligned} \tag{11}$$

Two of roots are $\lambda = H = \lambda_1(1 - \frac{2S^*}{K}) - \mu_s$ and $\lambda = -\mu_I$, which are both negative. In the following, we will investigate the rest of the roots.

If both time delays be zero, the rest of the roots of characteristic equation are obtained from the following polynomial

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{12}$$

where

$$\begin{aligned} a_1 &= \alpha S^* + \beta + \lambda_1 + \mu_M, \\ a_2 &= (\alpha S^* + \beta)(\lambda_1 + \mu_M) + \alpha\beta S^*, \\ a_3 &= \alpha\beta S^*(\lambda_1 + \mu_M) - \frac{\alpha\beta S^* \mu_P \gamma}{\gamma_1 + \lambda_1 + \mu_P}. \end{aligned}$$

Theorem 3.1. *Suppose that the rate of gluing is more than the rate of the medfly egg laying,* $\lambda_1 > \gamma$*, then the disease free equilibrium point is asymptotically stable.*

Proof. Using the above relationships and the assumption of the theorem $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$. Thus, using the Ruth- Hurwitz theorem, the disease free equilibrium point is asymptotically stable. ☐

As follows, the stability of the disease free equilibrium point is investigated by changing the time delay, τ_2. If $\tau_1 = 0$ and $\tau_2 > 0$ the characteristic equation is

$$(H - \lambda)(\mu_I + \lambda)[P_0(\lambda) + P_1(\lambda)e^{-\tau_2 \lambda}] = 0.$$

Where,
$P_0(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$,
and
$P_1(\lambda) = \mu_M(\lambda^2 + b_1\lambda + b_0)$.
Which,
$a_0 = \alpha\beta S^*(\lambda_1 - \frac{\gamma\mu_P}{\gamma_1+\lambda_1+\mu_P})$,
$a_1 = \alpha\beta S^* + \beta\lambda_1 + \alpha S^*\lambda_1$
$a_2 = \beta + \alpha S^* + \lambda_1$,
$b_0 = \alpha\beta S^*$,
$b_1 = \alpha S^* + \beta$.
If $\lambda_1 > \gamma$, all coefficients are positive. Hence, the equation

$$P_0(\lambda) + P_1(\lambda)e^{-\tau_2\lambda} = 0 \tag{13}$$

does not have nonnegative real roots.
As follows, if the root of the characteristic equation (13), λ is a complex number. The following theorem is established.

Theorem 3.2. *Consider* $\lambda_1 > \mu_M$*, then* $sign\{\frac{d(Re\lambda)}{d\tau_2}\}_{\lambda=i\theta} > 0$.

Proof. The proof of the Theorem is given in the appendix A. ☐

By theorem 3.2, τ_2^* is the value of τ_2 for which the roots of characteristic equation 13 cross the imaginary axis. Let $\lambda(\tau_2) = \alpha(\tau_2) + i\theta(\tau_2)$ be the root of 13 at $\tau_2 = \tau_2^*$ that satisfying $\alpha(\tau_2^*) = 0$ and $\theta(\tau_2^*) = \theta)$. Since, $sign\{\frac{d(Re\lambda)}{d\tau_2}\}_{\lambda=i\theta} > 0$, $\alpha(\tau_2) = Re(\lambda)$ is increasing function near τ_2^*. Hence, $\alpha(\tau_2) < 0$ for $0 < \tau_2 < \tau_2^*$ and $\alpha(\tau_2) > 0$ for $\tau_2 > \tau_2^*$. Therefore, the disease free equilibrium point is asymptotically stable for $0 < \tau_2 \leq \tau_2^*$ and is unstable for $\tau_2 > \tau_2^*$. Therefore, Hopf bifurcation is ocurred at a critical value of the time delay $\tau_2 = \tau_2^*$.
According to the above results, if the lifespan of an adult insect is less than τ_2^*, the disease free equilibrium point is asymptotically stable. Therefore, agricultural products are not affected by Medflies and remain healthy. Otherwise, an epidemic will occur and the agriculture product will be seriously damaged.

If $\tau_2 = 0$ and $\tau_1 > 0$, the characteristic equation is

$$\Delta(\tau_1, \lambda) = P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_1} + P_2(\lambda)e^{-2\lambda\tau_1} + P_3(\lambda)e^{-3\lambda\tau_1} = 0. \tag{14}$$

Where
$P_0(\lambda) = \lambda^2(\gamma_1 + \lambda_1)(\lambda^2 + (2\lambda_1 + \mu_p)\lambda + \lambda_1(\lambda_1 + \mu_P))$,
$P_1(\lambda) = (\lambda + \lambda_1 + \mu_P)(\beta\lambda(\lambda_1 + \gamma_1) + \alpha\lambda S^*(\lambda_1 + \gamma_1) + \mu_P\lambda^2)$,
$P_2(\lambda) = \mu_P\lambda^3 + \lambda^2(\mu_P(\lambda_1 + \mu_P) + (\lambda_1 + \gamma_1)(\alpha S^* + \beta)) + \lambda(\lambda_1 + \mu_P)(\lambda_1 + \gamma_1)(\alpha S^* + \beta)$,
and
$P_3(\lambda) = \alpha\beta S^*\mu_P(\lambda + \lambda_1 + \mu_P - \gamma)$.
Obviously, $P_i(\lambda) > 0$ for $i = 0, 1, 2$. If $\lambda_1 > \gamma$, $P_3(\lambda) > 0$. Therefore, all real roots of the above equation are negative.
The theorem that defines the stability conditions is given in Appendix A. To investigate of stability, we use of iterative procedure [18].
By Theorem 6.1 that is given in Appendix A, the disease free equilibrium point is unstable for $0 < \tau_1 \leq \tau_1^*$ and is stable for $\tau_1 > \tau_1^*$. Therefore, Hopf bifurcation is accrued a critical value of the time delay $\tau_1 = \tau_1^*$. Hence, if the duration of larva- to- pupa transmission, τ_1, is less than τ_1^*, the disease free equilibrium point is unstable. Thus, the epidemic is happened. If $\tau_1 > \tau_1^*$, the disease free equilibrium point is stable. Hence, the agricultural product remains healthy.

4. Parameter estimation

Since it is difficult to check the dynamic behavior of the model without specifying the parameters. The values of all parameters except γ_1 and λ_1 are obtained using the biological information of the Medfly [11], [4] and [18]. They are given in the table1. In this section, we estimate the values of the parameters γ_1 and λ_1. These are control parameters. Since Medfly is one of the pests that cause great damage to agricultural products, spatially garden crops. Control factors are needed to reduce damage. There are two effective methods, gluing and plowing. The value of these parameters is not fixed. Hence, using data obtained from a garden, they are approximated. These parameters are estimated according to the statistical data obtained from a peach garden in Kerman province. The yield of each peach tree is 50 to 70 kilogram. An average of 65 tons of fruit is harvested per hectare, so $K = 65$.
As follows, parameters λ_1 and γ_1 are estimated in the framework of control theory that is introduced in [2].

Based on the data obtained during 13 weeks, the values of the unknown parameters have been estimated.
According to the values of the parameters in the Table1, using the explanation provided in the Appendix B, the approximations of the parameter vector $p = (\gamma_1, \lambda_1)$ is given in Table2. Therefore, the value of the parameters $\gamma_1 = 0.1$ and $\lambda_1 = 0.09$ is approximated.

