Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

GENERALIZATIONS OF MANN'S ITERATIVE ALGORITHM FOR BEST PROXIMITY POINTS

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Article type: Research Article
(Received: 29 December 2024, Received in revised form 22 July 2025)
(Accepted: 13 August 2025, Published Online: 13 August 2025)

Abstract. The main purpose of this paper is to consider the convergence of iterative algorithms for finding the best proximity points for cyclic contraction mappings, that is a new extension of the Mann iteration by dropping some additional assumptions. To this end, the convergence behavior of the new algorithms is compared with a numerical example.

Keywords: Best proximity point, Cyclic relatively nonexpansive, Mann iteration, Iterative algorithm.
2020 MSC: 90C48, 47H09, 46B20.

1. Introduction

Let Ω be a Banach space and Θ be a nonempty subset of Ω. We remember that a mapping $\Lambda : \Theta \to \Theta$ is *nonexpansive* if $\|\Lambda\tau - \Lambda\upsilon\| \leq \|\tau - \upsilon\|$ for all $\tau, \upsilon \in \Theta$. Browder [2] considered that for the uniformly convex Banach space Ω, the nonexpansive mapping Λ has a fixed point, if Θ is a bounded, closed and convex subset.

Mann [11] in 1953 gave a new iteration algorithm for finding fixed points of nonexpansive mappings as follows:

$$\tau_{n+1} = (1 - \varsigma_n)\tau_n + \varsigma_n\Lambda\tau_n, \tag{1}$$

where $\{\varsigma_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \varsigma_n = 0$ and $\sum_{n=1}^{\infty} \varsigma_n = \infty$.

In the following, we suppose that Θ and Δ are two nonempty and disjoint subsets of a metric space Ω. A mapping $\Lambda : \Theta \cup \Delta \to \Theta \cup \Delta$ is called *cyclic* if $\Lambda(\Theta) \subseteq \Delta$ and $\Lambda(\Delta) \subseteq \Theta$.

Let Θ and Δ be subsets of a complete metric space Ω. A mapping $\Lambda : \Theta \cup \Delta \to \Theta \cup \Delta$ is a cyclic contraction if $\Lambda(\Theta) \subset \Delta$ and $\Lambda(\Delta) \subset \Theta$, and also for a $k \in (0, 1)$

$$d(\Lambda\upsilon, \Lambda\tau) \leq kd(\upsilon, \tau) + (1 - k)\mathrm{d}(\Theta, \Delta) \ \forall \upsilon \in \Theta, \tau \in \Delta.$$

where $\mathrm{d}(\Theta, \Delta) := \inf\{d(\upsilon, \tau) : (\upsilon, \tau) \in \Theta \times \Delta\}$. Also, a mapping $\Lambda : \Theta \cup \Delta \to \Theta \cup \Delta$ is cyclic relatively nonexpansive if $\Lambda(\Theta) \subset \Delta$ and $\Lambda(\Delta) \subset \Theta$, and also

$$d(\Lambda\upsilon, \Lambda\tau) \leq d(\upsilon, \tau) \ \forall \upsilon \in \Theta, \tau \in \Delta.$$

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https://doi.org/ 10.22103/jmmr.2025.24603.1743

Publisher: Shahid Bahonar University of Kerman
How to cite: M.R. Haddadi, M. Aliyari, *Generalizations of Mann's iterative algorithm for best proximity points*, J. Mahani Math. Res. 2026; 15(1): 227-238.

In this case, we consider a solution for an optimal problem

$$(2) \qquad \min_{\upsilon\in\Theta\cup\Delta} \|\upsilon - \Lambda\upsilon\|.$$

A point $\upsilon \in \Theta\cup\Delta$ is called a *best proximity point* for the cyclic mapping $\Lambda : \Theta\cup\Delta \to \Theta\cup\Delta$ if υ is a solution for the minimization problem (2), i.e., $\|\upsilon - \Lambda\upsilon\| = \mathrm{d}(\Theta, \Delta)$.

Inspired by cyclic mappings, the mapping $\Lambda : \Theta\cup\Delta \to \Theta\cup\Delta$ is called *noncyclic* whenever $\Lambda(\Theta) \subseteq \Theta$ and $\Lambda(\Delta) \subseteq \Delta$. In this case, we can consider the following, optimal problem:

$$(3) \qquad \min_{\upsilon\in\Theta} \|\upsilon - \Lambda\upsilon\|, \quad \min_{\tau\in\Delta} \|\tau - \Lambda\tau\|, \quad \min_{(\upsilon,\tau)\in\Theta\times\Delta} \|\upsilon - \tau\|.$$

We know that a pair $(\upsilon^\star, \tau^\star) \in \Theta\times\Delta$ for the noncyclic mapping Λ is called a *best proximity pair* if it is a solution of (3), hence, $\upsilon^\star = \Lambda\upsilon^\star, \tau^\star = \Lambda\tau^\star$ and $\|\upsilon^\star - \tau^\star\| = \mathrm{d}(\Theta, \Delta)$ (see [1]). Let Θ and Δ be nonempty subsets of a normed space Ω. We put

$$\Theta_0 := \{\upsilon \in \Theta : \|\upsilon - \tau\| = \mathrm{d}(\Theta, \Delta), \text{ for some } \tau \in \Delta\},$$

$$\Delta_0 := \{\tau \in \Delta : \|\upsilon - \tau\| = \mathrm{d}(\Theta, \Delta), \text{ for some } \upsilon \in \Theta\}.$$

Best proximity point theory of nonself mappings was recently introduced by Fan [4]. In recent years, the best proximity point problem for nonself mappings is an interesting topic in optimization theory and many authors have researched on existence of the best proximity point, see [1, 3, 7, 8, 12, 13].

