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SOME FUNDAMENTAL THEOREMS IN THE INTERVAL-VALUED FUNCTIONS AND TIME SCALES

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Abstract. This research investigates the topological structure of the space of interval-valued functions equipped with an order relation defined by open balls. The aim is to establish analogues of fundamental theorems in calculus, including Bolzano's theorem, the intermediate value theorem, and Rolle's theorem, within this interval-valued function setting. Moreover, we introduce a novel extension of the Mean Value Theorem to the context of interval-valued functions on time scales. Our findings contribute to the development of interval analysis and its applications in various fields.

Keywords: Interval analysis, Interval-valued function, Positive interval-valued function, Negative interval-valued function, Neutral interva-valued function, gH-diffrence, Intermediate value, time scales.
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1. Introduction

Many dynamic systems in engineering and applied sciences are inherently affected by uncertainty in their parameters. Such uncertainties may arise due to measurement errors, environmental fluctuations, model simplifications, or limitations in experimental accuracy. Therefore, the development of rigorous mathematical frameworks capable of incorporating and managing these uncertainties has become a critical area of research. One of the most effective approaches in this regard is interval analysis, which models data using intervals instead of precise numerical values. Initially introduced in the context of constrained optimization problems under uncertainty (e.g., [3, 4, 14]), interval analysis has significantly evolved and expanded into the study of interval differential equations (IDEs) [6, 9, 13]. These equations extend classical differential equations by allowing coefficients, initial conditions, and even solutions to be interval-valued, thus providing a more robust and realistic modeling of systems with incomplete or imprecise data. The foundational principles of interval arithmetic were established by Moore [8]. However, it soon became clear that the set of interval numbers, under classical arithmetic operations, does not form a vector space or an additive group. For instance, the identity $a - a = 0$, valid

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in the real number system, does not generally hold in the interval setting. To address this issue, Hukuhara [3] introduced a novel subtraction operation, now known as the Hukuhara difference ($\ominus$), in 1967. This difference is well-defined when one interval is a subset of the other, but it fails in the general case. To overcome these limitations, Markov [7] proposed an alternative, set-based definition of subtraction, and more recently, Stefanini [11] extended this concept by introducing the generalized Hukuhara difference ($\ominus_g$), which is applicable to arbitrary intervals while preserving desirable algebraic properties. These generalized differences have enabled the development of interval-valued vector spaces with operations that satisfy commutative and distributive laws, laying a solid foundation for further functional and topological analysis. Meanwhile, time scale calculus, originally introduced to unify the theories of discrete and continuous dynamical systems, has emerged as a valuable tool for modeling hybrid time structures. When integrated with interval analysis, it enables a refined study of dynamic systems that are subject to both temporal variability and parametric uncertainty, particularly in applications where data may be imprecise or sampled irregularly. In this work, we aim to contribute to the theoretical foundation of interval-valued analysis by investigating the behavior of interval-valued functions defined on time scales. In particular, we extend classical theorems such as the mean value theorem to this generalized setting. Our results provide a systematic and rigorous approach for studying interval differential equations on time scales, with potential applications in robust modeling, control theory, and the qualitative analysis of uncertain systems.

2. Preliminaries

2.1. **Introduction and fundamental properties of gH-difference.** Let $I\mathbb{R}$ denote the set of all bounded closed intervals on the real line $\mathbb{R}$. An interval is said to be degenerate if its left and right endpoints coincide; that is, for $a = [a^L, a^R]$, we have $a^L = a^R$. Two intervals are considered equal if their corresponding endpoints are identical.

Interval numbers exhibit subtraction properties that differ significantly from those of real numbers. In particular, subtraction does not function as the inverse operation of Minkowski addition. Although the cancellation law ($a + c = b + c$ implies $a = b$) holds for interval addition, the absence of additive inverses for most intervals—except for degenerate ones—prevents interval subtraction from mirroring classical arithmetic behavior. As a consequence, several foundational principles of number theory do not directly apply in the context of interval analysis.

To overcome these limitations, Hukuhara introduced a new subtraction operation called the *Hukuhara difference* (H-difference), denoted by $\ominus$. Specifically, $a \ominus b = c$ if and only if $a = b + c$. This definition establishes that subtracting an interval from itself yields the zero interval. However, its applicability is limited to cases where the width of a is greater than or equal to the width of b.

Stefanini [11] later extended this concept and established a more general framework for interval subtraction.

Definition 2.1 ([11])**.** The generalized Hukuhara difference (gH-difference) of two intervals a and b is defined as

$$a \ominus_g b = c,$$

where c satisfies one of the following conditions:

(i) $a = b + c$ if $\omega(a) \geq \omega(b)$, where $\omega(a)$ and $\omega(b)$ denote the widths of a and b, respectively;
(ii) $b = a + (-1)c$ if $\omega(a) < \omega(b)$, where $(-1)c = [-c^R, -c^L]$.

There exists a notable relationship between the H-difference and the gH-difference. When $\omega(a) \geq \omega(b)$, both operations yield the same result. In contrast, if $\omega(a) < \omega(b)$, the H-difference is not defined, but the gH-difference remains well-defined. Therefore, the gH-difference can be regarded as a generalization of the H-difference, offering enhanced flexibility in interval computations.