Table 1. The values of parameters

Parameter	Value	Definitions
γ	22	The number of eggs laid by an adult insect in each oviposition
α	0.01	The rate of larva that change to pupa
ω	0.01	The rate of damage hurt fruits by larva
μ_M	0.12	The mortality rate of adult insect
β	0.8	The rate of eggs that change to larva
μ_P	0.69	The rate of pupa that change to adult insect
μ_S	0.5	The rate of harvesting healthy fruits
μ_I	0.5	The rate of harvesting damage fruits
ν	0.01	The rate of damage healthy fruits by larva
λ_2	0.146	The growth rate of healthy fruits

Table 2. The values of parameters are obtained from biological information

pupulation	$E(k)$	$L(k)$	$P(k)$	$M(k)$	$S(k)$	$I(k)$
$k=0$	0	0	0	150	120	0
$k=1$	960	0	0	150	170	0
$k=2$	1781	679	0	120	250	0
$k=3$	1766	1264	450	90	290	80
$k=4$	959	1300	846	265	300	92
$k=5$	1310	700	820	501	240	66
$k=6$	2700	706	401	610	227	65
$k=7$	2851	717	388	700	108	51
$k=8$	1750	1960	469	475	50	33
$k=9$	700	908	520	301	24	12
$k=10$	300	410	443	209	10	4
$k=11$	171	122	178	100	6	1
$k=12$	0	50	53	70	2	0
$k=13$	0	0	0	55	0	0

Table 3. Data obtained during 13 weeks from a peach garden in Kerman province

Also, using of data in Table 3 and Matlab's curve fitting, the equation of γ_1 is $\gamma_1(t) = aexp(bt)$ where $a = 34.0751$ and $b = -0.8463$ with confidence interval $a = (18.7641, 49.3861)$, $b = (-1.1833, -0.5094)$ and $R^2 = 0.9208$. The equation of λ_1 is $\lambda_1(t) = p_4t^4 + p_3t^3 + p_2t^2 + p_1t + p_0$, where $p_4 = 0.001$, $p_3 = 0.135$, $p_2 = -0.0793$, $p_1 = 0.1921$ and $p_0 = -0.0796$. The interval confidence of coefficients are $p_4 = (-0.0015, -0.0005)$, $p_3 = (0.0069, 0.02)$,

Table 4. Estimated values of the parameters

Estimated values of the parameters vector $p = (\gamma_1, \lambda_1)$
$p_1 = (3.97, 0.1365)$
$p_2 = (3.3467, 0.1413)$
$p_3 = (0.9001, 0.1341)$
$p_4 = (0.3457, 0.0660)$
$p_5 = (0.0725, 0.0531)$
$p_6 = (0.1039, 0.0682)$
$p_7 = (0.0980, 0.0735)$
$p_8 = (0.0947, 0.0970)$
$p_9 = (0.0942, 0.1021)$
$p_{10} = (0.0980, 0.1021)$
$p_{11} = (0.1035, 0.1027)$
$p_{12} = (0.1018, 0.1013)$
$p_{13} = (0.1006, 0.0978)$
$p_{14} = (0.1, 0.09)$

$p_2 = (-0.1171, -0.0416)$, $p_1 = (0.0987, 0.2856)$ and $p_0 = (-0.1546, -0.0046)$ with $R^2 = 0.9304$.

5. Experimental calculations

In this section, according to the parameter values specified in the previous section, the solution curves of the model are drawn. MATLAB software was used to draw the solution curves. As you see, in the first weeks, due to the presence of enough eggs and larvae, the population of pupae increases. As a result, the population of adult insects increases. Then, in the following weeks, the number of eggs increases with the laying of eggs by adult insects and reaches its peak in the sixth week. In the seventh and eighth week, the number of larvae reaches its maximum number. But it is the peak of the harvest time. Hence, most of the larvae die and only a few become pupae. Therefore, after that, the population of adult insects, as well as the number of eggs, larvae and pupae, tends to zero.

According to the data in the Table 2, the initial value of healthy and damaged fruits usable for Medflies are 120 and 0 respectively. Overtime, the fruits will ripen over the following weeks. Also, enough food is provided for the Medfly. Hence, the number of damaged fruits are increased. After four weeks, the harvest season comes and most of healthy and damaged fruits are being picked. Therefore, the population of fruits decreases. It reaches its lowest value from the eighth week onwards. The reduction has a direct effect on the insect population.

According to the results, the fruits are harvested before the excessive increase

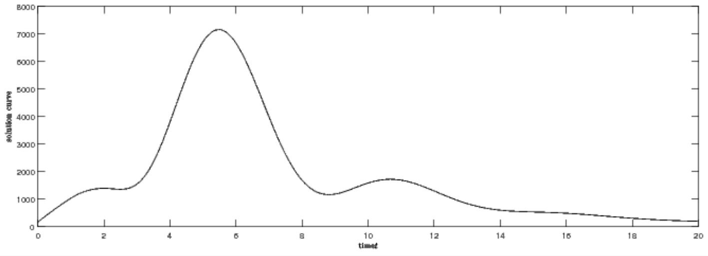

FIGURE 2. Solution curve for eggs

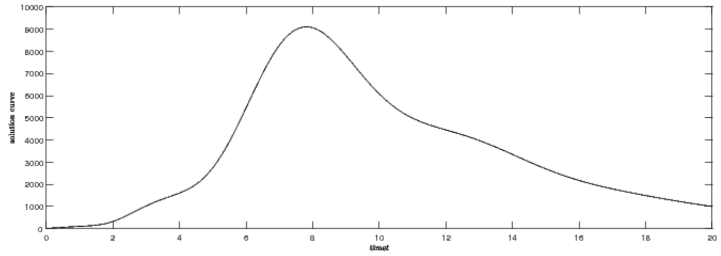

FIGURE 3. Solution curve for larva

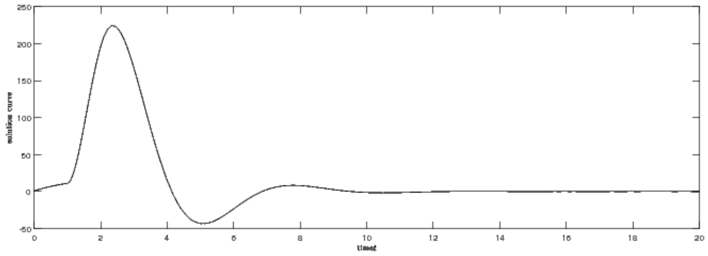

FIGURE 4. Solution curve for pupa

of insects. Therefore, the epidemic does not occur. It is difficult to check the stability of the endemic equilibrium point of the model analytically. Then, solving the characteristic equation at the endemic equilibrium point and according to the parameter values, a positive eigenvalue is obtained, so the equilibrium point is unstable. Therefore, according to the data obtained from this garden, the garden production is not seriously damaged.

Hence, the disease free equilibrium point is asymptotically stable and endemic equilibrium point is unstable.

