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A NOTE ON SENSITIVITY AND CHAOS ON HYPERSPACES OF UNIFORM SPACES

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Abstract. In this paper, we examine the dynamical characteristics of actions on the set of compact subsets within the phase space. Specifically, if X is identified as a uniform space, we denote $\mathscr{K}(X)$ as the collection of non-empty closed subsets of X equipped with the Hausdorff topology. If f represents a continuous self-map on X, there exist several naturally induced continuous self-maps on $\mathscr{K}(X)$. The principal focus of our investigation is the relationship between the dynamics of f and these induced mappings. For this purpose, we present topological notions of sensitivity and mixing properties pertinent to dynamical systems derived from uniform hyperspaces.

Keywords: sensitivity, transitivity, distal, hyperspaces, uniform space
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1. Introduction

A discrete dynamical system is typically characterized by a compact metric space X alongside a continuous function f that maps X to itself. Many significant properties of these systems are expressed in strictly topological terms, with transitivity, recurrence, and nonwandering points serving as pertinent examples. Conversely, other properties are defined in terms of the metric or the availability of an equivalent metric within the space, such as sensitivity, chain transitivity and recurrence, shadowing, and expansivity.

In certain instances, researchers prefer to investigate dynamical systems within broader topological frameworks. Uniform structures provide a method to generalize the notion of distance to topological spaces that do not necessarily possess a metric structure. By endowing a completely regular topological space with a uniformity, researchers can transcend the constraints imposed by a distance-based methodology in the topological theory of dynamical systems. The introduction of uniformity offers a systematic approach for regulating the distances between points within the space, even in the absence of a definable metric. Consequently, this allows for the articulation and examination of dynamical concepts within a more inclusive context.

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Utilizing this framework, researchers have achieved notable advancements in the generalization of various dynamical concepts to uniform spaces. For instance, Das et al. [5] have broadened the definitions of shadowing and chain recurrence to encompass topological spaces. Wu and Chen [1] established that a dynamical system situated in a uniform space is topologically chain mixing if and only if it possesses the property of being totally topologically chain transitive. Further findings include the demonstration that a dynamical system exhibiting ergodic shadowing is topologically chain transitive, and that a point-transitive dynamical system within a Hausdorff uniform space is either almost mean equicontinuous or mean sensitive [11]. Researchers have also made strides in the classification of topologically transitive dynamical systems, introducing innovative concepts such as sensitivity and syndetic sensitivity for non-autonomous dynamical systems situated in uniform spaces [9]. Moreover, the topological concept of average shadowing property has been articulated, with evidence presented to suggest that it implies topological chain transitivity [8].

The investigation of hyperspaces has its roots in early 20th century mathematics, with pioneering work by Hausdorff, Vietoris, Hahn, and Kuratowski. Bauer and Sigmund's 1975 contributions marked substantial progress, particularly in their examination of mappings between probability measures and hyperspaces. While research activity in hyperspace dynamics declined during the late 1900s, there has been renewed interest in recent years, with numerous studies utilizing Hausdorff metric topology approaches.

When considering a dynamical system (X, f) where X is a compact metric space and f is continuous, we naturally obtain associated mappings on related spaces. Important relationships exist between the system's dynamical features, the topological structure of its inverse limit space, and the corresponding shift map - connections that have been thoroughly explored in the literature [4, 6]

For any continuous map f on a compact metric space X, there is a canonical induced mapping on the hyperspace $\mathscr{K}(X)$ of nonempty closed subsets of X. This induced mapping remains continuous when $\mathscr{K}(X)$ is endowed with the Vietoris topology. This construction immediately raises a fundamental question: how do the dynamical properties of the original system (individual dynamics) relate to those of the induced system (collective dynamics)?

Recent years have seen growing attention to the study of hyperspace mappings [2, 6], building upon the foundational work of Bauer and Sigmund [3]. Their results demonstrate that the dynamics of $(\mathscr{K}(X), \tilde{f})$ exhibit greater complexity than those of the base system (X, f), as evidenced by various examples and theorems. These observations provide compelling motivation for further investigation of hyperspace dynamical systems.

The interplay between the orbital behaviors of the systems $(\mathscr{K}(X), \tilde{f})$ and (X, f) underscores the manner in which dynamics elucidate orbit evolution over time, particularly in relation to asymptotic behavior and resilience to minor perturbations. Under specific conditions, (X, f) and its corresponding subsystem situated within $(\mathscr{K}(X), \tilde{f})$ can be topologically conjugated by utilizing suitable hyperspace topologies. An invariant set derived from the original system is transformed into a fixed point within the hyperspace framework.