2.2. **Normed interval value space.** The set $I\mathbb{R}$ of all bounded closed intervals on $\mathbb{R}$ does not form a linear space under standard interval arithmetic, as defined in [11]. However, when subtraction is replaced by the generalized Hukuhara difference (gH-difference), $I\mathbb{R}$ exhibits quasi-linear structure. In this context, for each $a \in I\mathbb{R}$, there exists a unique $d \in I\mathbb{R}$ such that $a \ominus_g d = 0$.

To measure distances between interval elements, we use the Hausdorff-Pompeiu metric [7], defined by:

$$H(a, b) = \max\{|a^L - b^L|, |a^R - b^R|\},$$

where $a = [a^L, a^R]$ and $b = [b^L, b^R]$.

Aubin and Cellina [1] showed that $(I\mathbb{R}, H)$ is a complete metric space. This metric plays a key role in analyzing the properties of the H-difference and gH-difference.

Definition 2.2 ([15])**.** The set $\Omega = \{a \ominus a : a \in I\mathbb{R}\}$ is called the *null set* of $I\mathbb{R}$, serving as a generalized zero element.

Definition 2.3 ([15])**.** Two intervals $a, b \in I\mathbb{R}$ are said to be *almost identical*, denoted $a \overset{\Omega}{=} b$, if there exist $\omega_1, \omega_2 \in \Omega$ such that $a \oplus \omega_1 = b \oplus \omega_2$.

In particular, if $a \ominus b = c$, then generally $a \neq b \oplus c$, but $a \overset{\Omega}{=} b \oplus c$ holds, since there exists $\omega \in \Omega$ such that $a \oplus \omega = b \oplus c$.

Finally, define the interval norm $\|a\|_I := H(a, 0)$ for all $a \in I\mathbb{R}$. This function satisfies the properties of a norm, thereby endowing $I\mathbb{R}$ with the structure of a normed quasi-linear space. Here, $\| \cdot \|_I$ denotes the interval norm defined by the Hausdorff-Pompeiu distance on the space of intervals $I\mathbb{R}$.

2.3. **Space of interval-valued functions.** Let $J \subseteq \mathbb{R}$ be an interval. A function $f : J \to I\mathbb{R}$ is said to be continuous at $t_0 \in J$ if $\|f(t) \ominus_g f(t_0)\|_I \to 0$ as $t \to t_0$, where $\ominus_g$ denotes the generalized Hukuhara difference. The space of all continuous interval-valued functions on J is denoted by

$$C(J, I\mathbb{R}) = \{f : J \to I\mathbb{R} \mid f \text{ is continuous on } J\}.$$

A metric ρ on $C(J, I\mathbb{R})$ is defined by [14]

$$\rho(f, g) = \sup_{t \in J} H(f(t), g(t)),$$

where H denotes the Hausdorff-Pompeiu distance between intervals. It can be shown that ρ satisfies the properties of a metric, hence $(C(J, I\mathbb{R}), \rho)$ is a metric space.

Furthermore, a norm on $C(J, I\mathbb{R})$ is given by

$$\|f\|_C := \rho(f, 0) = \sup_{t \in J} H(f(t), 0) = \sup_{t \in J} |f(t)|,$$

which makes $C(J, I\mathbb{R})$ a normed quasi-linear space.

2.3.1. *Derivatives and order structure of interval-valued functions.* We now present various notions of differentiability for interval-valued functions, primarily based on the generalized Hukuhara difference (gH-difference) [12].

Definition 2.4 ([12])**.** Let $f : J \longrightarrow I\mathbb{R}$ be a function. We say that f is gH-differentiable at a point $t \in J$, if there exists an interval number $f'(t) \in I\mathbb{R}$ such that

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) \ominus_g f(t)}{h}, \tag{1}$$

where $\ominus_g$ denotes the generalized Hukuhara difference.

Definition 2.5 ([14])**.** For a function $f : J \longrightarrow I\mathbb{R}$ which is gH-differentiable at a point t within the interval $[a, b]$, f is termed (i)-gH-differentiable at t if

$$f'(t) = [(f^L)'(t), (f^R)'(t)],$$

and (ii)-gH-differentiable at t if

$$f'(t) = [(f^R)'(t), (f^L)'(t)].$$

We now introduce a partial ordering in the space $C(J, I\mathbb{R})$.

Definition 2.6. Let $f, g \in C(J, I\mathbb{R})$. We say $f \subseteq g$ if

$$f^L(t) \geq g^L(t) \quad \text{and} \quad f^R(t) \leq g^R(t), \quad \textit{for all } t \in J.$$

This defines a partial ordering on $C(J, I\mathbb{R})$.

Example 2.7. *Let* $f(t) = [t, 2 - t]$ *and* $g(t) = [t - 1, 3 - t]$ *for* $t \in [0, 1]$*. Then*

$$f^L(t) = t \geq t - 1 = g^L(t), \quad f^R(t) = 2 - t \leq 3 - t = g^R(t),$$

so $f \subseteq g$*.*

Definition 2.8. An interval-valued function $f \in C(J, I\mathbb{R})$ is called *positive* if $f^L(t) > 0$ for all $t \in J$, and *negative* if $f^R(t) < 0$ for all $t \in J$.