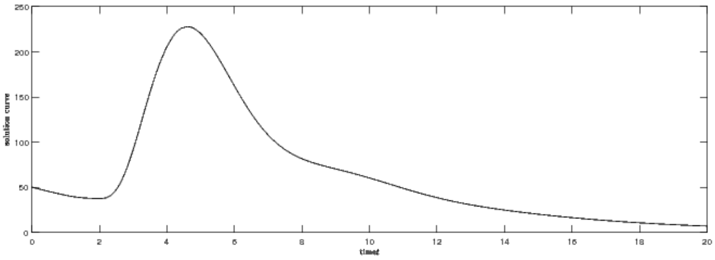

Figure 5. Solution curve for adults

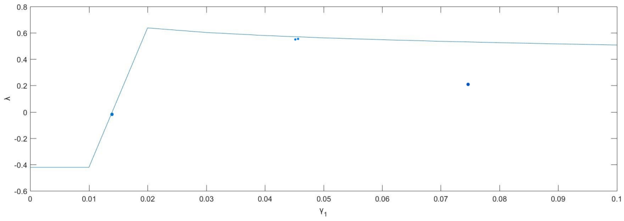

Figure 6. Changes in eigenvalue relative to parameter γ_1

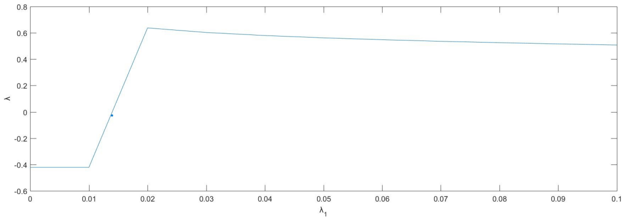

Figure 7. Changes in eigenvalue relative to parameter λ_1

5.1. **Parameter study.** As follow, the effect of changing parameters, γ_1 and λ_1 on the stability of the endemic equilibrium point is investigated.
In this section, we examine the effect of glue and plowing on the stability or instability of the endemic equilibrium point. The value of γ_1 and λ_1 is changed from 0 to 0.1. The following diagrams are drawn, that show the relationship between these two parameters with the eigenvalue of the characteristic equation at the endemic equilibrium point. As can be seen, the endemic equilibrium point is asymptotically stable for small values of the parameters, λ_1 and γ_1. Therefore, epidemic occurs. But by increasing the values of these two parameters, the point becomes unstable. Therefore, epidemic does not occur. This shows the importance of using these two control factors.

According to Figures 6 and 7, transcritical bifurcation occurs at $\gamma_1 = 0.024$ and $\lambda_1 = 0.014$.

6. **Conclusion**

One of the agricultural pests that damage more than 260 types of plants is the Mediterranean fly. This fly causes a lot of damage in many parts of Iran, including Kerman province. In this paper, considering the life cycle of Medfly, a dynamical model based on DDEs has been introduced. The model explains life cycle of fly and its effect on the fruit. In this model, two control parameters, glue to kill adult flies and plow to kill pupa are considered. These two parameters have effective role to control insect population. From dynamical point of view, the stability conditions of the disease free equilibrium point have been investigated analytically.

As follow, according to the biological properties of insect and data that is provided by Research Technology Institute of Plant Production in Kerman the values of most of parameters are estimated. To approximate two control parameters, glue and plow, the least square approach and a finite set of observation method is used. According to parameter values and initial conditions of insect population in the desired area, using Matlab software, solution curves are drawn.

The graphs show population changes for sixteen weeks. According to the diagrams, because of the presence of fruit, the insect population increase in the early weeks. But after a few weeks, due to the harvest and controlling factors, the population decrease. As a result, the production is not seriously damaged and epidemic is not occur. Also, it is determined that endemic equilibrium point is unstable.

Finally, the effect of glue and plow is investigated on stability of endemic equilibrium point. It can be seen that if the rate of trapping is low, the endemic equilibrium point is stable. Therefore, the epidemic will happened and the agriculture product is seriously damaged. Thus, a threshold value for trapping is obtained. And a transcritical bifurcation is occurred.

Appendix A

The proof of Theorem 3.2:

If $\lambda = i\theta$, where θ is a positive number. Separating real and imaginary part of equation (13) yields

$$a_0 - a_2\theta^2 + \mu_M((b_0 - \theta^2)cos\theta\tau_2 + b_1\theta sin\theta\tau_2) = 0$$

$$a_1\theta - \theta^3 + \mu_M(b_1\theta cos\theta\tau_2 - (b_0 - \theta^2)sin\theta\tau_2) = 0.$$

Squaring and adding these two equations give the following equation.

$$\theta^6 + (a_2^2 - 2a_1 - \mu_M^2)\theta^4 + (a_1^2 - 2a_0a_2 + (b_1^2 - 2b_0)\mu_M^2)\theta^2 + a_0^2 - \mu_M^2 b_0^2 = 0.$$

Let $\theta^2 = \omega > 0$. The equation reduces to

$$f(\omega) = \omega^3 + c_2\omega^2 + c_1\omega + c_0 = 0$$

where
$c_0 = a_0^2 - \mu_M^2 b_0^2$
$c_1 = a_1^2 - 2a_0a_2 + (b_1^2 - 2b_0)\mu_M^2$
and
$c_2 = a_2^2 - 2a_1 - \mu_M^2$.
We can check easily $c_2 > 0$. If $\lambda_1 > \mu_M$, $c_1 > 0$ and $f'(\omega) > 0$. Given that $f(0) = c_0$, two cases is accured. If $\lambda_1 - \mu_M > \frac{\gamma\mu_P}{\gamma_1+\lambda_1+\mu_P}$, $c_0 > 0$. Therefore, $f(\omega)$ has no root. Hence, the disease free equilibrium point is asymptotically stable. Otherwise, if $c_0 < 0$, $f(\omega)$ has a positive real root.

$$\omega = \frac{-1}{3}(c_1 + C + \frac{\Delta_0}{c_2}).$$

Where

$$C = (\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2})^{\frac{1}{3}};$$

$\Delta_0 = c_1^2 - 3c_2$,
and
$\Delta_1 = 2c_1^3 - 9c_1c_2 + 27c_0$.
Thus, $\theta = \sqrt{\omega}$. Substituting $\lambda = i\theta$ into equation (13) gives

$$\tau_2^* = \frac{1}{\theta} arctan(\frac{q_1(\theta)}{q_2(\theta)}).$$

Where
$q_1(\theta) = (b_0 - \theta^2)(b_1\theta(a_2\theta^2 - a_0) + (a_1\theta - \theta^3)(b_0 - \theta^2))$
and
$q_2(\theta) = (a_2\theta^2 - a_0)(b_1\theta^2 - (b_0 - \theta^2)^2) - b_1\theta(b_1\theta(a_2\theta^2 - a_0) + (a_1\theta - \theta^3)(b_0 - \theta^2)$.
To determine whether the steady state undergoes a Hopf bifurcation at $\tau_2 = \tau_2^*$, we have to compute the sign of $\frac{dRe[\lambda(\tau_2^*)]}{d\tau_2}$.
From (13), we have

$$(P_0'(\lambda) + (P_1'(\lambda) - \tau_2 P_1(\lambda))e^{-\tau_2\lambda})\frac{d\lambda}{d\tau_2} = \lambda P_1(\lambda)e^{-\tau_2\lambda}.$$