For the first time, iteration schemes for finding best proximity point on the cyclic contraction mapping were introduced by Eldred and Veeramani [3].Some authors presented a new iteration schemes for finding the best proximity point [5, 9, 10, 13, 14]. Suparatulatorn et al. gave new hybrid algorithms of finding the best proximity points in Hilbert spaces [15,16]. In a recent paper Aliyari et al. studied the convergence of best proximity points for cyclic relatively nonexpansive mappings in the setting of uniformly convex Banach spaces [1]. Also, Haddadi et al. [6] presented generalizations of the Mann and Ishikawa iteration algorithms and they got many of strong convergence to the best proximity point.

We remember that a mapping $\Lambda : \Theta\cup\Delta \to \Theta\cup\Delta$ is *relatively nonexpansive* if $\|\Lambda\upsilon - \Lambda\tau\| \le \|\upsilon - \tau\|$ for all $(\upsilon, \tau) \in \Theta\times\Delta$.

The motivation behind this paper stems from the need for efficient methods in solving optimization problems and fixed-point problems in various mathematical and applied fields. Finding the best proximity points can help in minimizing distances in optimization problems, which is essential in fields like operations research and economics. Also, algorithms for proximity points are vital in areas such as machine learning, where they can assist in clustering and classification tasks.

In this paper, we consider the convergence of the best proximity points for cyclic contractions by new iterations for the best proximity points. Hence, the paper introduces three new algorithms that extend the traditional Mann iteration by relaxing certain assumptions, thereby improving convergence rates. This advancement is significant as it provides more robust tools for researchers and practitioners facing complex problems where traditional methods may be insufficient.

It is noteworthy that new algorithms outperform the method of Theorem 2.5 and Theorem 2.6 in terms of both conditionality and convergence. Also, we introduce new hybrid algorithms for finding of the best proximity points in the setting of uniformly convex Banach spaces. At the end, we also give a numerical example to illustrate the convergence behavior of all algorithms and we compare the convergence behavior of iterations.

2. Preliminaries

In this section we give some preliminaries that needs for main results.

Theorem 2.1. ([3]) *Let Θ and Δ be nonempty closed convex subsets of a uniformly convex Banach space Ω and let $\Lambda : \Theta \cup \Delta \to \Theta \cup \Delta$ be a cyclic contraction. If Θ or Δ is boundedly compact, then Λ has a unique best proximity point.*

Lemma 2.2. ([17]) *Let Ω be a Banach space. Then Ω is uniformly convex if and only if for each $\epsilon > 0$, there is a continuous strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(s) = 0 \Leftrightarrow s = 0$ and*

$$\|\gamma\upsilon + (1-\gamma)\tau\|^2 \le \gamma\|\upsilon\|^2 + (1-\gamma)\|\tau\|^2 - \gamma(1-\gamma)\phi(\|\upsilon-\tau\|),$$

for all $\gamma \in [0,1]$ and $\upsilon, \tau \in \Omega$, where $\|\upsilon\| \le \epsilon$ and $\|\tau\| \le \epsilon$.

It is notable that if (Θ, Δ) is a nonempty, weakly compact and convex pair in Ω, then (Θ_0, Δ_0) is also nonempty, closed and convex pair. A set-valued mapping $\mathcal{P}_\Theta : \Omega \to 2^\Theta$ is called a *metric projection* if

$$\mathcal{P}_\Theta(\upsilon) := \{\tau \in \Theta : \|\upsilon - \tau\| = \mathrm{d}(\upsilon, \Theta)\}.$$

It is famous that if Θ is a nonempty, bounded, closed and convex of a uniformly convex Banach space Ω, then $\mathcal{P}_\Theta$ is a single-valued mapping. Furthermore, for a nonempty, closed and convex subset Θ of a Hilbert space $\mathbb{H}$, $\mathcal{P}_\Theta$ is nonexpansive. Hence, if $\upsilon \in \Delta_0$, then we have $\mathcal{P}_{\Delta_0}(\upsilon) \in \Theta$ such that $\|\upsilon - \mathcal{P}_{\Delta_0}(\upsilon)\| = d(\upsilon, \Theta)$. Also, if $\upsilon \in \Theta_0$, then we have $\mathcal{P}_{\Delta_0}(\upsilon) \in \Delta$ such that $\|\upsilon - \mathcal{P}_{\Theta_0}(\upsilon)\| = d(\upsilon, \Delta)$.