Example 2.9. *Let* $f(t) = \sin^2(\pi t) + 1$ *for* $t \in [0, 1]$. *Then* $f^L(t) = f^R(t) = \sin^2(\pi t) + 1 > 0$, *hence* f *is a positive interval-valued function.*

Definition 2.10. A function $f \in C(J, I\mathbb{R})$ is *maximal* if for all $g \in C(J, I\mathbb{R})$, $g \supseteq f$ implies $g = f$. Similarly, f is *minimal* if $g \subseteq f$ implies $g = f$.

Example 2.11. *Let* $f(t) = [t, 1]$ *for* $t \in [0, 1]$ *and consider the family*

$$\mathcal{F} = \{g \in C([0, 1], I\mathbb{R}) \mid g(t) \subseteq [t, 1]\}.$$

Then $f \in \mathcal{F}$ *is the maximal element with respect to the partial ordering* $\subseteq$.

We now introduce open balls and the induced topology on $C(J, I\mathbb{R})$.

Definition 2.12 ([15])**.** Given the non-negative real-valued function $\|\cdot\| : I\mathbb{R} \longrightarrow \mathbb{R}^+$, we consider the following conditions.
(i) $\|\alpha a\| = |\alpha|\|a\|$ for any $a \in I\mathbb{R}$ and $\alpha \in \mathbb{F}$;
(i') $\|\alpha a\| = |\alpha|\|a\|$ for any $a \in I\mathbb{R}$ and $\alpha \in \mathbb{F}$ with $\alpha \neq 0$.
(ii) $\|a \oplus b\| \leq \|a\| + \|b\|$ for any $a, b \in I\mathbb{R}$.
(iii) $\|a\| = 0$ implies $a \in \Omega$.
It is said that $\|\cdot\|$ satisfies the null condition when condition (iii) is replaced by $\|a\| = 0$ if and only if $a \in \Omega$.

Definition 2.13 ([15])**.** Different kinds of normed interval spaces are defined below.

- It is said that $(I\mathbb{R}, \|\cdot\|)$ is a pseudo-seminormed interval space if and only if conditions (i') and (ii) are fulfilled.
- It is said that $(I\mathbb{R}, \|\cdot\|)$ is a seminormed interval space if and only if conditions (i) and (ii) are fulfilled.
- It is said that $(I\mathbb{R}, \|\cdot\|)$ is a pseudo-normed interval space if and only if conditions (i'), (ii), and (iii) are satisfied.
- It is said that $(I\mathbb{R}, \|\cdot\|)$ is a normed interval space if and only if conditions $(i), (ii), (iii)$ are satisfied.

Definition 2.14 ([15])**.** Let $(J, \|\cdot\|)$ be a pseudo-seminormed interval space. Three types of open balls with radius ε are defined by

$$B^{\diamond}(a; \varepsilon) = \{a \oplus c : \|c\| < \varepsilon\},$$
$$B^{\star}(a; \varepsilon) = \{b : \|a \ominus b\| < \varepsilon\},$$
$$B(a; \varepsilon) = \{b : \|a \ominus_g b\| < \varepsilon\}.$$

We know that the inverse image of every continuous function is open. Let $f : J \longrightarrow I\mathbb{R}$ be a continuous interval-valued function. Then,

$$f^{-1}(B(a; \varepsilon)) = \{t | f(t) \in B(a; \varepsilon)\} = \{t | \|a \ominus_g f(t)\| < \varepsilon\}.$$

Since

$$\|a \ominus_g f(t)\| < \varepsilon \Longrightarrow H(a, f(t)) < \varepsilon,$$

it follows that

$$\max\{|a^L - f^L(t)|, |a^R - f^R(t)|\} < \varepsilon,$$

which implies

$$\begin{cases} |a^L - f^L(t)| < \varepsilon, \\ |a^R - f^R(t)| < \varepsilon, \end{cases}$$

and hence the preimage is open in both cases.

Topology on $C(J, I\mathbb{R})$ is defined as follows. For all $f \in C(J, I\mathbb{R})$

$$\begin{aligned} B_r^{\diamond}(f) &= \{f \oplus h : \ \|h\|_C \leq r, r \in \mathbb{R}\} \\ &= \{f \oplus h : \ \sup_{t \in I}\{H(h(t), 0)\} < r, r \in \mathbb{R}\} \\ &= \{f \oplus h : \ \sup_{t \in I}\{\max\{|h^L(t) - 0|, |h^R(t) - 0|\}\} < r\}. \end{aligned}$$

For all f in $C(J, I\mathbb{R})$

$$\begin{aligned} B_r(f) &= \{g | \ \rho(f, g) = \|f \ominus_g g\|_C \leq r, r \in R\} \\ &= \{g | \ \sup_{t \in I}\{H(f(t), g(t)) < r\}\} \\ &= \{g | \ \sup_{t \in I}\{\max\{|f^L(t) - g^L(t)|, |f^R(t) - g^R(t)|\}\} < r\}. \end{aligned}$$

3. Fundamental theorems for interval-valued functions

In this section, we investigate the adaptation of three fundamental results from differential calculus—namely, the Intermediate Value Theorem, Rolle's Theorem, and the Mean Value Theorem—in the setting of interval-valued functions. We begin by recalling the necessary definitions and then present the extended versions of these theorems along with rigorous proofs.