Therefore,

$$(\frac{d\lambda}{d\tau_2})^{-1} = \frac{-P_0'(\lambda)}{\lambda P_1(\lambda)}e^{\tau_2\lambda} + \frac{P_1'(\lambda) - \tau_2 P_1(\lambda)}{\lambda P_1(\lambda)},$$

and $e^{\tau_2\lambda} = \frac{-P_1(\lambda)}{P_0(\lambda)}$. Thus,
$sign\{\frac{d(Re\lambda)}{d\tau_2}\}_{\lambda=i\theta} = sign\{Re(\frac{d\lambda}{d\tau_2})^{-1}\}_{\lambda=i\theta}$
$= sign\{Re\frac{3\theta^2-a_1-i2a_2\theta}{(\theta^4-a_1\theta^2)+i(a_0\theta-a_0\theta^3)} + Re\frac{b_1+2i\theta}{-b_1\theta^2+i(b_0\theta-\theta^3)}\}$
$= sign\{\frac{(3\theta^2-a_1)(\theta^2-a_10-2a_2(a_0-a_1\theta^2)-b_1^2+2(b_0-\theta^2))}{\mu_M(b_0^2\theta^2+(b_0-\theta^2)^2}\}$
$= sign\{\frac{3\theta^4+2\theta^2(a+2\alpha\beta S^*(\alpha S^*+\beta+\lambda_1)\frac{\gamma\mu_P}{\gamma_1+\lambda_1+\mu_P}+(\alpha^2 S^{*2}+\beta^2)(\lambda_1^2-\mu_M^2))}{\mu_M^2(b_1^2\theta^2+(b_0-\theta^2)^2)}\}.$

Where $a = \alpha^2 S^{*2} + \beta^2 + \lambda^2 - \mu_M^2 + \alpha^2\beta^2 S^{*2}$.
Since, $\lambda_1 > \mu_M$, $sign\{\frac{d(Re\lambda)}{d\tau_2}\}_{\lambda=i\theta} > 0$. This implies that all the roots that cross the imaginary axis at $i\theta$ cross from left to right as τ_2^* increases.
Therefore, for $\tau_2 < \tau_2^*$, the equilibrium point is asymptotically stable and $\tau_2 > \tau_2^*$ the equilibrium point is unstable.

If $\tau_2 = 0$ and $\tau_1 > 0$, the characteristic equation is

$$\Delta(\tau_1, \lambda) = P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_1} + P_2(\lambda)e^{-2\lambda\tau_1} + P_3(\lambda)e^{-3\lambda\tau_1} = 0. \tag{15}$$

Where
$P_0(\lambda) = \lambda^2(\gamma_1 + \lambda_1)(\lambda^2 + (2\lambda_1 + \mu_p)\lambda + \lambda_1(\lambda_1 + \mu_P))$,
$P_1(\lambda) = (\lambda + \lambda_1 + \mu_P)(\beta\lambda(\lambda_1 + \gamma_1) + \alpha\lambda S^*(\lambda_1 + \gamma_1) + \mu_P\lambda^2)$,
$P_2(\lambda) = \mu_P\lambda^3 + \lambda^2(\mu_P(\lambda_1 + \mu_P) + (\lambda_1 + \gamma_1)(\alpha S^* + \beta)) + \lambda(\lambda_1 + \mu_P)(\lambda_1 + \gamma_1)(\alpha S^* + \beta)$,
and
$P_3(\lambda) = \alpha\beta S^*\mu_P(\lambda + \lambda_1 + \mu_P - \gamma)$.
Obviously, $P_i(\lambda) > 0$ for $i = 0, 1, 2$. If $\lambda_1 > \gamma$, $P_3(\lambda) > 0$. Therefore, all real roots of above equation are negative.
At the beginning of the appearance of the Medfly, the parameters λ_1 and γ_1 are considered zero. To investigate of stability, we use of iterative procedure [18]. If $\tau_1 \neq 0$, an iterative procedure can be used to find the equation $F(\omega)$, whose roots, ω, give the Hopf bifurcation with purely imaginary roots of the characteristic equation (13). To determine a function $F(\omega)$, which gives the Hopf bifurcation associated with purely imaginary roots of equation (15). Substituting $\lambda = i\omega$ into equation (15), since $P_0(i\omega) = 0$. Hence, the equation (15) reduces

$$\Delta(\tau_1, \lambda) = P_1(\lambda) + P_2(\lambda)e^{-\lambda\tau_1} + P_3(\lambda)e^{-2\lambda\tau_1} = 0 \tag{16}$$

Where
$P_1(\lambda) = P_2(\lambda) = (\lambda + \mu_P)\mu_P\lambda^2$,
and
$P_3(\lambda) = \alpha\beta S^*\mu_P(\lambda + \mu_P - \gamma)$.
Substituting $\lambda = i\omega$ into (16) and conjugating $\Delta(\tau_1, i\omega)$ gives
$\Delta(\tau_1, i\omega) = \sum_{k=1}^{3} P_k(i\omega)e^{-i\omega\tau_1}$, $\overline{\Delta(\tau_1, i\omega)} = \sum_{k=1}^{3} \overline{P_k(i\omega)}e^{-i\omega\tau_1}$.
Obviously, $\Delta(\tau_1, i\omega) = 0$ if and only if $\overline{\Delta(\tau_1, i\omega)} = 0$.
According iterative procedure, define $\Delta^{(1)}(\tau_1, i\omega)$.
$\Delta^{(1)}(\tau_1, i\omega) = \overline{P_1(i\omega)}\Delta(\tau_1, i\omega) - P_3(i\omega)\overline{\Delta(\tau_1, i\omega)}e^{-2i\omega\tau_1}$
$= P_1^{(1)}(i\omega) + P_2^{(1)}(i\omega)e^{-i\omega\tau_1}$.
Where

$$P_1^{(1)}(i\omega) = |P_1(i\omega)|^2 - |P_3(i\omega)|^2$$
$$P_2^{(1)}(i\omega) = \overline{P_1(i\omega)}P_2(i\omega) - \overline{P_2(i\omega)}P_3(i\omega).$$

Let $F(\omega) = |P_1^{(1)}(i\omega)|^2 - |P_2^{(1)}(i\omega)|^2$, $\Delta(\tau_1, i\omega) = 0$, whenever ω is a root of $F(\omega)$. The function $F(\omega)$ is
$a_8\omega^{16} + a_7\omega^{14} + a_6\omega^{12} + a_5\omega^{10} + a_4\omega^8 + a_3\omega^6 + a_2\omega^4 + a_1\omega^2 + a_0 = 0$.

Where,
$a_0 = \alpha^4\beta^4 S^{*4}\mu_P^4(\mu_P - \gamma)^4$,
$a_1 = 2\alpha^8\beta^8 S^{*8}\mu_P^8(\mu_P - \gamma)^8$,
$a_2 = 2\alpha^4\beta^4 S^{*4}\mu_P^4(\mu_P - \gamma)^2$,
$a_3 = 2\alpha^4\beta^4 S^{*4}\mu_P^6(\mu_P - \gamma)^2[2\alpha^2\beta^2 S^{*2} + 2\mu_P^2(\mu_P - \gamma)^2 + \alpha^3\beta^3 S^{*3}\mu_P^4(\mu_P - \gamma)^6(-2(\mu_P - \gamma) - \alpha\beta S^*(\gamma^2 - 4\mu_P\gamma + 2\mu_P^2))]$,
$a_4 = \alpha^2\beta^2 S^{*2}\mu_P^8[\alpha^2\beta^2 S^{*2}(\alpha^2\beta^2 S^{*2} + \mu_P^2(\mu_P - \gamma)^2)^2 + 4\alpha^3\beta^3 S^{*3}(\mu_P - \gamma)^2(-2(\mu_P - \gamma) - \alpha\beta S^*(\gamma^2 - 4\mu_P\gamma + 2\mu_P^2)) - 2(\mu_P - \gamma)^2(\alpha S^* + \beta)^2\gamma^2]$,
$a_5 = 2\alpha^2\beta^2 S^{*2}\mu_P^7[\alpha\beta S^*\mu_P(\alpha^2\beta^2 S^{*2} + \mu_P^2(\mu_P - \gamma)^2)(-2(\mu_P - \gamma) - \alpha\beta S^*(\gamma^2 - 4\mu_P\gamma + 2\mu_P^2)) - 2\gamma^2(\mu_P - \gamma)(\alpha S^* + \beta)(\mu_P^2 - \alpha S^* - \beta)]$,
$a_6 = \alpha^2\beta^2 S^{*2}\mu_P^6[\mu_P^2(2(\mu_P - \gamma) + \alpha\beta S^*(\gamma^2 - 4\mu_P\gamma + 2\mu_P^2)^2) - 2\gamma^2(\mu_P^2 - \alpha S^* - \beta)^2 + 4\mu_P\gamma(\mu_P - \gamma)^2(\alpha S^* + \beta)]$,
$a_7 = 2\alpha^2\beta^2 S^{*2}\mu_P^6\gamma(2\mu_P - \gamma)(\mu_P^2 - \alpha S^* - \beta)$,
$a_8 = -4\alpha^2\beta^2 S^{*2}\mu_P^7(\mu_P - \gamma)$.
Let $\theta = \omega^2$, the equation $F(\omega) = 0$ can rewritten as follows,