Definition 1.1. Let X be a non-empty set. A uniformity $\mathscr{U}$ on the set X is a family of subsets of the product $X \times X$ such that the following conditions hold:

(1) For any $E_1, E_2 \in \mathscr{U}$, the intersection $E_1 \cap E_2$ is also contained in $\mathscr{U}$, and if $E_1 \subset E_2$ and $E_1 \in \mathscr{U}$, then $E_2 \in \mathscr{U}$;
(2) Every set $E \in \mathscr{U}$ contains the diagonal $\Delta_X = \{(x, x) : x \in X\}$;
(3) If $E \in \mathscr{U}$, then $E^{-1} = \{(y, x) : (x, y) \in E\} \in \mathscr{U}$;
(4) For any $E \in \mathscr{U}$ there exists $\hat{E} \in \mathscr{U}$ such that $\hat{E} \circ \hat{E} \subset E$, where

$$\hat{E} \circ \hat{E} = \{(x, y) : (x, z) \in \hat{E}, (z, y) \in \hat{E}, \text{ for some } z \in X\}.$$

The set X with a uniformity $\mathscr{U}$ on it is called a *uniform space* and denoted by $(X, \mathscr{U})$ [7].

A subfamily $\mathscr{B}$ of $\mathscr{U}$ is called a base for $\mathscr{U}$ if each member of $\mathscr{U}$ contains a member of $\mathscr{B}$. Any compact topological space can be considered as a uniform space by equipping it with the uniform structure induced by the open coverings. This uniform structure allows us to study uniform properties of functions on compact topological spaces. An element of $\mathscr{U}$ is called an *entourage* of X. An entourage E is said to be *symmetric* if $E = E^{-1}$. One can easily check that symmetric entourages of $\mathscr{U}$ constitute a base for $\mathscr{U}$. Let $(X, \mathscr{U})$ be a uniform space. For $x \in X$ and $E \in \mathscr{U}$, the set $E[x] = \{y \in X : (x, y) \in E\}$ is called the *cross-section* of E at the point x.

Let $\tau_{\mathscr{U}} = \{A \subset X : \text{for each } a \in A, \text{ there exists } E \in \mathscr{U} \text{ with } E[a] \subset A\}$, then $\tau_{\mathscr{U}}$ is a topology on X defined by the uniformity $\mathscr{U}$ that is called the *uniform topology* on X. A mapping $f : X \longrightarrow Y$ from a uniform space X into a uniform space Y is said to be *uniformly continuous* if the inverse image $(f \times f)^{-1}(E)$ is an entourage of X for each entourage E of Y.

2. Expansivity on uniform spaces

Certain dynamical systems, like the doubling map on the circle, possess the characteristic that nearby points diverge over a positive time interval. Scholars have developed numerous methods to characterize this divergence of forward orbits. In our context, a continuous map or homeomorphism defined on a uniform space is deemed expansive if two points become separated by more than a specified distance after some time. Let $f : X \to X$ be a map on a uniform space $(X, \mathscr{U})$ and let D be an entourage of X. Define the following notations:

$$\Gamma^+(x, D, f) = \left(\bigcap_{i\in\mathbb{N}} F^{-i}(D)\right)[x]; \qquad \text{and } \Gamma(x, D, f) = \left(\bigcap_{i\in\mathbb{Z}} F^{-i}(D)\right)[x],$$

where $F = f \times f$.

Definition 2.1. We say that a map $f : X \to X$ on a uniform space $(X, \mathscr{U})$ is *positively expansive* (resp., expansive) if there exists an entourage D such that $\Gamma^+(x, D, f) = \{x\}$ (resp $\Gamma(x, D, f) = \{x\}$) for all $x \in X$.

It is straightforward to see that $f : X \to X$ is positively expansive if there exists an entourage D such that for all distinct points $x, y \in X$ there exists $n \geq 0$ such that $(f^n(x), f^n(y)) \notin D$. We shall call such a D a positive expansivity neighborhood (resp., expansivity neighborhood) of f. As one of the notions that are weaker than expansivity, we capture the notion which is called sensitive dependence on initial conditions.

Definition 2.2. [10] Given an entourage $E \in \mathscr{U}$, a point $x \in X$ is said to be an *E-sensitive point* of f if for any entourage $U \in \mathcal{U}$, there exist $y, z \in U[x]$ and $n \in \mathbb{N}$ such that $(f^n(y), f^n(z)) \notin E$. We denote by $\mathsf{sen}(E, f)$ the set of all E-sensitive points of f and define $\mathsf{sen}(f) = \bigcup_{E \in \mathscr{U}} \mathsf{sen}(f, E)$. The map f is said to be *topologically sensitive* if $\mathsf{sen}(f) = X$.

It is easy to see that (X, f) is topologically sensitive if there exists $D \in \mathscr{U}$ such that for any $x \in X$ and any $E \in \mathscr{U}$, there exists $y \in E[x]$ and $n \in \mathbb{N}$ such that $(f^n(x), f^n(y)) \notin D$.

Definition 2.3. [10] A continuous map $f : (X, \mathscr{U}_X) \to (Y, \mathscr{U}_Y)$ is called perfect if it is a closed map and $f^{-1}(y)$ is compact for all $y \in Y$.

Definition 2.4. [11] Let $f : X \longrightarrow X$ be a homeomorphism of a uniform compact Hausdorff space. Points $x, y \in X$ are called *proximal* if closure $\overline{\mathcal{O}((x, y))}$ of the orbit of (x, y) under $f \times f$ intersects the diagonal $\Delta = \{(z, z) \in X \times X : z \in X\}$.

