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ON NEW GENERALIZED CLOSURE OPERATOR IN THE FRAME OF IDEALS

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Abstract. This paper aims to propose and examine two new operators based on the fundamental structure "ideal" and the notion of "generalized" producing a new generalized ideal topological space. The produced generalized space is finer than the original spaces. Also, the introduced structures are explained in detail in terms of topological basic theories and some properties. Moreover, we obtain some results for the produced generalized ideal topological space. Further, we provide several essential results related to these new frameworks. We also provide some examples to further illustrate our discussions and related findings.

Keywords: ideal, $(\cdot)^{\bullet}$-operator, $\overrightarrow{\sum}$-operator, generalized topology.
2020 MSC: 54A05, 54B99, 54C60, 54C05

1. Introduction

The field of topology is essential to many areas of mathematics. Numerous scientific and social researchers have been interested in how various topological ideas might be applied to a variety of natural situations. Numerous new ideas have been added to topology, enriching it with a range of recently created fields of study. Closure space, proximity, filters [14], ideals [13], grills [6], and primals [3] are a few examples of the innovative structures that topologists have developed in an attempt to discover valid solutions to a number of these topological issues. Kuratowski created and studied the idea of ideals [14]. Additionally, a number of topologists have investigated this concept from different perspectives [15, 16]. The concept of ideals manifests as the dual structure of filters.

Kuratovski [14] introduced the concept of ideal in 1933, and several mathematicians have since examined it from a variety of angles. The notation $(\Xi, \nabla, \Im)$ indicates an ideal topological space(ITS, for short), which is a topological space (Ξ, ∇) with an ideal $\Im$ on Ξ. $\beth^{\diamond}$ represents the local function with regard to $\Im$ and ∇ for a subset $\beth \subseteq \Xi$. The definition of the local function $\beth^{\diamond}$ of $\beth \subseteq \Xi$ is $\beth^{\diamond}(\Im, \nabla) = \{\xi \in \Xi | (\forall \mathfrak{T} \in \nabla(\xi)) (\mathfrak{T} \cap \beth \notin \Im)\}$, where $\nabla(\xi)$ is the collection of all open subsets containing $\xi \in \beth$ (see [14]). Specifically, Vaidyanathaswamy [21] examined more detailed aspects of the local function

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in 1945. $*$-topology, a new topology introduced to the literature by the concept of the local function, was later studied by Samuel [18] in 1975 and a number of other researchers, such as Hayashi [11] in 1964 and Njastad [17] in 1966. After a 15-year gaps, Jankovic and Hamlett [13] revisited this topic in 1990. Along with compiling all of the information that is already accessible on the topic, they have also added some new findings. Around 2002, the generalized topological spaces (GTS, for short) were proposed [7]. Many of its features are examined in detail in [7,8,20], including the generalized neighbourhood, separation axioms, generalized interior of a set $\beth$, and generalized closure of a set $\beth$. The new terms "generalized continuity", "θ-continuity" and others draw a lot of attention to the idea of continuity (see [7,8]). The study examined a variety of generalized topologies, including "extremely disconnected generalized topology", "∇-and θ-modifications generalized topology," and others (see [1,5,7–10,20,22]).

Recently, Abbas *et al.* [2] introduced and examined new operators based on the structures "ideal" and "generalized" generating two GTSs. Moreover, they obtained bases for the generated GTSs. Further, they proposed the notion of topology suitable with an ideal.

In this paper, we define the operators $(\cdot)^{\bullet}_{\omega}$ and $\overrightarrow{\sum}_{\omega}$ with the help of the definition of GTSs [2] and the closure operator $\sum_{\omega}$ or $\omega - \sum$ [12] . We discuss innovative categories of operators namely ideal local operators. We generate a GTS via ideals by applying two new local functions. The produced generalized topological space $\nabla^{\bullet}_{\omega}$ is finer than both the original topological spaces ∇ and ∇_{ω}. Also, we study some essential features of these new structures as well as these new operators. According to the concept of ideals, the relationships between the generated GTS and the previous one are obtained. Further, we study the fundamental characteristics of the produced operators. Furthermore, we provide number of illustrative examples in addition to some connections.

Motivation and applications of the paper. The exploration of ideal generalized topological space (GTS) stems from the need to address limitations in classical topology and to provide more robust frameworks for analyzing complex mathematical structures. The motivation and applications behind developing GTS include:

(1) **Enhanced Flexibility:** GTS allows for the incorporation of ideals and other structures, enriching traditional topological concepts and enabling the study of new types of continuity and convergence.
(2) **Addressing Complex Problems:** The introduction of new operators and frameworks within GTS facilitates the resolution of various topological issues, particularly in the context of closure spaces and ideals.
(3) **Development of New Theories:** The exploration of GTS in the frame of ideals and their properties serves to extend existing theories in topology, leading to the formulation of new mathematical

(4) **Graph Theory:** The principles of GTS facilitate deeper explorations in graph theory, particularly in understanding connectivity and properties of networks.

2. Preliminaries

Throughout this paper, (Ξ, ∇) (briefly, Ξ) represents a topological space, unless specified otherwise. Moreover, $\beth, \daleth$ are two subsets of Ξ and 2^Ξ denotes the power set of Ξ. For any subset $\beth \subseteq \Xi$, the closure and interior are denoted by $\sum(\beth)$ and $int(\beth)$, respectively. The symbol $\nabla(\xi)$ or $O(\Xi, \xi)$ denotes the family of all subsets $\mathfrak{T} \in \nabla$ that contains ξ. The collection of all closed sets of Ξ is denoted $C(\Xi)$ and the collection of all closed sets of Ξ containing a point ξ of Ξ is denoted by $C(\Xi, \xi)$. The notion of ω-open was proposed in 2003 [4]. A subset $\beth \subseteq \Xi$ is said to be ω-open if there exists an open neighborhood $\mathfrak{T} \in \nabla$ such that $\xi \in \mathfrak{T}$ for all $\xi \in \beth$ and $\mathfrak{T} \cap \beth^c$ is countable. The symbol ∇_ω mentions the family of all ω-open sets in Ξ, that is, $\nabla_\omega = \{\beth \in \Xi |$ for all $\xi \in \beth$ there exists $\mathfrak{T} \in \nabla(\xi)$ such that $\xi \in \mathfrak{T}$ and $\mathfrak{T} \cap \beth^c$ is countable$\}$. The ω-interior of $\beth \subseteq \Xi$, denoted by $int_\omega(\beth)$, is the union of all subsets $\mathfrak{T} \in \nabla_\omega$. The set $[int_\omega(\beth)]^c = \omega - \sum(\beth^c) = \sum_\omega(\beth^c)$ is the ω-closure of $\daleth = \beth^c \subseteq \Xi$ and $\beth = \sum_\omega(\beth)$ means that $\beth$ is ω-closed. A set $\beth$ is ω-closed iff $\sum_\theta(\beth) = \{\xi \in \Xi : \overline{\mathfrak{T}} \cap \beth \neq \emptyset$ for all $\mathfrak{T} \in \nabla_\omega(\xi)\}$. Further, the family ∇_ω formed by all ω-open sets is a coarser topology on Ξ than the topology ∇. The symbol $\nabla_\omega(\xi)$ or $\omega O(\Xi, \xi)$ denotes the family of all subsets $\mathfrak{T} \in \nabla$ that contains ξ. The collection of all ω-closed sets of Ξ is denoted $\omega C(\Xi)$ and the collection of all ω-closed sets of Ξ containing a point ξ of Ξ is denoted by $\omega C(\Xi, \xi)$. The operator $\sum^* : 2^\Xi \to 2^\Xi$ denoted by $\sum^*(\beth) = \beth \cup \beth^*$ is a closure operator. The topology proposed by the closure operator $\sum^*$ is $\nabla^*(\Im, \nabla) = \{\beth \subseteq \Xi | \sum^*(\Xi \setminus \beth) = \Xi \setminus \beth\}$ and denoted by $*$-topology which is finer than the topology ∇. A subset $\beth$ of a space Ξ is said to be regular open if $\beth = \prod(\sum(\beth))$ [19]. A family $\mathfrak{G}$ is said to be GTS if $\emptyset \in \mathfrak{G}$ and arbitrary union of members of $\mathfrak{G}$ belongs to $\mathfrak{G}$ (see [7]). A subset $\beth \subseteq \Xi$ is said to be $\mathfrak{G}$-open if $\beth \in \mathfrak{G}$. $\beth$ is said to be $\mathfrak{G}$-closed if $\Xi - \beth \in \mathfrak{G}$. If $\mathfrak{G}$ is a GTS on Ξ and $\Xi \in \mathfrak{G}$, then $\mathfrak{G}$ is called a strong generalized topology or supra topology.