Theorem 3.1. *(Bolzano's Theorem)*

If $f : J \longrightarrow I\mathbb{R}$ is a countinuous interval-valued function, for any a and b in the interval J where $a < b$, the function f satisfies $f(a) < 0$ and $f(b) > 0$, then there is $J_1 \subseteq J$ such that for all $t \in J_1, 0 \in f(t)$.

Proof. It is evident that f^L, f^R satisfy the conditions of the theorem. Therefore, there are c_1, c_2 in J such that $f^L(c_1) = 0, f^R(c_2) = 0$. Since $f^L \leq f^R$ we have $c_2 \leq c_1$.

If $c_2 = c_1$ then $f(c_1) = 0$.

If $c_2 \neq c_1$ then for all $t \in [c_2, c_1]$ we have $0 \in f(t)$, because in this interval $f^L(t) \leq 0$ and $f^R(t) \geq 0$. □

Theorem 3.2. *(Intermediate Value Theorem)*

Let $f : J \longrightarrow I\mathbb{R}$ be a countinuous function, and $a, b \in J$ with $a < b$. If K is an interval satisfying $f(a) \subseteq K \subseteq f(b)$ then there is an interval $J_1 \subseteq [a, b]$ in which for any $t \in J_1$,

$$K \subseteq f(t).$$

Proof. For any $t \in J$ we set

$$R(t) = f(t) \ominus_g K.$$

Thus, since $f(a) < K, f(b) > K$, we have $R(a) < 0, R(b) > 0$. So by Theorem 3.1, there exists an interval $[a,b]$ such that for all $t \in [a,b]$, $0 \in R(t)$, so

$$0 \in f(t) \ominus_g K \Longrightarrow \begin{cases} f^L(t) - K^L \leq 0, \\ f^R(t) - K^R \geq 0, \end{cases}$$

which implies

$$\begin{cases} f^L(t) \leq K^L, \\ f^R(t) \geq K^R, \end{cases}$$

and thus

$$K \subseteq f(t).$$

Therefore, the proof is complete. □

Definition 3.3. Assume that $(I\mathbb{R}, \subseteq)$ represents a partially ordered set, $C = \{c_\alpha\}_{\alpha\in\Lambda} \subseteq I\mathbb{R}$ is a chain in $(I\mathbb{R}, \subseteq)$ if for all $\alpha, \beta \in \Lambda$,

$$C_\alpha \subseteq C_\beta \text{ or } C_\beta \subseteq C_\alpha.$$

Example 3.4. *Let $f : [0,1] \to I\mathbb{R}$ be an interval-valued function defined by*

$$f(t) = [0,t], \quad t \in [0,1].$$

Define the family $\mathcal{C} = \{f(t) \mid t \in [0,1]\} \subseteq I\mathbb{R}$. Then, for any $0 \leq t_1 < t_2 \leq 1$, we have

$$f(t_1) = [0,t_1] \subseteq [0,t_2] = f(t_2),$$

which shows that $\mathcal{C}$ is an ascending chain generated by the function f. Hence, $\mathcal{C}$ is a chain in $(I\mathbb{R}, \subseteq)$.

Theorem 3.5. *(Extremal Value Theorem)*
Let $f \in C(J, I\mathbb{R})$ then every closed chain $f(a_\lambda)_{\lambda\in\Lambda}$ has maximum and minimum points.

Proof. Let $f(a_\lambda)$ be a chain in $f(J)$, since f^L, f^R are continuous on J, $f^R(a_\lambda)$ has maximum in $\{a_\lambda\}$ and $f^L(a_\lambda)$ has minimum in $\{a_\lambda\}$, so every chain is bounded and closed so it has maximum and minimum. □

Corollary 3.6. *If two functions f^L, f^R are both strictly increasing (decreasing), then the function f has not maximal (minimal).*

Corollary 3.7. *Let $f \in C(J, I\mathbb{R})$ be gH-differentiable, if the derivative exists at the maximal or minimal points, then*

$$0 \in f^{'}(t).$$

Theorem 3.8. *(Rolle's Theorem)*
If an interval-valued function $f : J \longrightarrow I\mathbb{R}$ is continuous and gH-differentiable, for all $a, b \in J$, where $a < b$ and $f(a) = f(b)$, then there exists $t \in J$ such that $0 \in f^{'}(t)$.