Every point is proximal to itself. That is, $\overline{\mathcal{O}((x, x))} \cap \Delta \neq \emptyset$. If $(X, \mathscr{U})$ is a compact uniform space and let $m \in \mathbb{N}$, then $x, y \in X$ are *proximal*, if for any entourage $E \in \mathscr{U}$ there exists integer $n_k > m$ such that $(f^{n_k}(x), f^{n_k}(y)) \in E$.

Definition 2.5. [11] If two points $x, y \in X$ are not proximal, i.e., if $\overline{\mathcal{O}((x, y))} \cap \Delta = \emptyset$, they are called *distal*.

A homeomorphism $f : X \longrightarrow X$ is distal if every pair of distinct points $x, y \in X$ is distal. If $(X, \mathscr{U})$ is a compact Hausdorff space, then points $x, y \in X$ are *distal* if there is an entourage $E \in \mathscr{U}$ such that $(f^n(x), f^n(y)) \notin E$ for all $n \in \mathbb{Z}$.

Definition 2.6. A homeomorphism f of a compact uniform space $(X, \mathscr{U})$ is said to be *equicontinuous*, if for any entourage $E \in \mathscr{U}$, there exists an entourage $D \in \mathscr{U}$ such that $y \in D[x]$ implies that $(f^n(x), f^n(y)) \in E$ for all $n \in \mathbb{Z}$.

Lemma 2.7. Let X be a uniform space. Let M be a subset of $X \times X$, then $\overline{M} = \cap\{U \circ M \circ U : U \in \mathscr{U}\}$. In particular $\overline{M} = \cap\{U \circ \overline{M} \circ U : U \in \mathscr{U}\}$.

Proof.

$$\begin{aligned}(x,y) \in \overline{M} &\Leftrightarrow (V[x] \times V[y]) \cap M \neq \emptyset \; for\ each\ symmetric\ member\ V\ of \mathscr{U} \\ &\Leftrightarrow (a,b) \in V[x] \times V[y] \; for\ some\ (a,b) \in M \\ &\Leftrightarrow (x,y) \in V[a] \times V[b] \; for\ some\ (a,b) \in M \\ &\Leftrightarrow (x,y) \in \cup_{(a,b)\in M} V[a] \times V[b] \\ &\Leftrightarrow (x,y) \in V \circ M \circ V \; for\ every\ symmetric\ member\ V\ of \mathscr{U}.\end{aligned}$$

Also if $U \in \mathscr{U}$ is arbitrary, then there is a symmetric member V of $\mathscr{U}$ such that $V \subset U$. Hence, $(x,y) \in \overline{M} \Leftrightarrow (x,y) \in \cap\{U \circ M \circ U : U \in \mathscr{U}\}$. □

Proposition 2.8. Let f be a homeomorphism of an infinite compact Hausdorff uniform space X. Then for every entourage $E \in \mathscr{U}$ there are distinct points $x_0, y_0 \in X$ such that $(f^n(x_0), f^n(y_0)) \in \overline{E}$ for all $n \in \mathbb{N}_0$.

Proof. Fix $E \in \mathscr{U}$. Let A be the set of natural numbers m for which there is a pair $x, y \in X$ such that

(1) $$(x,y) \notin E \; and \; (f^n(x), f^n(y)) \in \overline{E} \; for \; n = 1, \dots, m.$$

Let $M = 0$, if $A = \emptyset$, and $M = \sup A$, if $A \neq \emptyset$.
If $M = \infty$, then for every $m \in \mathbb{N}$ there is a pair x_m, y_m satisfying 1 . By compactness, there is a sequence $m_k \in X$ such that for every $m_k > k$, $x_{m_k} \longrightarrow x', y_{m_k} \longrightarrow y'$. Since x_m, y_m is distinct points, hence by 1 $(x', y') \notin E$. To do this, choose $\hat{E}$ such that $\hat{E}^3 \subset E$, since $x_{m_k} \longrightarrow x'$ and $y_{m_k} \longrightarrow y'$ there is K such that for every $k \geq K$, $(x_{m_k}, x') \in \hat{E}$, $(y_{m_k}, y') \in \hat{E}$. For the sake of contradiction, let $(x', y') \in \hat{E}$, then $(x_{m_k}, y_{m_k}) \in \hat{E} \circ \hat{E} \circ \hat{E} \subset E$, which is contradiction, hence $(x', y') \notin \hat{E}$. Note that f^j is continuous and $(f^j(x_{m_k}), f^j(y_{m_k})) \in \overline{E}$ for every $j \in \mathbb{N}$. For any $U \in \mathscr{U}$ there exists $K_0 \in \mathbb{N}$ such that $(f^j(x'), f^j(x_{m_k})) \in U$ and $(f^j(y'), f^j(y_{m_k})) \in U$ for all $k \geq K_0$. Hence $(f^j(x'), f^j(y')) \in U \circ \overline{E} \circ U$. Since U is arbitrary by Lemma 2.7 we have $(f^j(x'), f^j(y')) \in \overline{E}$. Thus, $x_0 = f(x')$, $y_0 = f(y')$ are the desired points. Suppose now that M is finite. Since any finite collection of iterates of f is equicontinuous, there is an entourage $D \in \mathscr{U}$ such that if $(x,y) \in D$, then $(f^n(x), f^n(y)) \in E$ for $0 \leq n \leq M$, the definition of M implies that $(f^{-1}(x), f^{-1}(y)) \in E$. For see this suppose that $(f^{-1}(x), f^{-1}(y)) \notin E$ then by 1 $(f^{n+1}(f^{-1}(x)), f^{n+1}(f^{-1}(y))) \in \overline{E}$ for $n = 1, \dots, n+1$, which is contradiction the maximality of M. By induction we see that $(f^{-j}(x), f^{-j}(y)) \in E$ for $j \in \mathbb{N}$ whenever $(x,y) \in D$. Let $\hat{D} \in \mathscr{U}$ such that $2\hat{D} \subset D$. Since X is compact and $\{Int_X(\hat{D}[x])\}$ is an open cover of X, there exist $x_1, x_2, \dots, x_m$ in X such that $\beta = \{Int_X(\hat{D}[x_j]))\}_{j=1}^m$ is a finite subcover with cardinality m. Since X is infinite, we can choose a set $F \subset X$ to be any collection of $m+1$ points. For each integer n the set $f^n(F)$ has $m+1$ points and so by the