Lemma 2.3. ([1]) *Let (Θ, Δ) be a nonempty, bounded, closed and convex pair of a uniformly convex Banach space Ω. Suppose that $\mathcal{P} : \Theta_0 \cup \Delta_0 \to \Theta_0 \cup \Delta_0$ is defined as follows*

$$\mathcal{P}(\upsilon) = \begin{cases} \mathcal{P}_{\Theta_0}(\upsilon) & \text{if } \upsilon \in \Delta_0, \\ \mathcal{P}_{\Delta_0}(\upsilon) & \text{if } \upsilon \in \Theta_0. \end{cases} \tag{4}$$

Then the following statements hold:

(1)*For every* $\upsilon \in \Theta_0 \cup \Delta_0, \|\upsilon - \mathcal{P}\upsilon\| = \mathrm{d}(\Theta, \Delta)$.

(2)*For every* $(\upsilon, \tau) \in \Theta_0 \times \Delta_0, \|\mathcal{P}\upsilon - \mathcal{P}\tau\| = \|\upsilon - \tau\|$.

(3)$\mathcal{P}$ *is affine.*

(4)$\mathcal{P}^2|_{\Theta_0} = i_{\Theta_0}$ *and* $\mathcal{P}^2|_{\Delta_0} = i_{\Delta_0}$.

(5)$\mathcal{P}|_{\Theta_0}$ *and* $\mathcal{P}|_{\Delta_0}$ *are continuous.*

Lemma 2.4. ([17]) *Let* $\{\alpha_n\} \subset [0, \infty)$*,* $\{\beta_n\} \subset [0, \infty)$ *and* $\{\gamma_n\} \subset [0, 1)$ *be sequences such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \beta_n \ \forall n \in \mathbb{N},$$
$$\sum_{n=1}^{\infty} \gamma_n = \infty \ \textit{and} \ \sum_{n=1}^{\infty} \beta_n < \infty.$$

Then, $\lim_{n\to\infty} \alpha_n = 0$.

Theorem 2.5. ([1]) *Let* Ω *be a uniformly convex Banach space,* (Θ, Δ) *be a nonempty, disjoint, bounded, closed and convex pair,* $\Lambda : \Theta \cup \Delta \to \Theta \cup \Delta$ *be cyclic relatively nonexpansive and* $\tau_0 \in \Theta$*. Define*

$$\tau_{n+1} = (1 - \gamma_n)\tau_n + \gamma_n \Lambda^2 \tau_n, \ (n \geq 0), \tag{5}$$

where $\gamma_n \in (\gamma, 1 - \gamma)$ *and* $\gamma \in (0, 1/2]$*. Then we have* $\|\tau_n - \Lambda^2\tau_n\| \to 0$*. Also, if* $\Lambda^2(\Theta)$ *lies in a compact set and* $\mathrm{d}(\tau_n, \Theta_0) \to 0$*, then* $\{\tau_n\}$ *strongly converges to a fixed point of* Λ^2*.*
Also, Λ *satisfies the following condition*

$$\textit{If for } \upsilon \in \Theta_0, \quad \|\upsilon - \Lambda\upsilon\| > \mathrm{d}(\Theta, \Delta), \quad \textit{then } \|\Lambda^2\upsilon - \Lambda\upsilon\| < \|\upsilon - \Lambda\upsilon\|. \tag{6}$$

Furthermore $\{\tau_n\}$ *converges to a best proximity point of* Λ*.*

Theorem 2.6. ([6]) *Let* (Θ, Δ) *be a nonempty, disjoint, bounded, closed and convex pair in a uniformly convex Banach space* Ω*,* $\Lambda : \Theta \cup \Delta \to \Theta \cup \Delta$ *be a cyclic contraction and* $\tau_0 \in \Theta$*. Put*

$$\tau_{n+1} = (1 - \gamma_n)\tau_n + \gamma_n \Lambda^2 \tau_n, \quad n \geq 0, \tag{7}$$

where $\gamma_n \in (\gamma, 1 - \gamma)$ *and* $\gamma \in (0, 1/2]$*. Then the sequence* $\{\tau_n\}$ *strongly converges to a best proximity point of* Λ*.*

3. Generalizations of Mann's iterative algorithm

In this section, we present new iterations for finding the best proximity points. Throughout this section, we assume that Ω is a uniformly convex Banach space and (Θ, Δ) is a nonempty, disjoint, bounded, closed, and convex pair in Ω. First, consider the convergence of iterative algorithms for finding best proximity points for cyclic contractive mappings that is a new extension of the

Mann iteration. We give a new convergent theorem that has fewer conditions with respect to Theorem 2.5. We will omit the conditions $\mathrm{d}(\tau_n, \Theta_0) \longrightarrow 0$ and

$$(8) \quad \|\Lambda^2 \upsilon - \Lambda \upsilon\| < \|\upsilon - \Lambda \upsilon\| \quad \text{whenever} \quad \|\upsilon - \Lambda \upsilon\| > \mathrm{d}(\Theta, \Delta), \quad \forall \upsilon \in \Theta_0.$$

Theorem 3.1. *Let $\Lambda : \Theta \cup \Delta \to \Theta \cup \Delta$ be a cyclic relatively nonexpansive mapping. Also, let $\{\gamma_n\} \subseteq (\gamma, 1-\gamma)$, $\gamma \in (0,1)$ and $\{\tau_n\}$ be a sequence generated by $\tau_0 \in \Theta$ so that*

$$(9) \qquad \tau_{n+1} = (1-\gamma_n)\Lambda^3 \tau_n + \gamma_n \Lambda \tau_n \ \forall n \in \mathbb{N} \cup \{0\}.$$

If $\Lambda(\Theta)$ is compact, then $\{\tau_{2n}\}$ converges to $\upsilon \in \Theta$ such that $\|\upsilon - \Lambda \upsilon\| = \mathrm{d}(\Theta, \Delta)$.