Proof. Since f^L, f^R are continuous and differentiable, by assumption $f^L(a) = f^L(b)$ and $f^R(a) = f^R(b)$, so by Roll's theorem in real number, there are $t_1, t_2 \in [a, b]$ so that $(f^L)^{'}(t_1) = 0$, $(f^R)^{'}(t_2) = 0$. So

$$0 \in f^{'}(t).$$

□

Example 3.9. *Take $f(t) = [\sin^2(t), \cos^2(t)]$ and set $a = 0, b = \pi$ so $f(a) = f(b) = [0, 1]$, then $f^{'}(t) = [2\sin(t)\cos(t), -2\sin(t)\cos(t)]$. When $t = \frac{2\pi}{3}$ we have*

$$f^{'}(\frac{2\pi}{3}) = [2\sin(\frac{2\pi}{3})\cos(\frac{2\pi}{3}), -2\sin(\frac{2\pi}{3})\cos(\frac{2\pi}{3})] = [-\frac{3}{2}, \frac{3}{2}].$$

That is,

$$0 \in f^{'}(\frac{2\pi}{3}) = [-\frac{3}{2}, \frac{3}{2}].$$

Example 3.10. *Take $f(t) = [\sin(t), 1]$ and set $a = 0, b = 2\pi$, $(a < b)$ so that $f(a) = f(b) = [0, 1]$, then $f^{'}(t) = [\cos(t), 0]$ for $t = \pi$, we have*

$$f^{'}(\pi) = [\cos(\pi), 0] = [-1, 0].$$

That is,

$$0 \in f^{'}(\pi) = [-1, 0].$$

In the following, we will present a proof for the mean value theorem in interval-valued function space, using Rolle's Theorem. Note that to prove this theorem, the function $f(x)$ and the expression $\frac{f(b) \ominus_g f(a)}{b-a}x$ must be gH-differentiable from two different types.

Theorem 3.11. *(Mean Value Theorem)*
If an interval-valued function $f : J \longrightarrow I\mathbb{R}$ is continuous and (ii)-gH-differentiable, then for any $a, b \in J$ with $a < b$ there exists a point $t \in [a, b]$ such that

$$f^{'}(t) \supseteq \frac{f(b) \ominus_g f(a)}{b-a}.$$

Proof. Define

$$g(x) = f(x) \ominus_g \frac{f(b) \ominus_g f(a)}{b-a}x \ , \ x \in J.$$

It is evident that the expression $\dfrac{f(b) \ominus_g f(a)}{b-a}x$ is (i)-gH-differentiable. Since the interval-valued function f is continuous and gH-differentiable over the interval $[a, b]$, it follows that g is also continuous and gH-differentiable.

We now show that $0 \in g^{'}(t)$. This implies that the interval-valued function g satisfies the hypotheses of Rolle's Theorem 3.8. Specifically,

$$\begin{aligned} g(a) &= f(a) \ominus_g \frac{f(b) \ominus_g f(a)}{b-a} \cdot a \\ &= f(a) \ominus_g \frac{a \cdot f(b) \ominus_g a \cdot f(a)}{b-a}, \end{aligned}$$

and hence,

$$(b-a)g(a) = (b-a)f(a) \ominus_g (a \cdot f(b) \ominus_g a \cdot f(a)).$$

If $\omega(f(b)) \geq \omega(f(a))$, then by Lemma 2.3 of [14], we obtain

$$\begin{aligned} (b-a)g(a) &= ((b-a)f(a) + a \cdot f(a)) \ominus_g a \cdot f(b) \\ &= (b \cdot f(a) - a \cdot f(a) + a \cdot f(a)) \ominus_g a \cdot f(b) \\ &= b \cdot f(a) \ominus_g a \cdot f(b) \\ &= g(b)(b-a), \end{aligned}$$

and therefore,

$$g(a) = g(b).$$

On the other hand, if $\omega(f(b)) < \omega(f(a))$, then by Lemma 2.3 of [14], we have

$$\begin{aligned} (b-a)g(a) &= a \cdot f(a) \ominus_g (a \cdot f(b) + (-1)(b-a)f(a)) \\ &= a \cdot f(a) \ominus_g (a \cdot f(b) + (a-b)f(a)) \\ &= a \cdot f(a) \ominus_g (a \cdot f(b) + a \cdot f(a) \ominus_g b \cdot f(a)) \\ &= b \cdot f(a) \ominus_g (a \cdot f(b) + a \cdot f(a) + (-1)a \cdot f(a)) \\ &= b \cdot f(a) \ominus_g a \cdot f(b) \\ &= g(b)(b-a), \end{aligned}$$

and again

$$g(a) = g(b).$$

Consequently, the interval-valued function g satisfies Rolle's Theorem, and there exists $t \in [a, b]$ such that $0 \in g^{'}(t)$. From Theorem 3.8 and Theorem 4.5 of [14], we have

$$\begin{aligned} 0 \in g^{'}(t) &= f^{'}(t) + (-1)\frac{f(b) \ominus_g f(a)}{b-a} \\ &= f^{'}(t) + \frac{f(a) \ominus_g f(b)}{b-a}. \end{aligned}$$