Definition 2.1. [13] Consider a non-empty set Ξ. An ideal on Ξ is a collection $\Im \subseteq 2^\Xi$ that meets the following criteria:

(1) $\emptyset \in \Im$,
(2) if $\beth \in \Im$ and $\daleth \subseteq \beth$, then $\daleth \in \Im$,
(3) if $\beth, \daleth \in \Im$, then $\beth \cup \daleth \in \Im$.

The triple $(\Xi, \nabla, \Im)$ formed by an ideal $\Im$ with a topological space (Ξ, ∇), is called an ideal topological space (briefly, ITS). Along this paper, all defined ideals are proper ideals, that is, $\Im \neq 2^\Xi$ and $\Im \neq \{\emptyset\}$.

Definition 2.2. [2] Let $(\Xi, \nabla, \Im)$ be an ITS. A map $(\cdot)^\bullet : 2^\Xi \to 2^\Xi$ defined by $\beth^\bullet(\Xi, \nabla, \Im) = \{\xi \in \Xi : ($ for all $\mathfrak{T} \in \nabla(\xi))(\beth^c \cup \mathfrak{T}^c \in \Im)\}$ is called the $\bullet$-local function of $\beth \subseteq \Xi$ utilizing an ideal $\Im$ and a topology ∇ on Ξ. We may write $\beth^\bullet_\Im$ as $\beth^\bullet(\Xi, \nabla, \Im)$ to specify the ideal as per our requirements.

Definition 2.3. [2] Let $(\Xi, \nabla, \Im)$ be an ITS. Then a map $\sum^\bullet : 2^\Xi \to 2^\Xi$ given by $\sum^\bullet(\beth) = \beth \cup \beth^\bullet$, is a generalized closure operator.

Definition 2.4. [2] Let $(\Xi, \nabla, \Im)$ be an ITS. Define $\nabla^\bullet$ as $\nabla^\bullet = \{\beth \subseteq \Xi : \sum^\bullet(\beth) = \beth^c\}$. Moreover, $\nabla^\bullet$ is a generalized topological space on Ξ formed by ∇.

3. The $(\cdot)^\bullet_\omega$ operator and its basic properties

In this section, we propose and study a novel operator in ITSs namely $(\cdot)^\bullet_\omega$ with the aid of several examples. Also, we derive some basic properties of the given operator. Furthermore, We define and study its generated generalized closure operator.

Definition 3.1. Let $(\Xi, \nabla, \Im)$ be an ITS. An operator $(\cdot)^\bullet_\omega : 2^\Xi \to 2^\Xi$ denoted by $\beth^\bullet_\omega = \{\xi \in \Xi : (\forall \mathfrak{T} \in \omega O(\Xi, \xi))(\beth^c \cup \mathfrak{T}^c \in \Im)\}$ is said to be the $\bullet_\omega$-local function of a subset $\beth \subseteq \Xi$ utilizing an ideal $\Im$ and a topological space ∇ on Ξ. Some times we use the notation $\beth^\bullet_\omega$ as $\beth^\bullet_\omega(\Xi, \nabla, \Im)$ to indicate the defined ideal and topology as per our requirements.

Remark 3.2. Let $(\Xi, \nabla, \Im)$ be an ITS and $\beth \subseteq \Xi$. The inclusions of $\beth^\bullet_\omega \subseteq \beth$ or $\beth \subseteq \beth^\bullet_\omega$ need not be true in general as shown by the following examples.

Example 3.3. *Let* $\Xi = \{\xi_1, \xi_2, \xi_3\}$, $\nabla = \{\emptyset, \{\xi_2\}, \{\xi_3\}, \{\xi_2, \xi_3\}, \{\xi_1, \xi_3\}, \Xi\}$ *and* $\Im = \{\emptyset, \{\xi_2\}, \{\xi_3\}, \{\xi_2, \xi_3\}\}$. *For the subset* $\beth = \{\xi_2, \xi_3\}$, *we get* $\beth^\bullet_\omega = \emptyset$ *but* $\beth = \{\xi_2, \xi_3\} \nsubseteq \emptyset = \beth^\bullet_\omega$.

Example 3.4. *Let* $\Xi = \{\xi_1, \xi_2, \xi_3\}$, $\nabla = \{\emptyset, \Xi\}$ *and* $\Im = \{\emptyset, \{\xi_2\}, \{\xi_3\}, \{\xi_2, \xi_3\}\}$. *For the subset* $\beth = \{\xi_1\}$, *we get* $\beth^\bullet_\omega = \Xi$ *but* $\beth^\bullet_\omega = \Xi \nsubseteq \{\xi_1\} = \beth$.

Theorem 3.5. *Let* $(\Xi, \nabla, \Im)$ *be an ITS with* $\Im \neq 2^\Xi$ *and* $\beth, \daleth \subseteq \Xi$. *Then, the following hold:*

(a) $\beth^\bullet \supseteq \beth^\bullet_\omega$,
(b) *if* $\beth \in \omega C(\Xi)$, *then* $\beth^\bullet_\omega \subseteq \beth$,
(c) $\emptyset^\bullet_\omega = \emptyset$,
(d) $\beth^\bullet_\omega \in \omega C(\Xi)$,
(e) $(\beth^\bullet_\omega)^\bullet_\omega \subseteq \beth^\bullet_\omega$,
(f) *if* $\beth \subseteq \daleth$, *then* $\beth^\bullet_\omega \subseteq \daleth^\bullet_\omega$,
(g) $\beth^\bullet_\omega \cup \daleth^\bullet_\omega \subseteq (\beth \cup \daleth)^\bullet_\omega$,
(h) $(\beth \cap \daleth)^\bullet_\omega = \beth^\bullet_\omega \cap \daleth^\bullet_\omega$.

Proof. (a) The proof is straightforward from the definition of ω-open and Definition 3.1.

(b) Let $\beth \in \omega C(\Xi)$ and $\xi \notin \beth$. Our aim is to show that $\xi \notin \beth^{\bullet}_{\omega}$.

$$\begin{aligned}
\beth \in \omega C(\Xi) \quad &\Rightarrow \quad \beth^c \in \omega O(\Xi) \Rightarrow (\exists\, \beth \subseteq \omega O(\Xi))\,(\beth^c = \bigcup \beth)\,(\xi \notin \beth \Rightarrow \xi \in \beth^c) \\
&\Rightarrow \quad (\exists \daleth \in \omega O(\Xi))(\xi \in \daleth \subseteq \beth^c) \\
&\Rightarrow \quad (\daleth \in \omega O(\Xi,\xi))(\daleth^c \cup \beth^c \supseteq (\beth^c)^c \cup \beth^c = \beth \cup \beth^c = \Xi \notin \Im) \\
&\Rightarrow \quad (\daleth \in \omega O(\Xi,\xi))(\daleth^c \cup \beth^c \notin \Im) \\
&\Rightarrow \quad \xi \notin \beth^{\bullet}_{\omega}.
\end{aligned}$$

(c) $\emptyset^{\bullet}_{\omega} = \{\xi \in \Xi : (\forall \mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup \emptyset^c = \mathfrak{T}^c \cup \Xi = \Xi \in \Im)\} = \emptyset$.