Proof. From Theorem 2.1, we have a unique $q \in \Delta_0$ such that $\|q - Tq\| = \mathrm{d}(\Theta, \Delta)$. Since Λ is relatively nonexpansive, $\|\Lambda^2 q - \Lambda q\| = \mathrm{d}(\Theta, \Delta)$. Since (Θ, Δ) has P-property, we have $q = \Lambda^2 q$. Hence we have

$$\begin{aligned}
\|\tau_n - q\| &= \|(1-\gamma_{n-1})\Lambda \tau_{n-1} + \gamma_{n-1}\Lambda^3 \tau_{n-1} - (1-\gamma_{n-1})\Lambda^2 q - \gamma_{n-1}\Lambda^4 q\| \\
&\le (1-\gamma_{n-1})\|\tau_{n-1} - \Lambda q\| + \gamma_{n-1}\|\tau_{n-1} - \Lambda q\| \\
&\le \|\tau_{n-1} - \Lambda q\|.
\end{aligned}$$

Also,

$$\begin{aligned}
\|\tau_{n-1} - \Lambda q\| &= \|(1-\gamma_{n-2})\Lambda \tau_{n-2} + \gamma_{n-2}\Lambda^3 \tau_{n-2} - (1-\gamma_{n-2})\Lambda q - \gamma_{n-2}\Lambda^3 q\| \\
&\le (1-\gamma_{n-2})\|\Lambda \tau_{n-2} - \Lambda q\| + \gamma_{n-2}\|\Lambda^3 \tau_{n-2} - \Lambda^3 q\| \\
&\le \|\tau_{n-2} - q\|.
\end{aligned}$$

Thus $\{\|\tau_{2n} - q\|\}_{n \ge 1}$ is a nonnegative decreasing sequence. Suppose that

$$\lim_{n\to\infty} \|\tau_{2n} - q\| \ge \mathrm{d}(\Theta, \Delta).$$

By Lemma 2.2 there are strictly increasing and continuous functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ with $\psi(0) = \varphi(0) = 0$ such that

$$\begin{aligned}
\|\tau_{2n} - q\|^2 &= \|(1-\gamma_{2n-1})\Lambda \tau_{2n-1} + \gamma_{2n-1}\Lambda^3 \tau_{2n-1} - (1-\gamma_{2n-1})\Lambda q \\
&\quad - \gamma_{2n-1}\Lambda^2 q\|^2 \\
&= \|(1-\gamma_{2n-1})(\Lambda \tau_{2n-1} - \Lambda^2 q) + \gamma_n(\Lambda^3 \tau_{2n-1} - \Lambda^4 q)\|^2 \\
&\le (1-\gamma_{2n-1})\|\Lambda \tau_{2n-1} - \Lambda^2 q\|^2 + \gamma_{2n-1}\|\Lambda^3 \tau_{2n-1} - \Lambda^4 q\|^2 \\
&\quad - \gamma_{2n-1}(1-\gamma_{2n-1})\varphi(\|\Lambda \tau_{2n-1} - \Lambda^3 \tau_{2n-1}\| \\
&\le \|\tau_{2n-1} - \Lambda q\|^2 - \gamma_{2n-1}(1-\gamma_{2n-1})\varphi(\|\Lambda \tau_{2n-1} - \Lambda^3 \tau_{2n-1}\|).
\end{aligned}$$

On the other hand

$$\begin{aligned}\|\tau_{2n-1} - \Lambda q\|^2 &= \|(1-\gamma_{2n-2})(\Lambda\tau_{2n-2} - Tq) + \gamma_n(\Lambda^3\tau_{2n-2} - \Lambda^3 q)\|^2 \\ &\leq (1-\gamma_{2n-2})\|\Lambda\tau_{2n-2} - \Lambda q\|^2 + \gamma_{2n-2}\|\Lambda^3\tau_{2n-2} \\ &- \Lambda^3 q\|^2 - \gamma_{2n-2}(1-\gamma_{2n-2})\psi(\|\Lambda\tau_{2n-2} - \Lambda^3\tau_{2n-2}\|) \\ &\leq \|\tau_{2n-2} - q\|^2 - \gamma_{2n-2}(1-\gamma_{2n-2})\psi(\|\Lambda\tau_{2n-2} - \Lambda^3\tau_{2n-2}\|).\end{aligned}$$