pigeon-hole principle, for each $j \in \mathbb{Z}$, there exists a pair $a_j, b_j \in F$ such that $f^j(a_j)$ and $f^j(b_j)$ belong to the same subcover $B_j \in \beta$, so $(f^j(a_j), f^j(b_j)) \in D$. Thus for any $n \le j$, $(f^n(a_j), f^n(b_j)) \in E$. Since F is finite, there are distinct $x_0, y_0 \in F$ such that $a_j = x_0$ and $b_j = y_0$ for infinitely many positive j and hence $(f^n(x_0), f^n(y_0)) \in E$ for all $n \ge 0$. □

Proposition 2.9. Let f be an expansive homeomorphism of an infinite compact Hausdorff uniform space X. Then X contains a pair of distinct proximal points.

Proof. . Suppose that D is the expansive entourage of f. Choose $E \in \mathscr{U}$ such that $\overline{E} \subset D$. Then by the Proposition 2.8 there exist distict points $x, y \in X$ such that $(f^n(x), f^n(y)) \in E$ for all $n \ge 1$. Suppose that $\overline{\mathcal{O}((x,y))} \cap \Delta = \emptyset$. By compactness we can assume that $f^n(x) \to x'$ and $f^n(y) \to y'$, then $f^{m+n}(x) \to f^m(x')$ and $(f^{n+m}(y) \to f^m(y')$ for all $m \in \mathbb{Z}$. Let $U \in \mathscr{U}$ and choose large $n \ge -m$ such that $(f^{n+m}(x), f^m(x')) \in U$ and $f^{n+m}(y), f^m(y')) \in U$. Therefore $(f^m(x'), f^m(y')) \in U \circ E \circ U$ and by Lemma 2.7 we have $(f^m(x'), f^m(y')) \in \overline{E} \subset D$ which contradicts the definition of D. □

Corollary 2.10. Let f be an expansive homeomorphism of an infinite compact Hausdorff uniform space X. Then f is not distal.

Proposition 2.11. Let f be an equicontinuous homeomorphism of a compact Hausdorff uniform space X. Then f is distal.

Proof. Suppose the equicontinuous homeomorphism $f : X \longrightarrow X$ is not distal. Then there is a pair of distinct proximal points $x, y \in X$ and let $m \in \mathbb{N}$, so for any entourage $E \in \mathscr{U}$ there is integer $n_k > m$ such that $(f^{n_k}(x), f^{n_k}(y)) \in E$. Let $x_k = f^{n_k}(x)$ and $y_k = f^{n_k}(y)$. Suppose that $y \in E[x]$, then for any entourage $D \in \mathscr{U}$, there is some $k \in \mathbb{N}$ such that $(x_k, y_k) \in D$, but $(f^{-n_k}(x_k), f^{-n_k}(y_k)) \in E$, so f is not equicontinuous. □

3. Induced dynamics

In a uniform space $(X, \mathscr{U})$, we define hyperspaces of X as follows:

$$\mathscr{K}(X) = \{A \subset X : A \textit{ is closed and non} - \textit{empty}\},$$

$$\mathscr{C}(X) = \{A \in \mathscr{K}(X) : A \textit{ is compact}\},$$

$$\mathscr{F}_n(X) = \{A \in \mathscr{K}(X) : A \textit{ has at most } n \textit{ points}\}, n \in \mathbb{N},$$

$$\mathscr{F}(X) = \bigcup_{n=1}^{\infty} F_n(X) - \textit{the collection of all finite subsets of } X.$$

Let $(X, \mathscr{U})$ be a uniform space and $E \in \mathscr{U}$. If

$$2^E = \{(A, A') \in \mathscr{K}(X) \times \mathscr{K}(X) : A \subset E[A'],\ A' \subset E[A]\},$$

then it is easy to prove that the set $\mathfrak{B} = \{2^E : E \in \mathscr{U}\}$ is a base for a uniformity on $\mathscr{K}(X)$, that denoted by

$$2^{\mathscr{U}} = \{\mathcal{U} \subset \mathscr{K}(X) \times \mathscr{K}(X) : \text{there exists } E \in \mathscr{U} \text{ such that } 2^E \subset \mathcal{U}\}.$$