Hence

$$\begin{cases} (f^{'})^L(t) + (\dfrac{f(a)-f(b)}{b-a})^L \leq 0, \\ \\ (f^{'})^R(t) + (\dfrac{f(a)-f(b)}{b-a})^R \geq 0, \end{cases}$$

which implies

$$\begin{cases} (f^{'})^L(t) \leq (\dfrac{f(b)-f(a)}{b-a})^L, \\ \\ (f^{'})^R(t) \geq (\dfrac{f(b)-f(a)}{b-a})^R, \end{cases}$$

and thus

$$\frac{f(b) \ominus_g f(a)}{b-a} \subseteq f^{'}(t).$$

□

Example 3.12. *Take* $f(t) = [-2, \sin t]$ *and set* $a = \dfrac{\pi}{6}, b = \dfrac{2\pi}{3}$, *so that* $f(\dfrac{\pi}{6}) = [-2, \dfrac{1}{2}], f(\dfrac{2\pi}{3}) = [-2, \dfrac{\sqrt{3}}{2}]$ *and*

$$\begin{aligned} \frac{f(b) \ominus_g f(a)}{b-a} &= \frac{f(\frac{2\pi}{3}) \ominus_g f(\frac{\pi}{6})}{\frac{2\pi}{3} - \frac{\pi}{6}} \\ &= \frac{[-2, \frac{\sqrt{3}}{2}] \ominus_g [-2, \frac{1}{2}]}{\frac{\pi}{2}} \\ &= \frac{[\min\{-2+2, \frac{\sqrt{3}}{2} - \frac{1}{2}\}, \max\{-2+2, \frac{\sqrt{3}}{2} - \frac{1}{2}\}]}{\frac{\pi}{2}} = \frac{[0, \frac{\sqrt{3}-1}{2}]}{\frac{\pi}{2}} \\ &= [0, \frac{\sqrt{3}-1}{\pi}]. \end{aligned}$$

We have

$$f^{'}(t) = [0, \cos t].$$

Then, when $t = \dfrac{\pi}{3}$(*with* $a < t < b$), *we have* $f^{'}(\dfrac{\pi}{3}) = [0, \dfrac{1}{2}]$. *Consequently*

$$f^{'}(\frac{\pi}{3}) = [0, \frac{1}{2}] \supseteq \frac{f(\frac{2\pi}{3}) \ominus_g f(\frac{\pi}{6})}{\frac{2\pi}{3} - \frac{\pi}{6}} = [0, \frac{\sqrt{3}-1}{\pi}].$$

Remark 3.13. Note that in Mean Value Theorem where interval-valued function $f(x)$ is (i)-gH-differentiable, we have

$$f^{'}(t) \subseteq \frac{f(b) \ominus_g f(a)}{b-a}.$$

Example 3.14. *Take* $f(t) = [\sin^2(t), \cos^2(t)]$ *and set* $a = \frac{\pi}{4}, b = 2\pi,$ *we have* $f(\frac{\pi}{4}) = [\frac{1}{4}, \frac{1}{4}], f(2\pi) = [0,1]$ *and*

$$\begin{aligned}\frac{f(2\pi) \ominus_g f(\frac{\pi}{4})}{2\pi - \frac{\pi}{4}} &= \frac{[0,1] \ominus_g [\frac{1}{4}, \frac{1}{4}]}{\frac{7\pi}{4}} \\ &= \frac{[-\frac{1}{4}, \frac{3}{4}]}{\frac{7\pi}{4}} \\ &= [-\frac{1}{7\pi}, \frac{3}{7\pi}],\end{aligned}$$

then for $t = \frac{3\pi}{2}$ *we have*

$$f^{'}(\frac{3\pi}{2}) = [-2\sin(\frac{3\pi}{2})\cos(\frac{3\pi}{2}), 2\sin(\frac{3\pi}{2})\cos(\frac{3\pi}{2})] = [0,0].$$

Consequently

$$f^{'}(\frac{3\pi}{2}) = [0,0] \subseteq \frac{f(2\pi) \ominus_g f(\frac{\pi}{4})}{2\pi - \frac{\pi}{4}} = [-\frac{1}{7\pi}, \frac{3}{7\pi}].$$

And we have

$$\begin{aligned}\frac{f(2\pi) - f(\frac{\pi}{4})}{2\pi - \frac{\pi}{4}} &= \frac{[0,1] - [\frac{1}{4}, \frac{1}{4}]}{\frac{7\pi}{4}} \\ &= \frac{[-\frac{1}{4}, \frac{3}{4}]}{\frac{7\pi}{4}} \\ &= [-\frac{1}{7\pi}, \frac{3}{7\pi}].\end{aligned}$$

Consequently

$$f^{'}(\frac{3\pi}{2}) = [0,0] \subseteq \frac{f(2\pi) - f(\frac{\pi}{4})}{2\pi - \frac{\pi}{4}} = [-\frac{1}{7\pi}, \frac{3}{7\pi}].$$

Remark 3.15. Note that the above theorem holds only for the gH-difference and ordinary difference, and it does not hold for the Hokuhara difference of intervals.

We will now present an example that demonstrates the failure of the Mean Value Theorem for the Hokuhara difference.