(d) Our aim is to show that $\beth^{\bullet}_{\omega} = \omega\text{-}\sum(\beth^{\bullet}_{\omega})$. We have always $\beth^{\bullet}_{\omega} \subseteq \omega\text{-}\sum(\beth^{\bullet}_{\omega})$.

Conversely, now let $\xi \in \omega\text{-}\sum(\beth^{\bullet}_{\omega})$ and $\mathfrak{T} \in \omega O(\Xi,\xi)$.

$$\begin{aligned}
(\mathfrak{T} \in \omega O(\Xi,\xi))(\xi \in \omega\text{-}\textstyle\sum(\beth^{\bullet}_{\omega})) \quad &\Rightarrow \quad \mathfrak{T} \cap \beth^{\bullet}_{\omega} \neq \emptyset \Rightarrow (\exists y \in \Xi)(y \in \mathfrak{T} \cap \beth^{\bullet}_{\omega}) \\
&\Rightarrow \quad (\exists y \in \Xi)(y \in \mathfrak{T})(y \in \beth^{\bullet}_{\omega}) \\
&\Rightarrow \quad (\mathfrak{T} \in \omega O(\Xi,y))(y \in \beth^{\bullet}_{\omega}) \\
&\Rightarrow \quad \mathfrak{T}^c \cup \beth^c \in \Im.
\end{aligned}$$

Then, we have $\xi \in \beth^{\bullet}_{\omega}$. Thus, $\omega\text{-}\sum(\beth^{\bullet}_{\omega}) \subseteq \beth^{\bullet}_{\omega}$. Therefore, $\omega\text{-}\sum(\beth^{\bullet}_{\omega}) = \beth^{\bullet}_{\omega}$. Hence, $\beth^{\bullet}_{\omega} \in \omega C(\Xi)$.

(e) It is clear from (b) and (d).

(f) Let $\beth \subseteq \daleth$ and $\xi \in \beth^{\bullet}_{\omega}$. Our aim is to show that $\xi \in \daleth^{\bullet}_{\omega}$.

$$\begin{aligned}
\xi \in \beth^{\bullet}_{\omega} \quad &\Rightarrow \quad \left.\begin{array}{r} (\forall \mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup \beth^c \in \Im) \\[2ex] \beth \subseteq \daleth \Rightarrow \daleth^c \subseteq \beth^c \end{array}\right\} \\
&\Rightarrow \quad \left.\begin{array}{r} (\forall \mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup \daleth^c \subseteq \mathfrak{T}^c \cup \beth^c \in \Im) \\[2ex] \Im \text{ is an ideal on } \Xi \end{array}\right\} \\
&\Rightarrow \quad (\forall \mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup \daleth^c \in \Im) \Rightarrow \xi \in \daleth^{\bullet}_{\omega}.
\end{aligned}$$

(g) Let $\beth, \daleth \subseteq \Xi$. Then, we have

$$\left.\begin{array}{l} \beth, \daleth \subseteq \Xi \Rightarrow \beth \subseteq \beth \cup \daleth \Rightarrow \beth^{\bullet}_{\omega} \subseteq (\beth \cup \daleth)^{\bullet}_{\omega} \\[2ex] \beth, \daleth \subseteq \Xi \Rightarrow \daleth \subseteq \beth \cup \daleth \Rightarrow \daleth^{\bullet}_{\omega} \subseteq (\beth \cup \daleth)^{\bullet}_{\omega} \end{array}\right\} \Rightarrow \beth^{\bullet}_{\omega} \cup \daleth^{\bullet}_{\omega} \subseteq (\beth \cup \daleth)^{\bullet}_{\omega}.$$

(h) It is clear from (f) that $\beth^{\bullet}_{\omega} \cap \daleth^{\bullet}_{\omega} \supseteq (\beth \cap \daleth)^{\bullet}_{\omega}$.

Conversely, let $\xi \in \beth^{\bullet}_{\omega} \cap \daleth^{\bullet}_{\omega}$.

$$\begin{aligned}
\xi \in \beth^{\bullet}_{\omega} \cap \daleth^{\bullet}_{\omega} \quad &\Rightarrow \quad (\xi \in \beth^{\bullet}_{\omega})(\xi \in \daleth^{\bullet}_{\omega}) \Rightarrow (\forall \mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup \beth^c \in \Im)(\mathfrak{T}^c \cup \daleth^c \in \Im) \\
&\Rightarrow \quad (\mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup \beth^c \in \Im)(\mathfrak{T}^c \cup \daleth^c \in \Im) \\
&\Rightarrow \quad (\mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup (\beth \cap \daleth)^c = (\mathfrak{T}^c \cup \beth^c) \cup (\mathfrak{T}^c \cup \daleth^c) \in \Im) \\
&\Rightarrow \quad \xi \in (\beth \cap \daleth)^{\bullet}_{\omega}.
\end{aligned}$$

Therefore $(\beth \cap \daleth)^{\bullet}_{\omega} \supseteq \beth^{\bullet}_{\omega} \cap \daleth^{\bullet}_{\omega}$. Hence, we have $(\beth \cap \daleth)^{\bullet}_{\omega} = \beth^{\bullet}_{\omega} \cap \daleth^{\bullet}_{\omega}$.

□

Theorem 3.6. *Let* $(\Xi, \nabla, \Im)$ *and* $(\Xi, \nabla, \mathcal{Q})$ *be two ITSs and* $\beth \subseteq \Xi$. *If* $\Im \subseteq \mathcal{Q}$, *then* $\beth^{\bullet}_{\omega}(\Im) \subseteq \beth^{\bullet}_{\omega}(\mathcal{Q})$.

Proof. Let $\xi \in \beth^{\bullet}_{\omega}(\Im)$ and $\Im \subseteq \mathcal{Q}$.

$$\xi \in \beth^{\bullet}_{\omega}(\Im) \Rightarrow (\forall U \in \omega O(\Xi, \xi))\,(U^c \cup \beth^c \in \Im) \Rightarrow \xi \in \beth^{\bullet}_{\omega}(\mathcal{Q}).$$

□

Theorem 3.7. *Let* $(\Xi, \nabla, \Im)$ *and* $(\Xi, \sigma, \Im)$ *be two ITSs and* $\beth \subseteq \Xi$. *If* $\nabla \subseteq \sigma$, *then* $\beth^{\bullet}_{\omega}(\Xi, \sigma, \Im) \subseteq \beth^{\bullet}_{\omega}(\Xi, \nabla, \Im)$.

Proof. Let $\xi \in \beth^{\bullet}_{\omega}(\Xi, \sigma, \Im)$ and $\nabla \subseteq \sigma$.

$$\left.\begin{array}{r} \xi \in \beth^{\bullet}_{\omega}(\Xi, \sigma, \Im) \Rightarrow (\forall U \in \omega O_{\sigma}(\Xi, \xi))\,(U^c \cup \beth^c \in \Im) \\ \nabla \subseteq \sigma \end{array}\right\}.$$

Thus, for all $U \in \omega O_{\nabla}(\Xi, \xi)$, we have $U^c \cup \beth^c \in \Im$. Hence, $\xi \in \beth^{\bullet}_{\omega}(\Xi, \nabla, \Im)$.