$$h(\theta) = a_8\theta^8 + a_7\theta^7 + a_6\theta^6 + a_5\theta^5 + a_4\theta^4 + a_3\theta^3 + a_2\theta^2 + a_1\theta + a_0 = 0.$$

Without loss of generality, suppose that $h(\theta)$ has eight distinct positive roots denoted by $\theta_1, \theta_2, ..., \theta_8$, so $F(\omega)$ has eight positive roots

$$\omega_i = \sqrt{\theta_i}, i = 1, 2, ..., 8.$$

Substituting $\lambda_k = i\omega_k$ into equation (16) and then separating real and imaginary parts gives

$$-\mu_p\omega^2 cos\omega\tau_1 - \omega^3 sin\omega\tau_1 + \alpha\beta S^*(\mu_p - \gamma)cos2\omega\tau_1 + \alpha\beta S^*\omega sin2\omega\tau_1 = \mu_p\omega^2,$$

$$\mu_p\omega^3 sin\omega\tau_1 - \omega^3 cos\omega\tau_1 - \alpha\beta S^*(\mu_p - \gamma)sin2\omega\tau_1 + \alpha\beta S^*\omega cos2\omega\tau_1 = \omega^3.$$

Squaring two equations and adding them gives

$$\tau_{1k}^j = tan^{-1}\left(\frac{d_3(c_1d_1 - c_2d_2)}{d_1(c_3d_1 - c_2d_3)} - \frac{d_2}{d_1}\right) + j\pi. \quad k = 1, 2, ..., 8, \quad j = 0, 1, ...,$$

Where, $c_1 = -2\mu_p\omega_k^2\alpha\beta S^*(\mu_p - \gamma - \mu_p\omega_k^2)$,
$c_2 = 2\mu_p\omega_k^3\alpha\beta S^*(\mu_p - \gamma - 1)$,
$c_3 = -\mu_p^4\omega_k^6 - \alpha^2\beta^2 S^{*2}((\mu_p - \gamma)^2 + \omega_k^2)$,
$d_1 = \mu_p\omega_k^2 + \alpha\beta S^{*2}(\mu_p - \gamma)$,
$d_2 = \omega_k^3 - \alpha\beta S^{*2}\omega_k$,
$d_3 = \mu_p^4\omega_k^6$.
The solutions of (16) are (τ_{1k}^j, ω_k). That means $\lambda = i\omega_k$ are pairs of purely imaginary roots of (16) with $\tau_1 = \tau_{1k}^j$.
Let $\tau_1^* = \tau_{1k_0}^0 = min_{1\leq k\leq 8}\{\tau_{1k_0}^0\}$ and $\omega_0 = \omega_{k_0}$. This is the first time delay for which the roots of characteristic equation (16) cross the imaginary axis. Let $\lambda(\tau_1) = \alpha(\tau_1) + i\omega(\tau_1)$ be the root of (16), using the above calculations, $\alpha(\tau_{1k}^*) = 0$ and $\omega(\tau_{1k}^*) = \omega_0$. As follows, to determine whether the equilibrium point undergoes a Hopf bifurcation at $\tau_1 = \tau_1^*$, we determine the sign of $\frac{dRe(\lambda(\tau_1^*))}{d\tau_1^*}$, for $\lambda(\tau_1^*) = i\omega_0$.

Theorem 6.1. *Suppose,* $3\omega_0^2+2\mu_p-2\gamma<0$*,* $\alpha\beta\gamma S^*-\mu_p(1+\omega_0^2)(2\mu_p^2+3\omega_0^2)>0$ *and* $\gamma-\omega_0(2\mu_p^2+3\omega_0^2)>0$ *then* $\frac{dRe(\lambda(\tau_1^*))}{d\tau_1^*}<0$.

Proof. Let $P_1(\lambda)$, $P_2(\lambda)$ and $P_3(\lambda)$ be coefficients of equation (16). Considering $P_1(i\omega_0)=P_2(i\omega_0)=x_1(\omega_0)+iy_1(\omega_0)$ and $P_3(i\omega_0)=x_2(\omega_0)+iy_2(\omega_0)$. Where
$x_1(\omega_0)=-\mu_p^2\omega_0^2$,
$y_1(\omega_0)=-\mu_p\omega_0^3$,
$x_2(\omega_0)=\alpha\beta S_0\mu_p(\mu_p-\gamma)$,
$y_2(\omega_0)=\alpha\beta S_0\mu_p\omega_0$.
Hence, $P_1^{(1)}(i\omega_0)=x_1^2+y_1^2-x_2^2-y_2^2$, and $P_2^{(1)}(i\omega_0)=x_1^2+y_1^2-x_1x_2-y_1y_2+i(x_2y_1-x_1y_2)$.
Substituting these expressions into $\Delta(\tau_1^*,i\omega_0)=0$ and $\Delta^{(1)}(\tau_1^*,i\omega_0)=0$, then separating real and imaginary parts gives

$$\left\{\begin{array}{l} x_1cos\omega_0\tau_1^*+y_1sin\omega_0\tau_1^*+x_2cos2\omega_0\tau_1^*+y_2sin2\omega_0\tau_1^*=x_1\\ y_2cos\omega_0\tau_1^*-x_1sin\omega_0\tau_1^*+y_2cos2\omega_0\tau_1^*-x_2sin2\omega_0\tau_1^*=-y_1\\ (x_1^2+y_1^2-x_1x_2-y_1y_2)cos\omega_0\tau_1^*+(x_2y_1-x_1y_2)sin\omega_0\tau_1^*=x\\ (x_2y_1-x_1y_2)cos\omega_0\tau_1^*-(x_1^2+y_1^2-x_1x_2-y_1y_2)sin\omega_0\tau_1^*=0 \end{array}\right. \tag{17}$$