Example 3.16. *Take $f(t) = [-t^2, t^2+1]$ and set $a = -2, b = 1$ where $a < b$ such that $f(-2) = [-4,5], f(1) = [-1,2]$ and*

$$\begin{aligned}\frac{f(b) \ominus_g f(a)}{b-a} &= \frac{f(1) \ominus_g f(-2)}{1-(-2)} \\ &= \frac{[-1,2] \ominus_g [-4,5]}{1+2} \\ &= \frac{[\min\{-1+4, 2-5\}, \max\{-1+4, 2-5\}]}{3} = \frac{[-3,3]}{3} \\ &= [-1,1],\end{aligned}$$

and

$$\begin{aligned}\frac{f(b) - f(a)}{b-a} &= \frac{f(1) - f(-2)}{1-(-2)} \\ &= \frac{[-1,2] - [-4,5]}{1+2} \\ &= \frac{[-1-5, 2+4]}{3} = \frac{[-6,6]}{3} \\ &= [-2,2].\end{aligned}$$

We have

$$f'(t) = [-2t, 2t].$$

Then, when $t = \frac{1}{2}$(with $a < t < b$), we have $f'(\frac{1}{2}) = [-1,1]$. Consequently, we have the following inclusion

$$f'(\frac{1}{2}) = [-1,1] \subseteq \frac{f(1) \ominus_g f(-2)}{1-(-2)} = [-1,1].$$

Alternatively, we can write

$$f'(\frac{1}{2}) = [-1,1] \subseteq \frac{f(1) - f(-2)}{1-(-2)} = [-2,2].$$

Or for $t = 0(a < t < b)$, we have $f'(0) = [0,0]$, consequently

$$f'(0) = [0,0] \subseteq \frac{f(1) \ominus_g f(-2)}{1-(-2)} = [-1,1],$$

and also, for the ordinary difference, we have

$$f'(0) = [0,0] \subseteq \frac{f(1) - f(-2)}{1-(-2)} = [-2,2].$$

However, $\dfrac{f(b) \ominus f(a)}{b-a}$ *cannot be defined.*

4. Foundations of differentiation and the Mean Value Theorem for interval-valued functions on time scales

In this section, we begin by revisiting the fundamental notions of delta (Δ) and nabla (∇) differentiability for interval-valued functions defined on time scales. These concepts serve as the foundation for establishing a mean value theorem tailored to interval-valued functions. Our goal is to develop a unified theoretical framework that integrates both discrete and continuous cases within the setting of interval analysis.
A time scale $\mathbb{T}$ is a non-empty closed subset of $\mathbb{R}$. To characterize the structure of a time scale, we introduce the following operators and classifications [2, 5, 10]:
Jump operators
(i) **The forward jump operator**, $\sigma : \mathbb{T} \to \mathbb{T}$, yields the smallest element in $\mathbb{T}$ that is strictly greater than t, in other words

$$\sigma(t) := inf\{s \in \mathbb{T} : s > t\}.$$

(ii) **The backward jump operator**, $\rho : \mathbb{T} \to \mathbb{T}$, determines the largest element in $\mathbb{T}$ that is strictly less than t, in other words

$$\rho(t) := sup\{s \in \mathbb{T} : s < t\}.$$

In this context, we define $\inf \emptyset = \sup \mathbb{T}$, (meaning if $\mathbb{T}$ has a maximum element, the forward jump operator $\sigma(t)$ equals t) and $\sup \emptyset = inf \mathbb{T}$ (implying that the backward jump operator $\rho(t)$ equals t when $\mathbb{T}$ has a minimum element).
A point t within a time scale $\mathbb{T}$ is classified as:

- **Right-scattered** if the forward jump operator $\sigma(t)$ is strictly greater than t.
- **Right-dense** if the forward jump operator $\sigma(t)$ equals t.
- **Left-scattered** if the backward jump operator $\rho(t)$ is strictly less than t.
- **Left-dense** if the backward jump operator $\rho(t)$ equals t.

By considering a time scale $\mathbb{T}$, the forward step size function $\mu : \mathbb{T} \longrightarrow [0, \infty)$ is defined as $\mu(t) := \sigma(t) - t$. Similarly, the backward step size function $\nu : \mathbb{T} \longrightarrow [0, \infty)$ is defined as $\nu(t) := t - \rho(t)$.
We define two subsets, $\mathbb{T}_k$ and $\mathbb{T}^k$, of the time scale $\mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$; in other cases set $\mathbb{T}_k = \mathbb{T}$. In conclusion,

$$\mathbb{T}_k = \begin{cases} \mathbb{T} \setminus [\inf \mathbb{T}, \sigma(\inf \mathbb{T})) \ if \ \inf \mathbb{T} > -\infty \\ \mathbb{T} \ if \ \inf \mathbb{T} = -\infty. \end{cases}$$

If $\mathbb{T}$ includes a left-scattered maximum M, then $\mathbb{T}^k$ is defined as $\mathbb{T} - \{M\}$; if not, then $\mathbb{T}^k$ is simply $\mathbb{T}$. In conclusion,

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] \; if \;\; \sup \mathbb{T} < \infty \\ \mathbb{T} \; if \;\; \sup \mathbb{T} = \infty. \end{cases}$$