□

Theorem 3.8. *Let* $(\Xi, \nabla, \Im)$ *be an ITS with* $\Im \neq 2^{\Xi}$. *Then, for any two subsets* $\beth$ *and* $\daleth$ *of* Ξ, *the following hold:*

(1) $\beth^{\bullet}_{\omega} \subseteq \sum(\beth)$,
(2) $\sum(\beth^{\bullet}_{\omega}) \subseteq \sum(\beth)$,
(3) $\beth^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega} \subseteq (\beth \setminus \daleth)^{\bullet}_{\omega}$,
(4) $\beth^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega} = (\beth \setminus \daleth)^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega}$.

Proof. (1) Let $\xi \notin \sum(\beth)$. Our aim is to show that $\xi \notin \beth^{\bullet}_{\omega}$.

$$\left.\begin{array}{r} \xi \notin \sum(\beth) \Rightarrow (\exists U \in O(\Xi, \xi))(U \cap \beth = \emptyset) \\ O(\Xi, \xi) \subseteq \omega O(\Xi, \xi) \end{array}\right\} \Rightarrow (\exists U \in \omega O(\Xi, \xi))\,(\beth \subseteq U^c).$$

So, there exists $U \in \omega O(\Xi, \xi))$ such that $\Xi = \beth \cup \beth^c \subseteq U^c \cup \beth^c \notin \Im$. As a result, $\xi \notin \beth^{\bullet}_{\omega}$.

(2) Let $\beth \subseteq \Xi$. $\beth \subseteq \Xi \overset{(1)}{\Rightarrow} \beth^{\bullet}_{\omega} \subseteq \sum(\beth) \Rightarrow \sum(\beth^{\bullet}_{\omega}) \subseteq \sum(\sum(\beth)) = \sum(\beth)$.

(3) Let $\beth, \daleth \subseteq \Xi$. Then, we have

$$\begin{aligned} \beth, \daleth \subseteq \Xi &\Rightarrow \beth \subseteq (\beth \setminus \daleth) \cup \daleth \\ &\Rightarrow \beth^{\bullet}_{\omega} \subseteq [(\beth \setminus \daleth) \cup \daleth]^{\bullet}_{\omega} = (\beth \setminus \daleth)^{\bullet}_{\omega} \cup \daleth^{\bullet}_{\omega} \\ &\Rightarrow \beth^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega} \subseteq (\beth \setminus \daleth)^{\bullet}_{\omega}. \end{aligned}$$

(4) Let $\beth, \daleth \subseteq \Xi$. Then,we have

$$\left.\begin{array}{r} \beth, \daleth \subseteq \Xi \Rightarrow \beth \setminus \daleth \subseteq \beth \Rightarrow (\beth \setminus \daleth)^{\bullet}_{\omega} \subseteq \beth^{\bullet}_{\omega} \Rightarrow (\beth \setminus \daleth)^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega} \subseteq \beth^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega} \\ \beth, \daleth \subseteq \Xi \overset{(3)}{\Rightarrow} \beth^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega} \subseteq (\beth \setminus \daleth)^{\bullet}_{\omega} \Rightarrow (\beth^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega}) \setminus \daleth^{\bullet}_{\omega} = \beth^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega} \subseteq (\beth \setminus \daleth)^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega} \end{array}\right\}.$$

Hence, $\beth^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega} = (\beth \setminus \daleth)^{\bullet}_{\omega} \setminus \daleth^{\bullet}_{\omega}$.

□

Theorem 3.9. *Let* $(\Xi, \nabla, \Im)$ *be an ITS and* $\beth, \daleth \subseteq \Xi$. *If* $\beth \in \omega O(\Xi)$, *then* $\beth \cap \daleth^{\bullet}_{\omega} \subseteq (\beth \cap \daleth)^{\bullet}_{\omega}$.

Proof. Let $\sqsupset \in \omega O(\Xi)$ and $\xi \in \sqsupset \cap \urcorner^{\bullet}_{\omega}$.

$$\left.\begin{array}{r} \xi \in \sqsupset \cap \urcorner^{\bullet}_{\omega} \Rightarrow (\xi \in \sqsupset)(\xi \in \urcorner^{\bullet}_{\omega}) \Rightarrow (\xi \in \sqsupset)(\forall \mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup \urcorner^c \in \Im) \\ \sqsupset \in \omega O(\Xi) \Rightarrow (\exists \sqsupset \subseteq \omega O(\Xi))\,(\sqsupset = \bigcup \sqsupset) \Rightarrow (\exists \urcorner \in \omega O(\Xi))(\urcorner \subseteq \sqsupset) \end{array}\right\}.$$

So, for all $\mathfrak{T} \in \omega O(\Xi,\xi)$, we have $\mathfrak{T} \cap \urcorner \in \omega O(\Xi,\xi)$. Therefore

$$\mathfrak{T}^c \cup (\sqsupset \cap \urcorner)^c = (\mathfrak{T} \cap \sqsupset)^c \cup \urcorner^c \subseteq (\mathfrak{T} \cap \urcorner)^c \cup \urcorner^c \in \Im.$$

It follows that

$$\mathfrak{T}^c \cup (\sqsupset \cap \urcorner)^c \in \Im \text{ for all } \mathfrak{T} \in \omega O(\Xi,\xi).$$

Hence $\xi \in (\sqsupset \cap \urcorner)^{\bullet}_{\omega}$.

□

Corollary 3.10. *Let* $(\Xi, \nabla, \Im)$ *be an ITS and* $\sqsupset, \urcorner \subseteq \Xi$. *If* $\sqsupset$ *is open in* Ξ, *then* $\sqsupset \cap \urcorner^{\bullet}_{\omega} \subseteq (\sqsupset \cap \urcorner)^{\bullet}_{\omega}$.

Theorem 3.11. *Let* $(\Xi, \nabla, \Im)$ *be an ITS. Then, the following statements are equivalent:*

(a) $\Xi^{\bullet}_{\omega} = \Xi$;
(b) $\omega C(\Xi) \setminus \{\Xi\} \subseteq \Im$;
(c) $\sqsupset \subseteq \sqsupset^{\bullet}_{\omega}$ *for all* ω*-open subsets* $\sqsupset$ *of* Ξ.

Proof. (a) $\Rightarrow (b)$: Let $\Xi^{\bullet}_{\omega} = \Xi$.

$$\begin{array}{rcl} \Xi^{\bullet}_{\omega} = \Xi & \Rightarrow & (\forall \xi \in \Xi)(\xi \in \Xi^{\bullet}_{\omega}) \\ & \Rightarrow & (\forall \xi \in \Xi)(\forall \mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup \Xi^c = \mathfrak{T}^c \in \Im) \\ & \Rightarrow & (\forall \urcorner \in \omega C(\Xi) \setminus \{\Xi\})(\urcorner \in \Im) \\ & \Rightarrow & \omega C(\Xi) \setminus \{\Xi\} \subseteq \Im. \end{array}$$

(b) $\Rightarrow (a)$: Let $\xi \in \Xi$ and $\mathfrak{T} \in \omega O(\Xi,\xi)$.

$$\left.\begin{array}{r} \mathfrak{T} \in \omega O(\Xi,\xi) \Rightarrow \mathfrak{T}^c \in \omega C(\Xi) \setminus \{\Xi\} \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \mathfrak{T}^c \cup \Xi^c = \mathfrak{T}^c \cup \emptyset = \mathfrak{T}^c \in \Im.$$

Then, we have $\xi \in \Xi^{\bullet}_{\omega}$. Thus, $\Xi \subseteq \Xi^{\bullet}_{\omega} \subseteq \Xi$ and so $\Xi^{\bullet}_{\omega} = \Xi$.