Where $x=-x_1^2-y_1^2-x_2^2-y_2^2$. Solving the system gives the values of $sin\omega_0\tau_1^*$, $cos\omega_0\tau_1^*$, $sin2\omega_0\tau_1^*$ and $sin2\omega_0\tau_1^*$.
$sin\omega_0\tau_1^*=\frac{-AC}{C^2-B^2}$, $cos\omega_0\tau_1^*=\frac{AB}{C^2-B^2}$, $sin2\omega_0\tau_1^*=\frac{-2A^2BC}{(C^2-B^2)^2}$ and $cos2\omega_0\tau_1^*=\frac{-A^2}{C^2-B^2}$. Where, $A=x_1^2+y_1^2+x_2^2+y_2^2$, $B=x_1^2+y_1^2-x_1x_2-y_1y_2$ and $C=x_2y_1-x_1y_2$. We can check easily, $sin\omega_0\tau_1^*>0$, $cos\omega_0\tau_1^*<0$, $sin2\omega_0\tau_1^*<0$ and $cos2\omega_0\tau_1^*>0$.
To compute the sign of $\frac{dRe[\lambda(\tau_1^*)]}{d\tau_1}$, from the equation (16), we have $(P_1'(\lambda)+P_2'(\lambda)e^{-\lambda\tau_1}+P_3'(\lambda)e^{-2\lambda\tau_1}-2P_3(\lambda)\tau_1e^{-2\lambda\tau_1}-\tau_1e^{-\lambda\tau_1}P_2(\lambda))\frac{d\lambda}{d\tau_1}=-\lambda e^{-\tau_1\lambda}P_2(\lambda)-2\lambda P_3(\lambda)e^{-2\lambda\tau_1}=0$.
Hence,
$(\frac{d\lambda}{d\tau_1})^{-1}=\frac{P_1'(\lambda)+P_2'(\lambda)e^{-\tau_1\lambda}+P_3'(\lambda)e^{-2\lambda\tau_1}}{\lambda(P_2(\lambda)e^{-\tau_1\lambda}+2P_3(\lambda)e^{-2\lambda\tau_1})}-\frac{\tau_1}{\lambda}$.
$sign\{\frac{d(Re\lambda)}{d\tau_1}\}_{\lambda=i\omega_0}=sign\{Re(\frac{d\lambda}{d\tau_1})^{-1}\}_{\lambda=i\omega_0}=signRe\frac{1}{\omega_0\Lambda}\{-\mu_p^2\omega_0^3(2\mu_p^2+3\omega_0^2)-\alpha^2\beta^2S_0^2\mu_p^2\omega_0+\mu_p^2\omega_0^2(\mu_p\omega_0^2+2\alpha\beta S_0(3\gamma-\mu_p)+\alpha\beta S_0\mu_p)sin\omega_0\tau_1^*+\mu_p^2\omega_0^2(-\omega(2\mu_p^2+3\omega_0^2)+\alpha\beta S_0(6\omega_0+4(\mu_p-\gamma)-\omega_0^2))cos\omega_0\tau_1^*+2\mu_p^2\omega_0^2\alpha\beta S_0(3\gamma-\mu_p)sin2\omega_0\tau_1^*+2\mu_p^2\alpha\beta S_0\omega_0(3\omega_0^2+2(\mu_p-\gamma))cos2\omega_0\tau_1^*\}$.
Where $\Lambda=x_1^2+y_1^2+x_2^2+y_2^2$. Using of the assumptions of the theorem, $Sign\{\frac{d(Re(\lambda)}{d\tau_1}\}_{\tau_1=\tau_1^*}<0$. This implies that all the roots that cross the imaginary axis at $i\omega_0$ cross from left to right as τ_1^* decreases. ☐

By Lemma 6.1, the disease free equilibrium point is unstable for $0<\tau_1\leq\tau_1^*$ and is stable for $\tau_1>\tau_1^*$. Therefore, Hopf bifurcation is occurred at a critical value of the time delay $\tau_1=\tau_1^*$.

Appendix B

To approximate the parameters, the method described in [2] is used. we assume that the time delay, τ_2, is zero. Let $\tau_1 = \tau$, the model (1) is discretized and the following equations are written.

$$
\left\{
\begin{array}{c}
E(k+1) = E(k) + \gamma M(k) - \beta E(k-\tau), \\
L(k+1) = \beta E(k-\tau) + L(k) - \alpha L(k-\tau)(S(k)+I(k)), \\
P(k+1) = \alpha L(k-\tau)(S(k)+I(k)) - \mu_P P(t-\tau_1) - \gamma_1 P(t), \\
M(k+1) = \mu_P P(k-\tau) + (1-\lambda_1)M(k) - \mu_M M(k), \\
S(k+1) = \lambda_2 S(k)(1 - \frac{S(k)}{K}) - \nu S(k)L(k) - \mu_S S(k), \\
I(k+1) = \nu S(k)L(k) - \omega L(k)I(k) - \mu_I I(k).
\end{array}
\right. \tag{18}
$$

This system can be linearized around the equilibrium point G_1, which is defined as the equilibrium point where no medfly is presented in garden. One knows that a difference equation with delay is of the form:

$$x(k+1) = A(p)x(k) + Bx(k-\tau), x(k-\tau) = 0, \text{ when } k-\tau \le 0$$

For example let $k = 4$, and $\tau = 2$ then:
$x(1) = Ax(0)$,
$x(2) = A^2x(0)$,
$x(3) = [A^3 + B]x(0)$,
$x(4) = [A^4 + (AB + BA)]x(0)$,
$x(5) = [A^5 + (A^2B + ABA + BA^2)]x(0)$,
$x(6) = [A^6 + (A^3B + A^2BA + ABA^2 + BA^3) + B^2]x(0)$,
.
.
.
$x(11) = [A^{11} + (A^8B + A^7BA + \cdots + BA^8) + (A^5B^2 + A^4B^2A + ... + B^2A^5) + (A^3BA^2B + A^2BA^3B + A^2BA^2BA + ... + BABA^4) + (B^3A^2 + B^2A^2B + BA^2B^2 + A^2B^3)]x(0)$.
In the problem, this linear system is of the type $x(k+1) = A(p)x(k) + B$, where

$$A(p) = \begin{bmatrix} 1 & 0 & 0 & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-\gamma_1 & 0 \\ 0 & 0 & 0 & 1-\lambda_1-\mu_M \end{bmatrix},$$

$$B = \begin{bmatrix} -\beta & 0 & 0 & 0 \\ \beta & -\alpha(S(t)+I(t)) & 0 & 0 \\ 0 & \alpha(S(t)+I(t)) & -\mu_P & 0 \\ 0 & 0 & \mu_P & 0 \end{bmatrix}$$

To solve the estimation problem, we rewrite the system 6. $x(k+1) = M'(k)p + N(k-\tau)$ where

$$M'(k)=\begin{bmatrix}\frac{dE}{d\gamma_1} & \frac{dE}{d\lambda_1}\\ \frac{dL}{d\gamma_1} & \frac{dL}{d\lambda_1}\\ \frac{dP}{d\gamma_1} & \frac{dP}{d\lambda_1}\\ \frac{dM}{d\gamma_1} & \frac{dM}{d\lambda_1}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\\ -p(k) & 0\\ 0 & -M(k)\end{bmatrix}.$$

Notice that p is the parameter matrix, and $p(k)$ is the value of parameters in the stage k. In the other words:

$$p=\begin{bmatrix}\gamma_1\\ \lambda_1\end{bmatrix}, N(k-\tau)=\begin{bmatrix}N_1(k-\tau)\\ N_2(k-\tau)\\ N_3(k-\tau)\\ N_4(k-\tau)\end{bmatrix}$$

Now let $x(k+1)=M'(k)p+N(k-\tau)$ and then $N(k-\tau)=x(k+1)-M'(k)p$.

$$x(k+1)=\begin{bmatrix}E(k+1)\\ L(k+1)\\ P(k+1)\\ M(k+1)\end{bmatrix}, M'(k)p=\begin{bmatrix}0 & 0\\ 0 & 0\\ -p(k) & 0\\ 0 & -M(k)\end{bmatrix}\times\begin{bmatrix}\gamma_1\\ \lambda_1\end{bmatrix}.$$