If

$$\theta := \gamma_{2n-1}(1-\gamma_{2n-1})\varphi(\|\Lambda\tau_{2n-1} - \Lambda^3\tau_{2n-1}\|),$$

then

$$\begin{aligned}\varepsilon^2\psi(\|\Lambda\tau_{2n-1} - \Lambda^3\tau_{2n-1}\|) &< \gamma_{2n-1}(1-\gamma_{2n-1})\psi(\|\Lambda\tau_{2n-1} - \Lambda^3\tau_{2n-1}\|) + \theta \\ &\leq \|\tau_{2n} - q\|^2 - \|\tau_{2n-2} - q\|^2.\end{aligned}$$

Now, if $n \to \infty$, we have $\lim_{n\to\infty} \psi(\|\Lambda\tau_{2n-1} - \Lambda^3\tau_{2n-1}\|) = 0$. Since ψ is strictly increasing, we have $\|\Lambda\tau_{2n-1} - \Lambda^3\tau_{2n-1}\| \to 0$. Since Ω is a uniformly convex Banach space and Δ is weakly compact, $\{\tau_{2n-1}\}_{n\geq 1}$ has a weak convergent subsequence $\{\tau_{2n_k-1}\}_{k\geq 1}$, converging to some point $x^\star \in \Theta$. Therefore,

$$\begin{aligned}\|\Lambda\tau_{2n-1} - \Lambda^2\tau_{2n-1}\| &= \|\Lambda\tau_{2n-1} - \Lambda^3\tau_{2n-1}\| + \|\Lambda^2\tau_{2n-1} - \Lambda^3\tau_{2n-1}\| \\ &\leq \|\Lambda\tau_{2n-1} - \Lambda^3\tau_{2n-1}\| + k\|\Lambda\tau_{2n-1} - \Lambda^2\tau_{2n-1}\| \\ &+ (1-k)\mathrm{d}(\Theta,\Delta)\end{aligned}$$

hence

$$(1-k)\|\Lambda\tau_{2n-1} - \Lambda^2\tau_{2n-1}\| \leq \|\Lambda\tau_{2n-1} - \Lambda^3\tau_{2n-1}\| + (1-k)\mathrm{d}(\Theta,\Delta),$$

and so $\|\tau_{2n-1} - \Lambda\tau_{2n-1}\| \to \mathrm{d}(\Theta,\Delta)$. Now, since $\{\tau_{2n_k-1}\}_{k\geq 1}$, converges to $\tau^\star \in \Theta$ we have

$$\begin{aligned}\|\tau^\star - \Lambda\tau^\star\| &\leq \liminf_{k\to\infty} \|\tau_{2n_k-1} - \Lambda\tau_{2n_k-1}\| \\ &= \lim_{k\to\infty} \|\tau_{2n_k-1} - \Lambda\tau_{2n_k-1}\| = \lim_{n\to\infty} \|\tau_{2n-1} - \Lambda\tau_{2n-1}\| = \mathrm{d}(\Theta,\Delta).\end{aligned}$$

Since (Θ,Δ) has P-property, from the uniqueness of the best proximity point, $q = \tau^\star$ and therefore $\tau_n \to \tau^\star$. □

Theorem 3.2. *Let* Θ_0 *be a nonempty compact set and* $\Lambda : \Delta \to \Theta$ *be a nonexpensive mapping such that* $\Lambda\Delta_0 \subseteq \Theta_0$ *and* $f : \Theta \to \Theta$ *is a contractive mapping. Also suppose*

$$\tau_{n+1} = \gamma_n f(\tau_n) + (1-\gamma_n)\Lambda\mathcal{P}\tau_n,$$

for all $n \in \mathbb{N} \cup \{0\}$*, where* $\{\gamma_n\} \subseteq [0,1]$ *so that* $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$*. Then there is a subsequence of* $\{\tau_n\}$ *that converges to a best proximity point of* Λ*.*

Proof. From compactness of Θ_0, we have $\{\mathcal{P}\tau_n\}$ is bounded, and since $\|\mathcal{P}\tau_n - \tau_n\| = \mathrm{d}(\Theta, \Delta)$, $\{\tau_n\}$ is bounded. If $\Lambda = max\{sup\|f(\tau_n)\|, sup\|T\mathcal{P}\tau_n\|\}$ we have

$$\begin{aligned}\|\tau_{n+1} - \tau_n\| &= \|\gamma_n f(\tau_n) + (1-\gamma_n)\Lambda\mathcal{P}\tau_n - (\gamma_{n-1} f(\tau_{n-1}) \\ &\quad + (1-\gamma_{n-1})\Lambda\mathcal{P}\tau_{n-1})\| \\ &= \|\gamma_n f(\tau_n) - \gamma_n f(\tau_{n-1}) + \gamma_n f(\tau_{n-1}) + (1-\gamma_n)\Lambda\mathcal{P}\tau_n) \\ &\quad - [\gamma_{n-1} f(\tau_{n-1}) + (1-\gamma_n)\Lambda\mathcal{P}\tau_{n-1} - (1-\gamma_n)\Lambda\mathcal{P}\tau_{n-1} \\ &\quad + (1-\gamma_{n-1})\Lambda\mathcal{P}\tau_{n-1}]\| \\ &\leq \gamma_n a\|\tau_n - \tau_{n-1}\| + 2\Lambda|\gamma_n - \gamma_{n-1}| + (1-\gamma_n)\|\mathcal{P}\tau_n - \mathcal{P}\tau_{n-1}\|,\end{aligned}$$