(b) $\Rightarrow (c)$: Let $\sqsupset \in \omega O(\Xi)$.

$$\left.\begin{array}{r} \sqsupset \in \omega O(\Xi) \overset{\text{Theorem 3.9}}{\Rightarrow} \sqsupset \cap \Xi^{\bullet}_{\omega} \subseteq (\sqsupset \cap \Xi)^{\bullet}_{\omega} = \sqsupset^{\bullet}_{\omega} \\ \omega C(\Xi) \setminus \{\Xi\} \subseteq \Im \Rightarrow \Xi^{\bullet}_{\omega} = \Xi \end{array}\right\} \Rightarrow \sqsupset \subseteq \sqsupset^{\bullet}_{\omega}.$$

(c) $\Rightarrow (b)$: Let $\sqsupset \in \omega C(\Xi) \setminus \{\Xi\}$.

$$\beth \in \omega C(\Xi)\setminus\{\Xi\} \Rightarrow \left.\begin{array}{r}\beth^c \in \omega O(\Xi)\setminus\{\emptyset\}\\ \text{Hypothesis}\end{array}\right\} \Rightarrow \beth^c \subseteq (\beth^c)^\bullet_\omega \Rightarrow (\forall \xi \in \beth^c)(\xi \in (\beth^c)^\bullet_\omega)$$

$$\Rightarrow \left.\begin{array}{r}(\forall \xi \in \beth^c)(\forall \mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup (\beth^c)^c = \mathfrak{T}^c \cup \beth \in \Im)\\ \beth \subseteq \mathfrak{T}^c \cup \beth\end{array}\right\}.$$

Then, $\beth \in \Im$. Hence, we have $\omega C(\Xi)\setminus\{\Xi\} \subseteq \Im$.

□

Theorem 3.12. *Let* $(\Xi, \nabla, \Im)$ *be an ITS and* $\beth \subseteq \Xi$. *If* $\beth^\bullet_\omega \neq \emptyset$, *then* $\beth^c \in \Im$.

Proof. Let $\beth^\bullet_\omega \neq \emptyset$.

$$\left.\begin{array}{r}\beth^\bullet_\omega \neq \emptyset \Rightarrow (\exists \xi \in \Xi)(\xi \in \beth^\bullet_\omega) \Rightarrow (\forall \mathfrak{T} \in \omega O(\Xi,\xi))(\beth^c \subseteq \mathfrak{T}^c \cup \beth^c \in \Im)\\ \Im \text{ is an ideal on } \Xi\end{array}\right\}.$$

Hence $\beth^c \in \Im$. □

Corollary 3.13. *Let* $(\Xi, \nabla, \Im)$ *be an ITS with* $\Im \neq 2^\Xi$ *and* $\beth \subseteq \Xi$. *If* $\beth^c \notin \Im$, *then* $\beth^\bullet_\omega = \emptyset$.

Proof. It is clear from Theorem 3.12. □

Theorem 3.14. *Let* $(\Xi, \nabla, \Im)$ *be an ITS and* $\beth, \daleth \subseteq \Xi$. *Then,* $\beth^\bullet_\omega \setminus \daleth^\bullet_\omega \supseteq (\beth\setminus\daleth)^\bullet_\omega \setminus \daleth^\bullet_\omega$.

Proof. Let $\beth, \daleth \subseteq \Xi$.

$$\begin{aligned}\beth, \daleth \subseteq \Xi &\Rightarrow \beth\setminus\daleth \subseteq \beth\\ &\Rightarrow (\beth\setminus\daleth)^\bullet_\omega \subseteq \beth^\bullet_\omega\\ &\Rightarrow (\beth\setminus\daleth)^\bullet_\omega \setminus \daleth^\bullet_\omega \subseteq \beth^\bullet_\omega \setminus \daleth^\bullet_\omega.\end{aligned}$$

□

Theorem 3.15. *Let* $(\Xi, \nabla, \Im)$ *be an ITS and* $\beth, \daleth \subseteq \Xi$. *If* $\daleth^c \notin \Im$, *then*

$$(\beth\cup\daleth)^\bullet_\omega \supseteq \beth^\bullet_\omega \supseteq \beth\setminus\daleth)^\bullet_\omega.$$

Proof. Let $\beth, \daleth \subseteq \Xi$.

$$\left.\begin{array}{r}\beth, \daleth \subseteq \Xi \overset{\text{Theorem 3.14}}{\Rightarrow} \beth^\bullet_\omega \setminus \daleth^\bullet_\omega \supseteq (\beth\setminus\daleth)^\bullet_\omega \setminus \daleth^\bullet_\omega\\ \daleth^c \notin \Im \overset{\text{Corollary 3.13}}{\Rightarrow} \daleth^\bullet_\omega = \emptyset\end{array}\right\} \Rightarrow \beth^\bullet_\omega \supseteq (\beth\setminus\daleth)^\bullet_\omega.$$

$$\left.\begin{array}{r}\beth, \daleth \subseteq \Xi \overset{\text{Theorem 3.5}}{\Rightarrow} (\beth\cup\daleth)^\bullet_\omega \supseteq \beth^\bullet_\omega \cup \daleth^\bullet_\omega\\ \daleth^c \notin \Im \overset{\text{Corollary 3.13}}{\Rightarrow} \daleth^\bullet_\omega = \emptyset\end{array}\right\} \Rightarrow (\beth\cup\daleth)^\bullet_\omega \supseteq \beth^\bullet_\omega.$$

Hence $(\beth\cup\daleth)^\bullet_\omega \supseteq \beth^\bullet_\omega \supseteq (\beth\setminus\daleth)^\bullet_\omega$. □

4. The operator $\overset{\rightrightarrows}{\sum}_{\omega}$ and its generalized topology

The purpose of this section is to present a new generalized closure operator denoted by $\overset{\rightrightarrows}{\sum}_{\omega}$. This structure's fundamental properties are illustrated. Further, we study the associated generalized topology $\nabla^{\bullet}_{\omega}$.

Definition 4.1. Let $(\Xi, \nabla, \Im)$ be an ITS with $\Im \neq 2^{\Xi}$. Define an operator $\overset{\rightrightarrows}{\sum}_{\omega} : 2^{\Xi} \to 2^{\Xi}$ as $\overset{\rightrightarrows}{\sum}_{\omega}(\beth) = \beth \cup \beth^{\bullet}_{\omega}$, where $\beth \neq \emptyset$ is any subset of Ξ.

Theorem 4.2. *Let $(\Xi, \nabla, \Im)$ be an ITS with $\Im \neq 2^{\Xi}$ and $\beth, \daleth \subseteq \Xi$. Then, the following statements hold:*

(a) $\overset{\rightrightarrows}{\sum}_{\omega}(\emptyset) = \emptyset$,
(b) $\overset{\rightrightarrows}{\sum}_{\omega}(\Xi) = \Xi$,
(c) $\beth \subseteq \overset{\rightrightarrows}{\sum}(\beth) \subseteq \overset{\rightrightarrows}{\sum}_{\omega}(\beth)$,
(d) *if* $\beth \subseteq \daleth$, *then* $\overset{\rightrightarrows}{\sum}_{\omega}(\beth) \subseteq \overset{\rightrightarrows}{\sum}_{\omega}(\daleth)$,
(e) $\overset{\rightrightarrows}{\sum}_{\omega}(\beth \cup \daleth) \supseteq \overset{\rightrightarrows}{\sum}_{\omega}(\beth) \cup \overset{\rightrightarrows}{\sum}_{\omega}(\daleth)$,
(f) $\overset{\rightrightarrows}{\sum}_{\omega}(\overset{\rightrightarrows}{\sum}_{\omega}(\beth)) = \overset{\rightrightarrows}{\sum}_{\omega}(\beth)$,
(g) $\overset{\rightrightarrows}{\sum}_{\omega}(\beth \cap \daleth) = \overset{\rightrightarrows}{\sum}_{\omega}(\beth) \cap \overset{\rightrightarrows}{\sum}_{\omega}(\daleth)$.