From the initial difference equations, one has:

$$x(k+1)=\begin{bmatrix}E(k)+\gamma M(k)-\beta E(k-\tau)\\ \beta E(k-\tau)+L(k)-\alpha L(k-\tau)(S(k)+I(k))\\ \alpha L(k-\tau)(S(k)+I(k))-\mu_P P(k-\tau)+P(k)(1-\gamma_1)\\ \mu_P P(k-\tau)+(1-\lambda_1)M(k)-\mu_M M(k)\end{bmatrix}.$$

Hence the matrix $N(k-\tau)$ can be obtained as follows:

$$N(k-\tau)=\begin{bmatrix}E(k)+\gamma M(k)-\beta E(k-\tau)\\ \beta E(k-\tau)+L(k)-\alpha L(k-\tau)(S(k)+I(k))\\ \alpha L(k-\tau)(S(k)+I(k))-\mu_P P(k-\tau)+P(k)\\ \mu_P P(k-\tau)+M(k)-\mu_M M(k)\end{bmatrix}$$

Let $ob(k)$ be the experimental data set in the stage k, and define:

$e(k)=(ob(k)-x(k)), k=1,2,...,K$ and $e_K=column(e(k))_{k=1}^{K}$

$d(k)=ob(k)-N(k-\tau), k=1,2,...,K$ and $d_k=column(d(k))_{k=1}^{K}$

$H_K=column(e(k))_{k=1}^{K}$ and $E(k-\tau)=L(k-\tau)=P(k-\tau)=0$, where $k\leq\tau$.

$$d(1)-N(0)=\begin{bmatrix}E(1)\\ L(1)\\ P(1)\\ M(1)\end{bmatrix}-\begin{bmatrix}E(1)+\gamma M(1)\\ L(1)\\ P(1)\\ M(1)-\mu_M M(1)\end{bmatrix}=\begin{bmatrix}-\gamma M(1)\\ 0\\ 0\\ \mu_M M(1)\end{bmatrix},$$

$$\vdots$$

$$d(\tau)=ob(\tau)-N(0)=\begin{bmatrix}E(\tau)\\ L(\tau)\\ P(\tau)\\ M(\tau)\end{bmatrix}-\begin{bmatrix}E(\tau)+\gamma M(\tau)\\ L(\tau)\\ P(\tau)\\ M(\tau)-\mu_M M(\tau)\end{bmatrix}=\begin{bmatrix}-\gamma M(\tau)\\ 0\\ 0\\ \mu_M M(\tau)\end{bmatrix},$$

$\vdots$

$$d(\tau+1) = ob(\tau+1) - N(1) = \\ \begin{bmatrix} E(\tau+1) \\ L(\tau+1) \\ P(\tau+1) \\ M(\tau+1) \end{bmatrix} - \begin{bmatrix} E(\tau+1) + \gamma M(\tau+1) - \beta E(1) \\ \beta E(1) + L(\tau+1) - \alpha L(1)(S(\tau+1) + I(\tau+1)) \\ \alpha L(1)(S(\tau+1) + I(\tau+1)) - \mu_P P(1) + P(\tau+1) \\ \mu_P P(1) + M(\tau+1) - \mu_M M(\tau+1) \end{bmatrix} = \\ \begin{bmatrix} \beta E(1) - \gamma M(\tau+1) \\ \alpha L(1)(S(\tau+1) + I(\tau+1)) - \beta E(1) \\ -\alpha L(1)(S(\tau+1) + I(\tau+1)) - \mu_P P(1) \\ -\mu_P P(1) + \mu_M M(\tau+1) \end{bmatrix}.$$

$\vdots$

$$d(\tau+m) = \begin{bmatrix} \beta E(m) - \gamma M(\tau+m) \\ \alpha L(1)(S(\tau+m) + I(\tau+m)) - \beta E(m) \\ -\mu_P P(m) + \mu_M M(\tau+m) \end{bmatrix}, \; m \geq 1.$$

$$H_K = column(M'(k)_{k=1}^{K}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -p(0) & 0 \\ 0 & -M(0) \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ -p(K) & 0 \\ 0 & -M(K) \end{bmatrix}$$

Therefore, for K observations one can survey the parameter vector p that optimizes the function $J_K(p)$, where $J_K(p)$ is calculated as follows:
$J_K(p) = \frac{1}{2}\sum_{k=1}^{K} e(k)^T e(k) = \frac{1}{2} {e_K}^T e_K = \frac{1}{2}(d_K - H_K p)^T (d_K - H_K p)$, where

$$d_K = \begin{bmatrix} d(1) \\ d(2) \\ \vdots \\ d(\tau) \\ d(\tau+1) \\ \vdots \\ d(K) \end{bmatrix} = \begin{bmatrix} -\gamma M(1) \\ 0 \\ 0 \\ \mu_M M(1) \\ \vdots \\ -\gamma M(\tau) \\ 0 \\ 0 \\ \mu_M M(\tau) \\ \beta E(1) - \gamma M(\tau+1) \\ \alpha L(1)(S(\tau+1) + I(\tau+1)) - \beta E(1) \\ -\alpha L(1)(S(\tau+1) + I(\tau+1)) - \mu_P P(1) \\ \mu_P P(1) - \mu_M M(\tau+1) \\ \vdots \\ \beta E(m) - \gamma M(\tau+m) \\ \alpha L(m)(S(\tau+m) + I(\tau+m)) - \beta E(m) \\ -\alpha L(m)(S(\tau+m) + I(\tau+m)) - \mu_P P(m) \\ \mu_P P(m) - \mu_M M(\tau+m) \end{bmatrix},$$

$K = \tau + m.$

In order to find the parameter vector, one should acquire $H_K p$.

$$H_K p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -p(0) & 0 \\ 0 & M(0) \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ -p(K-1) & 0 \\ 0 & M(K-1) \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\gamma_1 p(0) \\ \lambda_1 M(0) \\ \vdots \\ 0 \\ 0 \\ -\gamma_1 p(K-1) \\ \lambda_1 M(K-1) \end{bmatrix}.$$

Now let's obtain $d_K - H_K p$ as follows:

$$d_K - H_K p = \begin{bmatrix} -\gamma M(1) \\ 0 \\ \gamma_1 P(0) \\ \mu_M M(1) - \lambda_1 M(0) \\ \vdots \\ -\gamma M(\tau) \\ 0 \\ \gamma_1 P(\tau) \\ \mu_M M(\tau) - \lambda_1 M(\tau - 1) \\ \beta E(1) - \gamma M(\tau + 1) \\ \alpha L(1)(S(\tau+1) + I(\tau+1)) - \beta E(1) \\ -\alpha L(1)(S(\tau+1) + I(\tau+1)) - \mu_P P(1) + \gamma_1 P(\tau) \\ \mu_P P(1) - \mu_M M(\tau+1) - \lambda_1 M(\tau) \\ \vdots \\ \beta E(m) - \gamma M(\tau + m) \\ \alpha L(m)(S(\tau+m) + I(\tau+m)) - \beta E(m) \\ -\alpha L(m)(S(\tau+m) + I(\tau+m)) - \mu_P P(m) + \gamma_1 P(\tau+m-1) \\ \mu_P P(m) - \mu_M M(\tau+m) - \lambda_1 M(\tau+m-1) \end{bmatrix}$$

and

$$H_K^T H_K p = \begin{bmatrix} \gamma_1 \sum_{k=1}^{K-1} p(k)^2 \\ \lambda_1 \sum_{k=1}^{K-1} M(k)^2 \end{bmatrix}.$$