It is known that the topology induced by $2^{\mathscr{U}}$ coincides with the *Vietoris topology.* The Vietoris topology on $\mathscr{K}(X)$ is generated by the following subbase:

1. *Upper Vietoris topology*: For each open $U \subseteq X$, define

$$U^+ = \{A \in \mathscr{K}(X) \mid A \subseteq U\}.$$

These sets form the *upper Vietoris topology.*

2. *Lower Vietoris topology*: For each open $U \subseteq X$, define

$$U^- = \{A \in \mathscr{K}(X) \mid A \cap U \neq \emptyset\}.$$

These sets form the *lower Vietoris topology.*

The Vietoris topology is the smallest topology containing both the upper and lower Vietoris topologies. A basis for this topology consists of finite intersections of such sets:

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{n} U_i^- \cap V^+ \;\middle|\; U_1, \ldots, U_n, V \text{ open in } X \right\}.$$

If (X, d) is a compact metric space, the Vietoris topology coincides with the topology induced by the Hausdorff metric:

$$d_H(A, B) = \max\left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.$$

If X is Hausdorff, then $\mathscr{K}(X)$ with the Vietoris topology is also Hausdorff. If X is compact, then $\mathscr{K}(X)$ is compact under the Vietoris topology. If X is a compact metric space, the Vietoris topology is metrizable via the Hausdorff metric.

Let $f : X \longrightarrow X$ be a map. We define $\tilde{f} : \mathscr{K}(X) \to \mathscr{K}(X)$ by $\tilde{f}(A) = f(A)$ for all $A \in \mathscr{K}(X)$. Then $(\mathscr{K}(X), \tilde{f})$, $(\mathscr{F}(X), \tilde{f})$ and $(\mathscr{C}(X), \tilde{f})$ are called induced dynamical systems. If $f : X \to X$ is continuous, then the induced map $\tilde{f} : \mathscr{K}(X) \to \mathscr{K}(X)$ (where $\tilde{f}(A) = f(A)$) is also continuous in the Vietoris topology. Also $\tilde{f} : \mathscr{F}(X) \to \mathscr{F}(X)$ and $\tilde{f} : \mathscr{C}(X) \to \mathscr{C}(X)$ are well-defined continuous maps.

This contributes to our understanding of the interplay between individual and collective dynamics in dynamical systems and provides valuable insights into the behavior of induced maps on uniform hyperspaces. A dynamical system (X, f) is called *topologically transitive* if $N_f(U, V) = \{n \in \mathbb{N} \mid U \cap f^{-n}(V) \neq \emptyset\}$ is a non-empty set for any pair of nonempty open subsets U, V of X. A dynamical system (X, f) is called *topologically weakly mixing* if the product system $(X \times X, f \times f)$ is topologically transitive. Akin et.al. [2] studied the dynamical properties of actions on the space of compact subsets of the phase space. They studied the properties of transitive points of the induced system $(\mathscr{K}(X), \tilde{f})$ both topologically and dynamically. They proved the following main theorem.

Theorem 3.1. *Let $\mathcal{C}$ be a Cantor set. If f is a continuous map on a metric space X, then the following are equivalent:*

(1) (X, f) *is weakly mixing.*
(2) $(\mathcal{C}, \tilde{f})$ *is topologically transitive.*

Wang et.al. [10] proved that the sensitivity of the induced hyperspace system on metric spaces is equivalent to the collective sensitivity of the original system.

Example 3.2. *[10] Let $(\Sigma(p), \sigma)$ be the full two-sided p-shift, where*

$$\Sigma(p) = \{1, 2, \ldots, p\}^Z = \{s = (\ldots, s_{-1}, s_0, s_1, \ldots) | s_n \in \{1, 2, \ldots, p\}\},$$

and $\sigma(s) = t$ where $t_n = s_{n+1}$. The metric $\tilde{d}$ of $(\Sigma(p), \sigma)$ is defined by

$$\tilde{d}(s, t) = \sum_{n=-\infty}^{\infty} \frac{\delta(s_n, t_n)}{2^{|n|}},$$

where $\delta(a, b) = 1$ if $a \neq b$ and $\delta(a, b) = 0$ if $a = b$.

Let $X = \Sigma(p) \setminus \{(\ldots, 1, 1, 1, \ldots)\}$ and $f = \sigma|_X$. As $\Sigma(p)$ is compact, X is Hausdorff locally compact second countable and $d = \tilde{d}|_{X \times X}$ is a metric. Since $(\Sigma(p), \sigma)$ is weakly mixing and (X, f) is a dense subsystem of $(\Sigma(p), \sigma)$, (X, f) is weakly mixing. Since $(\Sigma(p), \sigma)$ has a dense set of periodic points, so does (X, f). By Corollary 4.4 in [10], (X, f) is collectively sensitive.

Theorem 3.3. *[10] Let (X, f) be a dynamical system and Y be a dense subset of X, then (X, f) is sensitive if and only if (Y, f) is sensitive.*

Theorem 3.4. *[10] Let (X, f) be a dynamical system. Then $(\mathscr{C}(X), \tilde{f})$ is sensitive if and only if $(\mathscr{F}(X), \tilde{f})$ is.*

Proposition 3.5. Let (X, f) be a dynamical system. Then (X, f) is sensitive if and only if $(\mathscr{F}(X), \tilde{f})$ is.