Proof. (a) The proof is straightforward from Definition 4.1.

(b) Since $\Xi^{\bullet}_{\omega} \subseteq \Xi$, we have $\overset{\rightrightarrows}{\sum}_{\omega}(\Xi) = \Xi \cup \Xi^{\bullet}_{\omega} = \Xi$.

(c) Since $\overset{\rightrightarrows}{\sum}_{\omega}(\beth) = \beth \cup \beth^{\bullet}_{\omega}$, we have $\beth \subseteq \overset{\rightrightarrows}{\sum}_{\omega}(\beth)$. Also, since $\beth^{\bullet} \subseteq \beth^{\bullet}_{\omega}$, we have $\overset{\rightrightarrows}{\sum}(\beth) \subseteq \overset{\rightrightarrows}{\sum}_{\omega}(\beth)$.

(d) Let $\beth \subseteq \daleth$.

$$\beth \subseteq \daleth \Rightarrow \beth^{\bullet}_{\omega} \subseteq \daleth^{\bullet}_{\omega} \Rightarrow \beth \cup \beth^{\bullet}_{\omega} \subseteq \daleth \cup \daleth^{\bullet}_{\omega} \Rightarrow \overset{\rightrightarrows}{\sum}_{\omega}(\beth) \subseteq \overset{\rightrightarrows}{\sum}_{\omega}(\daleth).$$

(e) It is straightforward from the definition of the operator $\overset{\rightrightarrows}{\sum}_{\omega}$ in Definition 4.1 and Theorem 3.5 (g).

(f) Let $\beth \subseteq \Xi$. It is clear from (c) that $\overset{\rightrightarrows}{\sum}_{\omega}(\beth) \subseteq \overset{\rightrightarrows}{\sum}_{\omega}(\overset{\rightrightarrows}{\sum}_{\omega}(\beth))$.

Conversely, let $\xi \in \overset{\rightrightarrows}{\sum}_{\omega}(\overset{\rightrightarrows}{\sum}_{\omega}(\beth)) = \overset{\rightrightarrows}{\sum}_{\omega}(\beth) \cup \left(\overset{\rightrightarrows}{\sum}_{\omega}(\beth)\right)^{\bullet}_{\omega}$ and $\mathfrak{T} \in \nabla(\xi)$. Then, we have the following two cases:

(i) $\xi \in \overset{\rightrightarrows}{\sum}_{\omega}(\beth)$. This means that $\overset{\rightrightarrows}{\sum}_{\omega}(\overset{\rightrightarrows}{\sum}_{\omega}(\beth)) \subseteq \overset{\rightrightarrows}{\sum}_{\omega}(\beth)$.

(ii) Let $\xi \notin \overrightarrow{\sum}_\omega(\sqsupset)$. Then, we have

$$\begin{aligned}
\xi \in (\sqsupset \cup \sqsupset^\bullet_\omega)^\bullet_\omega &\Rightarrow [\mathfrak{T}^c \cup (\sqsupset \cup \sqsupset^\bullet_\omega)^c \in \Im\ \forall \mathfrak{T} \in \nabla(\xi)][\exists\ \mathfrak{T} \in \nabla(\xi) :\ \mathfrak{T}^c \cup (\sqsupset)^c \notin \Im] \\
&\Rightarrow \mathfrak{T}^c \in \Im \text{ and } \sqsupset^c \notin \Im \\
&\Rightarrow \mathfrak{T}^c \cup \sqsupset^c \notin \Im \text{ for all } \mathfrak{T} \in \nabla(\xi^*), \xi^* \in \Xi \\
&\Rightarrow \sqsupset^\bullet_\omega = \emptyset \Rightarrow \xi \in \overrightarrow{\sum}_\omega(\sqsupset), \text{ which is a contradiction} \\
&\Rightarrow \overrightarrow{\sum}_\omega(\overrightarrow{\sum}_\omega(\sqsupset)) \subseteq \overrightarrow{\sum}_\omega(\sqsupset).
\end{aligned}$$

Thus, from (i), (ii) we have $\overrightarrow{\sum}_\omega(\overrightarrow{\sum}_\omega(\sqsupset)) \subseteq \overrightarrow{\sum}_\omega(\sqsupset)$. Consequentially, $\overrightarrow{\sum}_\omega(\overrightarrow{\sum}_\omega(\sqsupset)) = \overrightarrow{\sum}_\omega(\sqsupset)$.

(g) Let $\sqsupset, \urcorner \subseteq \Xi$.

$$\begin{aligned}
\overrightarrow{\sum}_\omega(\sqsupset \cap \urcorner) &= (\sqsupset \cap \urcorner) \cup (\sqsupset \cap \urcorner)^\bullet_\omega \\
&= (\sqsupset \cap \urcorner) \cup (\sqsupset^\bullet_\omega \cap \urcorner^\bullet_\omega) \\
&\subseteq (\sqsupset \cup \sqsupset^\bullet_\omega) \cap (\urcorner \cup \urcorner^\bullet_\omega) \\
&= \overrightarrow{\sum}_\omega(\sqsupset) \cap \overrightarrow{\sum}_\omega(\urcorner).
\end{aligned}$$

Conversely, let $\xi \in \overrightarrow{\sum}_\omega(\sqsupset) \cap \overrightarrow{\sum}_\omega(\urcorner)$.

$$\begin{aligned}
\xi \in \overrightarrow{\sum}_\omega(\sqsupset) \cap \overrightarrow{\sum}_\omega(\urcorner) &\Rightarrow (\xi \in \sqsupset)(\xi \in \sqsupset^\bullet_\omega) \text{ and } (\xi \in \urcorner)(\xi \in \urcorner^\bullet_\omega) \\
&\Rightarrow (\forall \mathfrak{T} \in \omega O(\Xi, \xi))(\xi \in \sqsupset \text{ and } \xi \in \urcorner)(\mathfrak{T}^c \cup \sqsupset^c \in \Im)(\mathfrak{T}^c \cup \urcorner^c \in \Im) \\
&\Rightarrow (\mathfrak{T} \in \omega O(\Xi, \xi))(\xi \in \sqsupset \text{ and } \xi \in \urcorner)(\mathfrak{T}^c \cup \sqsupset^c \in \Im)(\mathfrak{T}^c \cup \urcorner^c \in \Im) \\
&\Rightarrow (\mathfrak{T} \in \omega O(\Xi, \xi))(\xi \in \sqsupset \text{ and } \xi \in \urcorner) \\
&\Rightarrow (\mathfrak{T}^c \cup (\sqsupset \cap \urcorner)^c = (\mathfrak{T}^c \cup \sqsupset^c) \cup (\mathfrak{T}^c \cup \urcorner^c) \in \Im) \\
&\Rightarrow \xi \in \sqsupset \cap \urcorner \text{ and } \xi \in (\sqsupset \cap \urcorner)^\bullet_\omega.
\end{aligned}$$

Thus $\xi \in \overrightarrow{\sum}_\omega(\sqsupset \cap \urcorner)$. Therefore $\overrightarrow{\sum}_\omega(\sqsupset \cap \urcorner) \supseteq \overrightarrow{\sum}_\omega(\sqsupset) \cap \overrightarrow{\sum}_\omega(\urcorner)$. Hence, we have $(\sqsupset \cap \urcorner)^\bullet_\omega = \sqsupset^\bullet_\omega \cap \urcorner^\bullet_\omega$.