Now it's remained to compute $H_K^T d_K$.

$$H_K^T d_K = \begin{bmatrix} (-\alpha L(1)(S(\tau+1) + I(\tau+1)) - \mu_P P(1))p(\tau) \\ M + [-\mu_P P(1) + \mu_M M(\tau+1)]M(\tau) \end{bmatrix} + ... +$$
$$\begin{bmatrix} (-\alpha L(m)(S(\tau+m) + I(\tau+m)) - \mu_P p(m))P(\tau+m-1) \\ (-\mu_P P(m) + \mu_M M(\tau+m))M(\tau+m-1) \end{bmatrix}$$

where $M = \mu_M(M(0)M(1) + M(1)M(2) + ... + M(\tau-1)M(\tau))$.
$\frac{\partial J_K(p)}{\partial p} = H_K^T H_K p - H_K^T d_K =$
$$\begin{bmatrix} \gamma_1 \sum_{k=0}^{K-1} p(k)^2 - \alpha a \\ \lambda_1 \sum_{k=0}^{K-1} M(k)^2 - \mu_M \sum_{k=0}^{\tau-1} M(k)M(k+1) - b \end{bmatrix}$$
where $a = \sum_{k=1}^{m}(L(k)(S(k+\tau) + I(k+\tau)) + \mu_P P(k))P(k+\tau-1)$ and $b = \sum_{k=1}^{m}(\mu_P P(k) - \mu_M M(k+\tau))M(k+\tau-1)$.
Hence one has λ_1, γ_1 as above:

$$\gamma_1 = \frac{\alpha \sum_{k=1}^{m}(L(k)(S(k+\tau)+I(k+\tau))+\mu_P P(k))P(k-\tau)}{\sum_{k=0}^{K} -1P(k)^2},$$

$$\lambda_1 = \frac{\mu_M \sum_{k=0}^{\tau-1} M(k)M(k-1)+\sum_{k=1}^{m}(-\mu_P P(k)+\mu_M M(k+\tau))}{\sum_{k=0}^{K-1} M(k)^2}.$$

$$H_K^T H_K = \begin{bmatrix} \sum_{k=0}^{K-1} p(k)^2 & 0 \\ 0 & \sum_{k=0}^{K-1} M(k)^2 \end{bmatrix}.$$

Therefore one knows that $H_K^T H_K$, is non-singular, $H_K^T H_K p = H_K^T H_K$, and then $p = (H_K^T H_K)^{-1}(H_K^T H_K)$, and hence p exists and is unique.

Definition 6.2. If A is a square matrix, then the spectral norm of A, is defined by:

$$\|A\|_2 = (maximum \ of \quad eigenvalues \ of \ \ A^T A)^{\frac{1}{2}}.$$

Definition 6.3. Let A be a square matrix, and $\lambda_1, \lambda_2, ..., \lambda_n$ be its eigenvalues. Then the spectral radius of A is defined as follows:

$$\rho(A) = max(|\lambda_1|, |\lambda_2|, ..., |\lambda_n|).$$

Theorem 6.4. [2]*Let* $S_K = H_K^T H_K$, $p_K = S_K^{-1} H_K^T d_K$, *and let* $x(k+1) = Ax(k) + Bx(k-\tau)$. *If* $\|d_K\|_2 \le \frac{1}{\rho(S_K^{-1})})(\rho(S_K))^{\frac{-1}{2}}$, *then* $\| p_K \|_2 < 1$.

One can see that

$$S_K = H_K^T H_K = \begin{bmatrix} \sum_{k=0}^{K-1} (p(k))^2 & 0 \\ 0 & \sum_{k=0}^{K-1} (M(k))^2 \end{bmatrix}$$

Let $E^2 = \sum_{k=0}^{K-1} (p(k))^2, F^2 = \sum_{k=0}^{K-1} (M(k))^2$, and then

$$p_K = S_K^{-1} H_K^T d_K = \frac{1}{F^2} \begin{bmatrix} 0 \\ \mu_M M(0)M(1) + ... + \mu_M M(\tau-1)M(\tau) \end{bmatrix} + \frac{1}{F^2} \begin{bmatrix} 0 \\ n(1)M(\tau+1) + ... + n(m)M(\tau+m) \end{bmatrix}$$

where $n(i) = \mu_P p(i) - \mu_M M(\tau+i)$ for $i = 1, ..., m$.
$p_K^T p_K = \frac{1}{F^4}[(\mu_M \sum_{k=1}^{\tau} M(k-1)M(k)) + \sum_{k=1}^{m} n(k)M(\tau+k)]^2$.
One may remember that $K = \tau + m, F^2 = \sum_{k=1}^{K-1} M(k)^2$.

Corollary 6.5. [2] *If* $[(\mu_M \sum_{k=1}^{\tau} M(k-1)M(k)) + \sum_{k=1}^{m} n(k)M(\tau+k)]^2 \le F^4$. *Then* $\|p_K\|_2 < 1$

Now by assuming $\|p_K\|_2 < 1$, one can add one observation $ob(k+1)$, and show that there is a fitting between the mathematical model to the $k+1$ data of the observed data set.
$H_{K+1}^T H_{K+1} = H_K^T H_K + {M'}^T(K)M'(K)$,
and $H_{K+1}^T d_{K+1} = H_K^T d_K + {M'}^T(K)d(K+1)$.
On the other hand,
$p_{K+1} = S_{K+1}^{-1} H_{K+1}^T d_{K+1} = S_{K+1}^{-1}(H_K^T d_K + {M'}^T(K)d(K+1))$
$= S_{K+1}^{-1} S_K p_K + S_{K+1}^{-1} M\prime^T(K) = A_K p_K + BK.$

Where
$A_K = S_{K+1}^{-1} S_K$
and
$B_K = S_{K+1}^{-1} M'^{T}(K) d(K+1)$.
Then
$p_{K+1} = A_K p_K + B_K, K \geq 1$.
Easily one can acquire that:
$p_K = A_{K-1}...A_{k_0} p_{k_0} + \sum_{i=k_0+1}^{K-1} (A_{K-1} A_{K-2}...A_{i+1} B_i)$, if $K > k_0$, and $p_K = p_{k_0}$, if $K = k_0$.
The spectral radius of A_K, that has been defined as:

$$\rho(A_K) = max\{|\lambda_1|, ..., |\lambda_n|\}, \text{ where } \lambda_i \text{ is the eigenvalue of } A_K.$$

If $\rho(A_K) < 1$, for all $K > k_0$, since S_K is symmetric, $\rho(A_{K-1}...A_{k_0}) < \Pi_{i=k_0}^{K-1} \rho(A_i) < 1$.
Therefore $A_{K-1}...A_{k_0}$ is asymptotically stable and hence p_K exists, and if B_K is bounded , then p_K is bounded.

Theorem 6.6. [2]*Let* $p_K = A_{K-1}...A_{k_0} p_{k_0} + \sum_{j=k_0}^{K-1} A_K...A_{j+1} B_j, K > k_0$. *Furthermore let* $\rho(A_K) < 1, for\ all\ K > k_0$, *and* $\|ob(K+1) - x(K+1)\|_2 < \frac{\epsilon}{\rho(S_{K+1}^{-1})\rho(M^T(K)M(K))}$, *for some* $\epsilon > 0$. *Then* $\|p_{K+1} - p_K\|_2 < \epsilon$.

Competing interests:
The authors declare no competing interests.

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