where $a \in [0,1)$. We have $\|\tau_n - \mathcal{P}\tau_n\| = \|\tau_{n-1} - \mathcal{P}\tau_{n-1}\| = \mathrm{d}(\Theta, \Delta)$, and so from P-property $\|\tau_n - \tau_{n-1}\| = \|\mathcal{P}\tau_n - \mathcal{P}\tau_{n-1}\|$. Hence

$$\|\tau_{n+1} - \tau_n\| \leq (1 - \gamma_n(1-a))\|\tau_n - \tau_{n-1}\| + 2\Lambda|\gamma_n - \gamma_{n-1}|.$$

By Lemma 2.4, we have

$$\lim_{n\to\infty} \|\tau_{n+1} - \tau_n\| = 0 \textit{ and } \lim_{n\to\infty} \|\mathcal{P}\tau_{n+1} - \mathcal{P}\tau_n\| = 0,$$

also

$$\begin{aligned}\|\tau_n - \Lambda\mathcal{P}\tau_n\| &= \|\gamma_{n-1}\tau_{n-1} + (1-\gamma_{n-1})\Lambda\mathcal{P}\tau_{n-1} - \Lambda\mathcal{P}\tau_n\| \\ &\leq \gamma_{n-1}\|\tau_{n-1} - \Lambda\mathcal{P}\tau_n\| + (1-\gamma_{n-1})\|\Lambda\mathcal{P}\tau_n - \Lambda\mathcal{P}\tau_{n-1}\|\end{aligned}$$

and so

$$\lim_{n\to\infty} \|\tau_n - \Lambda\mathcal{P}\tau_n\| = 0. \tag{10}$$

Therefore

$$\begin{aligned}\|\tau_n - \Lambda\tau_n\| &\leq \|\tau_n - \Lambda\mathcal{P}\tau_n\| + \|\Lambda\tau_n - \Lambda\mathcal{P}\tau_n\| \\ &\leq \|\tau_n - \Lambda\mathcal{P}\tau_n\| + \|\tau_n - \mathcal{P}\tau_n\|.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|\tau_n - \Lambda\tau_n\| = \mathrm{d}(\Theta, \Delta). \tag{11}$$

Since Θ is compact, there is a subsequence $\{\tau_{n_k}\}$ of $\{\tau_n\}$ that converges to $\upsilon \in \Theta_0$ and so by (11) we have $\|\upsilon - \Lambda\upsilon\| = \mathrm{d}(\Theta, \Delta)$ and $\tau_n \to \upsilon$. □

In the following we consider the convergence of best proximity points for cyclic relatively nonexpansive.

Theorem 3.3. *Let $\Lambda : \Theta \cup \Delta \to \Theta \cup \Delta$ be a relatively nonexpansive mapping such that $\Lambda(\Theta) \subset \Delta$ and $\Lambda(\Delta) \subset \Theta$ and $d(a_0, b_0) = d(\Theta, \Delta)$. Suppose*

$$\tau_0 \in \Theta, \tau_{n+1} = \begin{cases} \gamma a_0 + (1-\gamma)\Lambda\tau_n, \ if\ n\ odd, \\ \gamma b_0 + (1-\gamma)\Lambda\tau_n, \ if\ n\ even, \end{cases}$$

where $n \in \mathbb{N} \cup \{0\}$ and $0 < \gamma < 1$. If either Θ or Δ is boundedly compact, then there is $\tau \in \Theta \cup \Delta$ such that $\tau_{2n} \to \tau$ and $\|\tau - \Lambda\tau\| = \mathrm{d}(\Theta, \Delta)$.

Proof. Choose $\tau_0 \in \Theta$ and let n be odd. Then we have

$$\begin{aligned} \|\tau_{n+1} - \tau_n\| &\leq \gamma\|a_0 - b_0\| + (1-\gamma)\|\Lambda\tau_n - \Lambda\tau_{n-1}\| \\ \|\tau_{n+1} - \tau_n\| &\leq \gamma\|a_0 - b_0\| + (1-\gamma)\|\tau_n - \tau_{n-1}\| \\ &\leq (1-(1-\gamma)^2)\|a_0 - b_0\| + (1-\gamma)^2\|\tau_n - \tau_{n-1}\| \\ &\vdots \\ &\leq (1-(1-\gamma)^n)\|a_0 - b_0\| + (1-\gamma)^n\|\tau_1 - \tau_0\| \\ &= (1-(1-\gamma)^n)\mathrm{d}(\Theta, \Delta) + (1-\gamma)^n\|\tau_1 - \tau_0\|. \end{aligned}$$