□

Corollary 4.3. *Let $(\Xi, \nabla, \Im)$ be an ITS. Then the operator $\overrightarrow{\sum}_\omega : 2^\Xi \to 2^\Xi$ defined by $\overrightarrow{\sum}_\omega(\sqsupset) = \sqsupset \cup \sqsupset^\bullet_\omega$, where $\sqsupset \subseteq \Xi$, is a generalized closure operator.*

Definition 4.4. Let $(\Xi, \nabla, \Im)$ be an ITS. Then, the collection $\nabla^\bullet_\omega = \{\sqsupset \subseteq \Xi : \overrightarrow{\sum}_\omega(\sqsupset^c) = \sqsupset^c\}$ is a generalized topology on Ξ introduced by the generalized topology ∇ and ideal $\Im$. Also, we can use the notation $\nabla^\bullet_{\omega(\Im)}$ instead of $\nabla^\bullet_\omega$ to illustrate the ideal as per our requirements.

Theorem 4.5. *Let $(\Xi, \nabla, \Im)$ be an ITS. Then, we have $\nabla \subseteq \nabla^\bullet \subseteq \nabla^\bullet_\omega$.*

Proof. We have $\nabla \subseteq \nabla^\bullet$ from [2]. Now, let $\beth \in \nabla^\bullet$. We will prove that $\beth \in \nabla^\bullet_\omega$.

$$\left.\begin{array}{r} \beth \in \nabla^\bullet \Rightarrow \sum^\bullet(\beth^c) = \beth^c \\ \beth \subseteq \Xi \Rightarrow (\beth^c)^\bullet_\omega \subseteq (\beth^c)^\bullet \Rightarrow \overrightarrow{\sum}_\omega(\beth^c) \subseteq \sum^\bullet(\beth^c) \end{array}\right\} \Rightarrow \left.\begin{array}{r} \overrightarrow{\sum}_\omega(\beth^c) \subseteq \beth^c \\ \beth^c \subseteq \overrightarrow{\sum}_\omega(\beth^c) \end{array}\right\}.$$

Therefore, $\beth^c = \overrightarrow{\sum}_\omega(\beth^c$. Hence, $\beth \in \nabla^\bullet_\omega$. □

Theorem 4.6. *Let $(\Xi, \nabla, \Im)$ be an ITS. Then, we have $\nabla \subseteq \nabla_\omega \subseteq \nabla^\bullet_\omega$.*

Proof. We have $\nabla \subseteq \nabla_\omega$ from [4]. Now, let $\beth \in \nabla_\omega$. We will prove that $\beth \in \nabla^\bullet_\omega$.

$$\beth \in \nabla_\omega \overset{\text{Theorem 3.5}}{\Rightarrow} (\beth^c)^\bullet_\omega \subseteq \beth^c \quad \Rightarrow \overrightarrow{\sum}_\omega(\beth^c) = \beth^c \cup (\beth^c)^\bullet_\omega \subseteq \beth^c \cup \beth^c = \beth^c.$$

Therefore, $\beth^c = \overrightarrow{\sum}_\omega(\beth^c)$. Hence, $\beth \subseteq \nabla^\bullet_\omega$. □

Remark 4.7. We have the following diagram from the definitions in the literature (see [2]) and Definition 4.4:

$$\begin{array}{ccccc} \nabla^\bullet_R\text{-open} & \rightarrow & \nabla^\bullet\text{-open} & \rightarrow & \nabla^\bullet_\omega\text{-open} \\ \uparrow & & \uparrow & & \uparrow \\ \nabla_\delta\text{-open} & \rightarrow & \nabla\text{-open} & \rightarrow & \nabla_\omega\text{-open} \end{array}$$

Remark 4.8. Let $(\Xi, \nabla, \Im)$ be an ITS. The following examples indicate that the converse of the implications stated in the diagram in Corollary 4.7 need not be true.

Example 4.9. *Define the topology $\nabla = \{U \mid 0 \notin U\} \cup \{\mathbb{R}\}$ with the ideal $\Im = 2^{\mathbb{R}\setminus\{0\}}$ on $\mathbb{R}$. Then, $[0, \infty) \in \nabla^\bullet_\omega$ but $[0, \infty) \notin \nabla_\omega$.*

Example 4.10. *Let $\Xi = \{\xi_1, \xi_2, \xi_3\}$ with the topology $\nabla = \{\emptyset, \Xi, \{\xi_1, \xi_2\}\}$. Define the ideal $\Im = \{\{\xi_1\}, \{\xi_3\}, \{\xi_1, \xi_3\}\}$ on Ξ. Then, $\{\xi_1, \xi_3\} \in \nabla^\bullet_\omega = \nabla_\omega = 2^\Xi$ but $\{\xi_1, \xi_3\} \notin \nabla^\bullet = \nabla$.*

Theorem 4.11. *Let $(\Xi, \nabla, \Im)$ be an ITS and $\beth \subseteq \Xi$. Then, the following statements hold:*

(a) *$\beth \in \nabla^\bullet_\omega$ iff for all ξ in $\beth$, there exists an ω-open set $\mathfrak{T}$ containing ξ such that $\mathfrak{T}^c \cup \beth \notin \Im$,*
(b) *if $\beth \notin \Im$, then $\beth \in \nabla^\bullet_\omega$.*

Proof. (a) Let $\beth \in \nabla^\bullet_\omega$.

$$\begin{aligned}\beth \in \nabla^\bullet_\omega &\Leftrightarrow \overrightarrow{\sum}_\omega(\beth^c) = \beth^c\\ &\Leftrightarrow \beth^c \cup (\beth^c)^\bullet_\omega = \beth^c\\ &\Leftrightarrow (\beth^c)^\bullet_\omega \subseteq \beth^c\\ &\Leftrightarrow \beth \subseteq ((\beth^c)^\bullet_\omega)^c\\ &\Leftrightarrow (\forall \xi \in \beth)(\xi \notin (\beth^c)^\bullet_\omega)\\ &\Leftrightarrow (\forall \xi \in \beth)(\exists \mathfrak{T} \in \omega O(\Xi,\xi))(\mathfrak{T}^c \cup (\beth^c)^c = \mathfrak{T}^c \cup \beth \notin \Im).\end{aligned}$$

(b) Let $\beth \notin \Im$ and $\xi \in \beth$.

$$\left.\begin{array}{r}(\mathfrak{T} := \Xi)(\xi \in \beth) \Rightarrow (\mathfrak{T} \in \omega O(\Xi,\xi))(\beth = \mathfrak{T}^c \cup \beth)\\ \beth \notin \Im\end{array}\right\} \Rightarrow \mathfrak{T}^c \cup \beth \notin \Im.$$

According to (a), $\beth \in \nabla^\bullet_\omega$.

□

Remark 4.12. Let $(\Xi, \nabla, \Im)$ be an ITS. The reverse of Theorem 4.11 (b) need not be true in general as shown by the following example.

Example 4.13. *Let* $\Xi = \{\xi_1, \xi_2, \xi_3\}$ *with the topology* $\nabla = \{\emptyset, \Xi, \{\xi_1\}, \{\xi_2\}, \{\xi_1, \xi_2\}\}$ *and* $\Im = \{\emptyset, \{\xi_1\}, \{\xi_2\}, \{\xi_1, \xi_2\}\}$. *Simple computations show that* $\nabla^\bullet_\omega = 2^\Xi$. *It is clear that the set* $\{\xi_2\}$ *belongs to both* $\nabla^\bullet_\omega$ *and* $\Im$.