Proof. If $(\mathscr{F}(X), \tilde{f})$ is sensitive then obviously (X, f) is sensitive. Conversely suppose that (X, f) is sensitive with sensitivity entourage D. Let $E \in \mathscr{U}$ be symetric and $A = \{x_1, \cdots, x_n\} \in \mathscr{F}(X)$, then for any i, there exists $y_i \in E[x_i]$ and n_i such that $(f^{n_i}(x_i), f^{n_i}(y_i)) \notin D$. Therefore $B = \{y_1, \cdots, y_n\} \in \mathscr{F}(X)$ and there exists $n \in \mathbb{N}$ such that $(\tilde{f}^n(A), f^n(B)) \notin 2^D$. Hence 2^D is a sensitivity entourage for $(\mathscr{F}(X), \tilde{f})$. □

The concepts of collective sensitivity and compact-type collective sensitivity are introduced as stronger conditions than traditional sensitivity for dynamical systems and Hausdorff locally compact second countable systems, respectively. Let (X, f) be a dynamical system, and let $D \in \mathscr{U}$ be an entourage. Dynamical system (X, f) is called *collectively sensitive* with collective sensitivity entourage D, if for any finitly many distinct points $x_1, x_2, \ldots, x_n$ and any entourage $E \in \mathscr{U}$, there exist the same number of distinct points $y_1, y_2, \ldots, y_n$ in X and a natural number $k \in \mathbb{N}$ such that $\cup_{i=1}^{n}(x_i, y_i) \subset E$ and there exist j with $1 \leq j \leq n$ such that

$$\cup_{i=1}^{n}(f^k(x_i), f^k(y_j)) \subset X \times X \setminus D$$

or

$$\cup_{i=1}^{n}(f^k(x_j), f^k(y_i)) \subset X \times X \setminus D.$$

Theorem 3.6. *Let (X, f) be a dynamical system. If (X, f) is weakly mixing, then (X, f) is collectively sensitive.*

Proof. Let p_1 , p_2 be two distinct points in X. Then there exists $U \in \mathscr{U}$ such that $U[p_1] \cap U[p_2] = \emptyset$. choose $D \in \mathscr{U}$ such that $D^{10} \subset U$. $D[p_1]$ and $D[p_2]$ are disjoint.We show that f is collectively sensitive with collective sensitivity entourage D. Let $x_1, x_2, \cdots, x_n$ be distinct points in X and $E \in \mathscr{U}$.Let $\{U_i\}_{i=1}^n \subset \mathscr{U}$. Since f is weakly mixing, f^{2n} is topologically transitive. Therefore there exists $k \in \mathbb{N}$ such that $f^k(U_i[x_i]) \cap D[p_1] \neq \emptyset$ and $f^k(U_i[x_i]) \cap D[p_2] \neq \emptyset$ for $i = 1, 2, \cdots, n$. Hence there exist $w_i, w_i' \in U_i[x_i]$ such that $f^k(w_i) \in D[p_1]$ and $f^k(w_i') \in D[p_2]$. This implies that $(f^k(w_i), f^k(w_i')) \subset X \times X \setminus D^8$. In particular, we can see that $(f^k(x_1), f^k(w_1)) \in X \times X \setminus D^4$ or $(f^k(x_1), f^k(w_1')) \in X \times X \setminus D^4$. By adequate selection of $y_i = w_i$ or $y_i = w_i'$ we obtain $\bigcup_{i=1}^n \{f^k(x_1), f^k(y_i)\} \subset X \times X \setminus D$. This implies that (X, f) is collectively sensitive with collective sensitivity entourage D.

□

Theorem 3.7. *Let (X, f) be a dynamical system. Then $(\mathscr{F}(X), \tilde{f})$ is sensitive if and only if (X, f) is collectively sensitive.*

Proof. Suppose that $(\mathscr{F}(X), \tilde{f})$ be sensitive with sensivity entourage D. For any distinct points $x_1, x_2, \ldots, x_n$ of X. Put $A = \{x_1, x_2, \ldots, x_n\}$ and any entourage $E \in \mathscr{U}$, with $E^2 \cap \{(x_i, x_j) : 1 \leq i, j \leq n \textit{ and } i \neq j\} = \emptyset$. By the assumption, there exist $B \in \mathscr{F}(X)$ and $k \in \mathbb{N}$ satisfying $(A, B) \in 2^E$ and $(\tilde{f}^k(A), \tilde{f}^k(B)) \notin 2^D$. Since $(A, B) \in 2^E$, for any $y \in B$ there is only one index $1 \leq i \leq n$ such that $(y, x_i) \in E$. Put $B_i = \{y : y \in B \textit{ and } (y, x_i) \in E\}$ for $1 \leq i \leq n$. Obviously $B_i \neq \emptyset$ for each $1 \leq i \leq n$. Since $(\tilde{f}^k(A), \tilde{f}^k(B)) \notin 2^D$ we have two cases,

(i) $\tilde{f}^k(B) \nsubseteq D[\tilde{f}^k(A)]$

(ii) $\tilde{f}^k(A) \nsubseteq D[\tilde{f}^k(B)]$.