Thus $\|\tau_{n+1} - \tau_n\| \to \mathrm{d}(\Theta, \Delta)$. Now, we show that the sequences $\{\tau_{2n}\}$ and $\{\tau_{2n+1}\}$ are bounded. Since $\|\tau_{2n} - \tau_{2n+1}\|$ converges to $\mathrm{d}(\Theta, \Delta)$, it is enough to show that $\{\tau_{2n+1}\}$ is bounded. Suppose $\{\tau_{2n+1}\}$ is not bounded. Then for every $M > \mathrm{d}(\Theta, \Delta)$ there exists $n_0 \in \mathbb{N}$ such that

$$\|\tau_2 - \tau_{2n_0+1}\| > M \ and\ \|\tau_0 - \tau_{2n_0-1}\| < M.$$

Therefore

$$M < \|\tau_2 - \tau_{2n_0+1}\| \leq \|\tau_0 - \tau_{2n_0-1}\|,$$

and so $M < d(\Theta, \Delta)$ which is a contradiction. Since Θ is boundedly compact, there is a convergent subsequence $\{\tau_{2n_k}\}$ of $\{\tau_{2n}\}$ that $\{\tau_{2n_k}\}$ converges to some $\tau \in \Theta$. Now

$$\mathrm{d}(\Theta, \Delta) \leq \|\tau - \tau_{2n_k-1}\| \leq \|\tau - \tau_{2n_k}\| + \|\tau_{2n_k} - \tau_{2n_k-1}\|.$$

Thus, we have $\|\tau - \tau_{2n_k-1}\|$ converges to $\mathrm{d}(\Theta, \Delta)$. Since

$$\begin{aligned} \mathrm{d}(\Theta, \Delta) &\leq \|\tau_{2n_k} - \gamma\Theta_0 - (1-\gamma)\Lambda\tau\| \\ &\leq \gamma\|a_0 - b_0\| + (1-\gamma)\|\Lambda\tau_{2n_k-1} - \Lambda\tau\| \\ &\leq \gamma\mathrm{d}(\Theta, \Delta) + (1-\gamma)\|\tau_{2n_k-1} - \tau\|, \end{aligned}$$

and, since Λ is nonexpansive, we have $\|\tau - \Lambda\tau\| = \mathrm{d}(\Theta, \Delta)$.

□

4. **Numerical example**

In this section, our goal is to consider the convergence of the new iterative algorithms that is presented in this paper and we show that they converge to the best proximity point faster than algorithm of Theorem 2.5 and Theorem 2.6. Consider $\Omega = \mathbb{R}^2$ with the usual metric, $\Theta = [1, 2] \times [0, 2], \Delta = [-2, -1] \times [0, 2]$ and $\tau_0 \in \Theta$. Define the mapping $\Lambda : \Theta \cup \Delta \to \Theta \cup \Delta$ by

$$\Lambda(x) = \begin{cases} (-\frac{\tau_1}{3} - \frac{2}{3}, 2 - \sin\frac{\pi}{4}\tau_2), & x = (\tau_1, \tau_2) \in \Theta, \\ (-\frac{\tau_1}{3} + \frac{2}{3}, 2 - \sin\frac{\pi}{4}\tau_2), & x = (\tau_1, \tau_2) \in \Delta. \end{cases} \tag{12}$$

Obviously, Λ is cyclic on $\Theta \cup \Delta$ which satisfies the condition (6). Initially, we show that Λ is a cyclic contraction. For $\tau = (\tau_1, \tau_2) \in \Theta$ and $\nu = (\nu_1, \nu_2) \in \Delta$ we have

$$\begin{aligned} \|\Lambda\tau - \Lambda\nu\| &= \|(-\frac{\tau_1}{3} - \frac{2}{3} - [-\frac{\nu_1}{3} + \frac{2}{3}], sin\frac{\pi}{4}\nu_2 - \sin\frac{\pi}{4}\tau_2)\| \\ &\leq \sqrt{|\frac{\nu_1 - \tau_1}{3} - \frac{4}{3}|^2 + [\sin\frac{\pi}{4}\nu_2 - sin\frac{\pi}{4}\tau_2]^2} \\ &\leq \frac{1}{3}\|\nu - \tau\| + \frac{4}{3} \\ &= \frac{1}{3}\|\tau - \nu\| + \frac{2}{3}\mathrm{d}(\Theta, \Delta),\ k = \frac{1}{3}, \mathrm{d}(\Theta, \Delta) = 2 \\ &\leq k\|\nu - \tau\| + (1 - k)\mathrm{d}(\Theta, \Delta), \end{aligned}$$

that is, Λ is a cyclic contraction mapping and $\nu^* = (1, 1)$ is the best proximity point $\|\nu^* - \Lambda\nu^*\| = \|(1, 1) - (-1, 1)\| = \mathrm{d}(\Theta, \Delta)$. Now, all four iterative algorithms are presented.