Theorem 4.14. *Let* $(\Xi, \nabla, \Im)$ *be an ITS. If* $\Im = \{\emptyset\}$, *then* $\nabla^\bullet_\omega = 2^\Xi$.

Proof. We have always $\nabla^\bullet_\omega \subseteq 2^\Xi$. Now, let $\beth \in 2^\Xi$. Our aim is to show that $\beth \in \nabla^\bullet_\omega$.

$$\left.\begin{array}{l}\overrightarrow{\sum}_\omega(\beth^c) = \beth^c \cup (\beth^c)^\bullet_\omega\\ \Im = \{\emptyset\} \Rightarrow (\beth^c)^\bullet_\omega = \emptyset\end{array}\right\} \Rightarrow \overrightarrow{\sum}_\omega(\beth^c) = \beth^c \Rightarrow \beth \in \nabla^\bullet_\omega.$$

Then, we have $2^\Xi \subseteq \nabla^\bullet_\omega$. Hence $\nabla^\bullet_\omega = 2^\Xi$. □

Remark 4.15. Let $(\Xi, \nabla, \Im)$ be an ITS. The reverses of Theorem 4.11 (a) and Theorem 4.14 need not be true as shown by the following example.

Example 4.16. *Let* $\Xi = \{\xi_1, \xi_2, \xi_3\}$ *with the topology* $\nabla = \{\emptyset, \Xi, \{\xi_1\}, \{\xi_2\}, \{\xi_1, \xi_2\}\}$ *and* $\Im = \{\emptyset, \{\xi_1\}, \{\xi_2\}, \{\xi_1, \xi_2\}\}$. *Simple computations show that* $\nabla^\bullet_\omega = 2^\Xi$, *but* $\Im \neq \{\emptyset\}$.

Theorem 4.17. *Let* $(\Xi, \nabla, \Im)$ *be an ITS. Then, the family* $\beta = \{T \cap P | (T \in \nabla_\omega)(P \notin \Im)\}$ *is a base for the topology* $\nabla^\bullet_\omega$ *on* Ξ.

Proof. Let $\urcorner \in \beta$.

$$\left.\begin{array}{r}\urcorner \in \beta \Rightarrow (\exists T \in \nabla_\omega)(\exists P \notin \Im)(\urcorner = T \cap P) \\ \nabla_\omega \subseteq \nabla^\bullet_\omega\end{array}\right\} \overset{\text{Theorem 4.11}}{\Rightarrow} (T, P \in \nabla^\bullet_\omega)(\urcorner = T \cap P) \Rightarrow \urcorner \in \nabla^\bullet_\omega.$$

Then, we have $\beta \subseteq \nabla^\bullet_\omega$. Now, let $\sqsupset \in \nabla^\bullet_\omega$ and $\xi \in \sqsupset$. Our aim is to find $\urcorner \in \beta$ such that $\xi \in \urcorner \subseteq \sqsupset$.

$$\left.\begin{array}{r}\xi \in \sqsupset \in \nabla^\bullet_\omega \overset{\text{Theorem 4.11}}{\Rightarrow} (\exists \mathfrak{T} \in \omega O(\Xi, \xi))(\mathfrak{T}^c \cup \sqsupset \notin \Im) \\ \urcorner := \mathfrak{T} \cap (\mathfrak{T}^c \cup \sqsupset)\end{array}\right\} \Rightarrow (\urcorner \in \beta)(\xi \in \urcorner \subseteq \sqsupset).$$

As a result, β is a base for the topology $\nabla^\bullet_\omega$ on Ξ. □

Theorem 4.18. *Let* $(\Xi, \nabla, \Im)$ *and* $(\Xi, \nabla, \mathcal{Q})$ *be two ITSs. If* $\Im \subseteq \mathcal{Q}$, *then* $\nabla^\bullet_{\omega(\mathcal{Q})} \subseteq \nabla^\bullet_{\omega(\Im)}$.

Proof. Let $\sqsupset \in \nabla^\bullet_{\omega(\mathcal{Q})}$.

$$\left.\begin{array}{r}\sqsupset \in \nabla^\bullet_{\omega(\mathcal{Q})} \Rightarrow (\forall \xi \in \sqsupset)(\exists \mathfrak{T} \in \omega O(\Xi, \xi))(\mathfrak{T}^c \cup \sqsupset \notin \mathcal{Q}) \\ \Im \subseteq \mathcal{Q}\end{array}\right\} \Rightarrow (\forall \xi \in \sqsupset)(\exists \mathfrak{T} \in \omega O(\Xi, \xi))\ (\mathfrak{T}^c \cup \sqsupset \notin \Im) \Rightarrow \sqsupset \in \nabla^\bullet_{\omega(\Im)}.$$

□

5. Conclusion

Generalized topology had a great deal of literature and several features and hypotheses have been investigated.

Applying the concepts of "ideal" and "generalized open set", we continue on the results of [2]. We create and examine a new local functions in this research, denoted by $(\cdot)^\bullet_\omega$ and $\overrightarrow{\sum}_\omega$ with the help of the definition of the closure operator $\sum_\omega$ or $\omega - \sum$ [12], to derive a new GTS. Moreover, we extend several fundamental results related to the operators $(\cdot)^\bullet_\omega$ and $\overrightarrow{\sum}_\omega$. Furthermore, we give not only some relationships but also several examples.

In future work, we will define more notions that are related to the generated operators and their ITS $(\Xi, \nabla, \Im)$. Based on the proposed GTSs in ideals frame, we will introduce new operators and examine their main properties. We will create new styles of GTSs. Additionally, we will propose some results related to suitability. We will also suggest additional topological concepts, such connectedness of these spaces and separation axioms. We would like to link the suggested operators with concepts such as infra-topology, if that is feasible. Applying these results to graph theory is also our aim in the future.

6. Data Availability Statement

No data were used to support this study.

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8. Conflict of interest

The authors declare that they have no conflicts of interest.

9. Nomenclature

A list of symbols used in the paper and their meaning are provided in Table 1.

Symbol	**Description**
Ξ	Universal set
2^{Ξ}	Power set of Ξ
$\beth, \daleth$	Two subsets of $\beth$
ξ	an element of Ξ
∇	Topology on Ξ
$\sum(\beth)$ and $int(\beth)$	Closure and interior of ∇, respectively
$\nabla(\xi)$ or $O(\Xi, \xi)$	Neighborhood of ξ in ∇
$C(\Xi)$	Closed sets in ∇
$C(\Xi, \xi)$	Closed sets of ∇ containing ξ
∇_ω	Family of all ω-open sets in Ξ
$\nabla_\omega(\xi)$ or $\omega O(\Xi, \xi)$	Family of all ω-open sets in Ξ containing ξ
$\omega C(\Xi)$	ω-closed sets in ∇
$\omega C(\Xi, \xi)$	ω-closed sets of ∇ containing ξ
$\Im$	An ideal on Ξ
$\beth^\bullet$	$\bullet$-local function of $\beth$ by $\Im$, ∇ on Ξ
$\nabla^\bullet$	Generalized topology by $(.)^\bullet$
$\sum^\bullet(\beth)$	Generalized closure operator by $(.)^\bullet$
$\beth^\bullet_\omega$	$\bullet_\omega$-local function of $\beth$ by $\Im$ and ∇ on Ξ
$\nabla^\bullet_\omega$	Generalized topology by $(.)^\bullet_\omega$
$\overrightarrow{\sum}_\omega$	Generalized closure operator by $(.)^\bullet_\omega$

Table 1. Symbols used in the paper and its meaning

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