We consider these cases, separately. If Case (i) holds, then there exists $z \in B_i$ satisfying

$$\cup_{i=1}^{n}(f^k(z), f^k(x_i)) \subset X \times X \setminus D$$

for each i, we choose $y_i \in B_i$. In particular, choose $y_i = z$. Consequently we obtain $\cup_{i=1}^n (x_i, y_i) \in E$ and there exists an index j with $1 \leq j \leq n$ satisfying

$$\cup_{i=1}^{n}(f^k(x_i), f^k(y_j)) \subset X \times X \setminus D.$$

If case (ii) holds, then there exists an index j with $1 \leq j \leq n$ satisfying

$$\cup_{y \in B}(f^k(x_j), f^k(y)) \subset X \times X \setminus D.$$

For each i, we choose $y_i \in B$, we obtain that $\cup_{i=1}^n (x_i, y_i) \subset E$ and $\cup_{i=1}^n (f^k(x_j), f^k(y_i)) \subset X \times X \setminus D$. Hence $(X, \tilde{f})$ is collectively sensitive.

Conversely, let $(X, \tilde{f})$ be collectively sensitive with a collective sensitivity entourage $D \in \mathscr{U}$, for any $A = \{x_1, x_2, \dots, x_n\} \in \mathscr{F}(X)$ and $E \in \mathscr{U}$, there exists n distinct points $y_1, y_2, \dots, y_n$ of X and $k \in \mathbb{N}$ such that $\cup_{i=1}^{n}(x_i, y_i) \subset E$ and there exist index j with $1 \leq j \leq n$ such that $\cup_{i=1}^{n}(f^k(x_i), f^k(y_j)) \subset X \times X \setminus D$ or $\cup_{i=1}^{n}(f^k(x_j), f^k(y_i)) \subset X \times X \setminus D$. Put $B = \{y_1, y_2, \dots, y_n\}$. Thus $(A, B) \subset 2^E$ and $\tilde{f}^k(B) \nsubseteq D[\tilde{f}^k(A)]$ or $\tilde{f}^k(A) \nsubseteq D[\tilde{f}^k(B)]$. This implies that $(\tilde{f}^k(A), \tilde{f}^k(B)) \notin 2^D$. Therefore $(\mathscr{F}(X), \tilde{f})$ is sensitive. □

A point $x \in X$ is a minimal point or almost periodic point if the subsystem $(\overline{orb_f(x)}, f)$ is minimal. A dynamical system (X, f) is called *pointwise minimal* if all points in X are minimal.

Proposition 3.8. Let (X, f) be a dynamical system. Then the following are equivalent:

(1) (X, f) is equicontinuous;
(2) $(\mathscr{K}(X), \tilde{f})$ is equicontinuous;
(3) $(\mathscr{K}(X), \tilde{f})$ is distal;
(4) $(\mathscr{K}(X), \tilde{f})$ is pointwise minimal.

Proof. $2 \Rightarrow 3$ and $3 \Rightarrow 4$ is clear.
$4 \Rightarrow 1$ Assume that $(\mathscr{K}(X), \tilde{f})$ is pointwise minimal. First we claim that (X, f) is distal. By the sake of contradiction let there exist $x, y \in X$ such that (x, y) is proximal, so $\overline{\mathcal{O}(x, y)} \cap \Delta \neq \emptyset$. Hence there are $z \in X$ such that

$$(z, z) \in \overline{\mathcal{O}(x, y)} = \cap_{E \in \mathscr{U}} E \circ \mathcal{O}(x, y) \circ E$$

So for each symmetric entourage E there exists $n \in \mathbb{N}$ such that $(f^n(x), z) \in E$ and $(f^n(y), z) \in E$. Note that $\{z\}$ and $\{x, y\}$ are minimal points in $\mathscr{K}(X)$. Hence $\{x, y\} \in \overline{orb(\{z, \tilde{f}\})}$ which implies that $x = y = z$, which is a contradiction, this implies that f is distal.
Now we aim to show that (X, f) is locally almost periodic. Fix any $x \in X$ and any open neighborhood U of x. Take any open set V such that $x \in V \subset \overline{V} \subset U$. Then $\overline{V} \in \langle U \rangle$ and since $(\mathscr{K}(X), \tilde{f})$ is pointwise minimal, there is a syndetic set $F \subseteq \mathbb{Z}$ such that $\tilde{f}^n(\overline{V}) \in \langle U \rangle$ for any $n \in F$. That is $f^n V \subset f^n \overline{V} \subset U$ for all $n \in F$, showing that x is locally almost periodic point, which implies that (X, f) is equicontinuous.
$1 \Rightarrow 2$ Suppose that f is equicontinuous. We show that $\tilde{f}$ is so. Let $2^E \in 2^{\mathscr{U}}$, then by equicontinuoity of f there exists an entourage $D \in \mathscr{U}$ such that $(f \times f)^i(D) \subset E$ for all i. Assume that $A, B \in \mathscr{K}(X)$ and $(A, B) \in 2^D$, then $A \subset D[B]$ and $B \subset D[A]$, therefore we conclude that $f^i(A) \subset E[f^i(B)]$ and $f^i(B) \subset E[f^i(A)]$ for all $i \geq 0$. This implies that $(f^i(A), f^i(B)) \in 2^E$, which completes the proof. □