(1) Suppose that $\gamma_n \in (\gamma, 1 - \gamma)$ and $\gamma \in (0, 1/2]$.

$$\tau_{n+1} = (1 - \gamma_n)\tau_n + \gamma_n\Lambda^2\tau_n, \quad \forall n \geq 0. \tag{I}$$

(2) Suppose $\{\gamma_n\} \subseteq (\gamma, 1 - \gamma)$, $\gamma \in (0, 1)$ and $\{\tau_n\}$ is a sequence generated by $\tau_0 \in \Theta$ and

$$\tau_{n+1} = (1 - \gamma_n)\Lambda^3\tau_n + \gamma_n\Lambda\tau_n,\ \forall n \geq 0. \tag{II}$$

(3) Suppose $n \geq 0$, $0 < \gamma < 1$,

$$\tau_{n+1} = \begin{cases} \gamma\Theta_0 + (1 - \gamma)\Lambda\tau_n,\ if\ n\ odd, \\ \gamma b_0 + (1 - \gamma)\Lambda\tau_n\ if\ n\ even, \end{cases} \tag{III}$$

where $a_0 = (1, 2)$ and $b_0 = (-1, 2)$.

(4) Suppose $f : \Theta \to \Theta$ is contractive and let $\{(\tau_n, \mathcal{P}\tau_n)\}$ in $\Theta_0 \times \Delta_0$ such that

$$\tau_{n+1} = \gamma_n f(\tau_n) + (1 - \gamma_n)\Lambda\mathcal{P}\tau_n, \tag{IV}$$

for all $n \geq 0$, where $\{\gamma_n\} \subseteq [0, 1]$ satisfies $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Let $\tau_0 \in \Theta$ be an arbitrary element and by considering the iterative sequence defined in (II), set

$$\tau_1 = (1 - \gamma_0)\Lambda^3\tau_0 + \gamma_0\Lambda\tau_0.$$

Since $\Lambda\tau_0 = -\frac{\tau_0}{3} - \frac{2}{3} \in \Delta$ *and* $\Lambda^3\tau_0 = -\frac{\tau_0}{3^3} - \frac{26}{27} \in \Delta$,

$$\tau_1 = -\frac{1}{3^3}[(1+8\gamma_0)\tau_0 - (8\gamma_0 - 26)]$$

and so

$$\tau_2 = \frac{(-1)^2}{(3^3)^2}[(1+8\gamma_1)^2\tau_0 - (8\gamma - 26)^2]$$

and

$$\tau_{2n-1} = \frac{(-1)^{2n-1}}{(3^3)^{2n-1}}[(1+8\gamma)^{2n-1}\tau_0 - (8\gamma - 26)^{2n-1}]$$
$$\tau_{2n} = \frac{(-1)^{2n}}{(3^3)^{2n}}[(1+8\gamma)^{2n}\tau_0 - (8\gamma - 26)^{2n}].$$

It can be easily proved that $\tau_{2n} \to x^\star = (1,1) \in \Theta$ and $\tau_{2n-1} \to y^\star = (-1,1) \in \Delta$ is a best proximity point of Λ in Θ. Similarly, we estimate τ_n for (III) and (IV). It is notable that stopping criterion for our testing method is $E(n) := \|\tau_n - x^*\| \leq 10^{-7}$, where x^* is the best proximity point. From Table 1 we see (II), (III) and (IV) algorithms converge to the best proximity point faster than (I).

Algorithms	Number of Iterations
(I)	N=39
(II)	N=14
(III)	N=16
(IV)	N=8

TABLE 1. The rate of convergence for iterative algorithms (I)-(IV)

The enhanced convergence of Algorithms (II), (III), and IV can be attributed to:

- These algorithms use adaptive steps that adjust based on proximity to the best point, facilitating quicker convergence.
- By leveraging information from prior iterations, the algorithms make more informed decisions, reducing redundancy.
- This flexibility allows for broader applicability, leading to faster convergence in various scenarios.
- The algorithms exploit the geometric properties of the space and the structure of the mappings, keeping iterates in regions that promote rapid convergence.

Overall, the innovative approaches of Algorithms (II)–(IV) significantly improve their efficiency compared to Algorithm (I), as shown in Theorem 2.5 and Theorem 2.6. The results of Table 1 is illustrated with Figure 1.

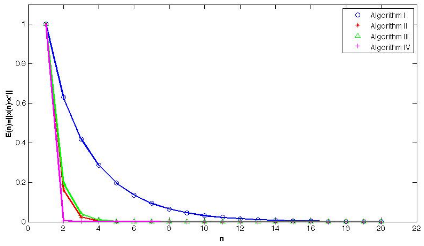

FIGURE 1. The rate of convergence for iterative algorithms (I)-(IV)

5. Conclusion and future work

In conclusion, this paper has successfully addressed the convergence of iterative algorithms for finding best proximity points in the context of cyclic contractive mappings. By extending the Mann iteration and relaxing certain assumptions, we have introduced three new algorithms (II), (III), and (IV) that demonstrate improved convergence rates compared to previous methods. The results indicate that these algorithms converge more quickly to the best proximity point, providing a significant advancement in the field.
Also looking ahead, several avenues for future research can be explored:

- Investigating additional relaxations of assumptions in the iterative processes could lead to even more robust algorithms applicable to a wider range of problems.
- Conducting extensive numerical experiments to compare the performance of the new algorithms against existing methods in various practical scenarios will help validate their efficiency and applicability.
- Exploring the application of these algorithms in diverse fields such as machine learning, optimization, and control systems could uncover new insights and enhance their utility.

By pursuing these directions, future research can build upon the foundation laid in this paper, contributing to the development of more effective algorithms for solving complex mathematical problems.

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