Example 3.9. Let $\Sigma = \{(x_j) : x_j \in \{0,1\},\ j = 0,1,2,\cdots\}$. For a subset $\mathbb{J}$ of $\mathbb{N}_0$, the upper density $\overline{d}$ of $\mathbb{J}$ is defined by

$$\overline{d}(\mathbb{J}) = \limsup_{n\to\infty} \frac{1}{n} |\mathbb{J} \cap \{0,1,\cdots,n-1\}|.$$

For each $\epsilon > 0$, let

$$U_\epsilon = \{\big((x_j),(y_j)\big) \in \Sigma \times \Sigma :\ \overline{d}\,(\{j \geq 0 :\ x_j \neq y_j\}) < \epsilon\}.$$

Then the family $\mathscr{B} = \{U_\epsilon : \epsilon > 0\}$ is a base for a non-metrizable uniformity $\mathscr{U}$ on Σ. Let $\sigma : \Sigma \to \Sigma$ be the shift map. Then $\sigma : (\Sigma, \mathscr{U}) \to (\Sigma, \mathscr{U})$ is uniformly continuous, and furthermore for any $\epsilon > 0$ and any $k \in \mathbb{N}$ we have

$$(\sigma(\mathbf{x}), \sigma(\mathbf{y})) \in U_\epsilon \Leftrightarrow (\mathbf{x}, \mathbf{y}) \in U_\epsilon$$

This fact shows that $\sigma : (\Sigma, \mathscr{U}) \to (\Sigma, \mathscr{U})$ is equicontinuous. Consider two points $\mathbf{x} = (0,0,0\cdots)$ and $\mathbf{y} = (y_j)$ with

$$\mathrm{y_j} = \begin{cases} 1 & j = 2^n \text{ for some } n \\ 0 & \text{o.w.} \end{cases}$$

Then points $\mathbf{x}$ and $\mathbf{y}$ are proximal. This implies that $\sigma : (\Sigma, \mathscr{U}) \to (\Sigma, \mathscr{U})$ is not distal.

conclusion

In this work, we have studied sensitivity in dynamical systems on hyperspaces, characterizing how chaotic behavior extends from the base space to its hyperspace. Our results establish sufficient conditions under which sensitivity is preserved, contributing to the understanding of complex dynamics in set-valued frameworks.

Furthermore, we have introduced a generalization to collective sensitivity in this setting, laying the groundwork for future research. Subsequent works will explore deeper connections between collective sensitivity, entropy, and other chaos-related properties in hyperspace dynamics, as well as potential applications in control and synchronization of set-valued systems.

References

[1] Ahmadi, S. A., Wu,X. & Chen, G. (2020) Topological chain and shadowing properties of dynamical systems on uniform spaces. *Topology Appl.*, 275, 107153 (11 pages).

[2] Akin, E., Auslander, J. & Nagar, A. (2017), Dynamics of induced systems. *Ergodic Theory Dynam. Systems* 37, no. 7, 2034–2059.

[3] Bauer, W. & Sigmund, K. (1975). Topological dynamics of transformations induced on the space of probability measures. *Monatsh. Math.* 79, 81–92.

[4] Darban Maghami, N., Ahmadi, S.A. & Shabani, Z. (2025). Strong chain transitivity in hyperspaces of uniform spaces, *Appl. Gen. Topol.* doi:10.4995/agt.2025.21100.

[5] Das, T., Lee,K., Richeson, D. & Wiseman, J. (2013). Spectral decomposition for topologically anosov homeomorphisms on noncompact and non-metrizable spaces. *Topology Appl.*, 160, no. 1, 149–158.

[6] Fernández, L., Good, C., Puljiz, M., & Ramírez, Á. (2015). Chain transitivity in hyperspaces. *Chaos Solitons Fractals* 81 part A, 83–90.
[7] Kelley, J. L. *General topology.* Springer-Verlag, New York-Berlin, 1975.
[8] Pirfalak,F., Ahmadi,S. A., Wu, X., & Kouhestani, N. (2021). Topological Average Shadowing Property on Uniform Spaces. *Qual. Theory Dyn. Syst.*, 20(2), no. 31 (15 pages).
[9] Salman, M., Wu, X., & Das, R. (2021). Sensitivity of nonautonomous dynamical systems on uniform spaces. *International Journal of Bifurcation and Chaos*, 31, no. 1, 2150017 (11 pages).
[10] Wang, Y., Wei, G. & Campbell, William H. (2009). Sensitive dependence on initial conditions between dynamical systems and their induced hyperspace dynamical systems. *Topology Appl.* 156, no. 4, 803–811.
[11] Wu,X., Liang, S., Ma, X., Lu, T. & Ahmadi, S. A. (2020). The mean sensitivity and mean equicontinuity in uniform spaces. *Int. J. Bifurcation and Chaos*, 2050122 (11 pages).

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