In particular, given a time scale interval $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} | a \leq t \leq b\}$, $(a, b]_{\mathbb{T}} = \{t \in \mathbb{T} | a < t \leq b\}$, observe that $[a, b]^k = [a, b]_{\mathbb{T}}$ when b is left-dense ($i.e.$ $a < \rho(b) = b$) and $[a, b]^k = [a, b)_{\mathbb{T}} = [a, \rho(b)]_{\mathbb{T}}$ when b is left-scattered ($i.e.$ $a < \rho(b) < b$). And also $[a, b]_k = [a, b]_{\mathbb{T}}$ when a is right-dense ($i.e.$ $\sigma(a) = a < b$) and $[a, b]_k = (a, b]_{\mathbb{T}} = [\sigma(a), b]_{\mathbb{T}}$ when a is right-scattered ($i.e.$ $a < \sigma(a) < b$).
Let g be a scalar-valued function defined on the time scale $\mathbb{T}$ (i.e., $g : \mathbb{T} \longrightarrow \mathbb{R}$), and let t be an element of $\mathbb{T}_k$. For all $\varepsilon > 0$, suppose there exist a real number α and a neighborhood U around t such that

$$|g(\rho(t)) - g(s) - \alpha(\rho(t) - s)| \leq \varepsilon |\rho(t) - s| \; for \; all \; s \in U \cap \mathbb{T},$$

then g is considered to be nabla(∇) differentiable at the point t. The real number α, denoted by $g^{\nabla}(t)$, is known as the ∇-derivative. In general terms, g is considered nabla(∇) differentiable on $\mathbb{T}_k$ if for all $t \in \mathbb{T}_k$ the ∇-derivative $g^{\nabla}(t)$ exists.

Definition 4.1 ([10])**.** Let $F : \mathbb{T} \longrightarrow I\mathbb{R}$ be an interval-valued function defined on a time scale $\mathbb{T}$, and let $A \in I\mathbb{R}$. We say that A is the $\mathbb{T}$-limit of F at $t_0 \in \mathbb{T}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d_H(F(t), A) \leq \varepsilon$ for all $t \in U_T(t_0, \delta)$, where $U_{\mathbb{T}}(t_0, \delta) = (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$. If such a $\mathbb{T}$-limit A exists, it is unique and denoted by $A = \mathbb{T} - \lim_{t \to t_0} F(t)$.

Definition 4.2 ([10])**.** Let $F : \mathbb{T} \to I\mathbb{R}$ be an interval-valued function on a time scale $\mathbb{T}$, and let $t \in \mathbb{T}_k$. $F^{\nabla}_{gH}(t)$ is called the nabla(∇) generalized Hukuhara derivative of F at t if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_H(F(\rho(t)) \ominus_g F(s), (\rho(t) - s)F^{\nabla}_{gH}(t)) \leq \varepsilon |\rho(t) - s|$$

for all $s \in U_{\mathbb{T}}(t, \delta)$. If this condition holds for all $t \in \mathbb{T}_k$, then F is said to be ∇_{gH}-differentiable on $\mathbb{T}_k$. In the case $\mathbb{T} = \mathbb{R}$, this derivative coincides with the classical gH-derivative $F^{'}_{gH}$.

The Mean Value Theorem on time scales for interval-valued functions

In what follows, we present a generalized version of the Mean Value Theorem for interval-valued functions defined on time scales. We begin by introducing the necessary definitions and properties of interval-valued functions on time scales, and then proceed to develop and prove the new version of the Mean Value Theorem in this context.

Definition 4.3. An interval-valued function $f : \mathbb{T} \longrightarrow I\mathbb{R}$ is considered to achieve its local left-maximum (its local left-minimum) at $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$ if

(i) for a left-scattered t_0, $f(\rho(t_0)) \subseteq f(t_0)$ ($f(\rho(t_0)) \supseteq f(t_0)$);
(ii) for a left-dense t_0, there is a neighborhood U of t_0 where

$$f(t_0) \supseteq f(t) \ (\ f(t_0) \subseteq f(t) \ )$$

for all $t \in U \cap \mathbb{T}$ with $t < t_0$.

Example 4.4. *Let $\mathbb{T} = \{0\} \cup [0.5, 1] \subset \mathbb{R}$. Note that $t = 0.5$ is left-scattered and every point in $(0.5, 1]$ is left-dense. Define the interval-valued function $f : \mathbb{T} \to I\mathbb{R}$ by*

$$f(t) = \begin{cases} [0,1], & t = 0, \\ [1,2], & t = 0.5, \\ [1.5,2], & t \in (0.5,1]. \end{cases}$$

Case 1 (Left-scattered point):
Consider $t_0 = 0.5$. Since $\rho(0.5) = 0$, we have

$$f(\rho(0.5)) = f(0) = [0,1] \subseteq [1,2] = f(0.5).$$

Therefore, by Definition 4.3, the function f attains a local left-maximum at $t_0 = 0.5$.
Case 2 (Left-dense point):
Now consider $t_0 = 1$. For all $t \in (0.5, 1)$, we have $f(t) = [1.5, 2] = f(1)$, so clearly

$$f(1) \supseteq f(t).$$

Hence, condition (ii) of Definition 4.3 is satisfied, and f also achieves a local left-maximum at $t_0 = 1$.

Theorem 4.5. *Assume $f : \mathbb{T} \longrightarrow I\mathbb{R}$ is differentiable at $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$. If zero is the interior point of $f^{\nabla}(t_0)$ and f is (i)-gH-differentiable then f achieves its local left-maximum at t_0. If zero is the interior point of $f^{\nabla}(t_0)$ and f is (ii)-gH-differentiable then f achieves its local left-minimum at t_0.*

Proof. Let $0 \in f^{\nabla}(t_0)$ and f be (i)-gH-differentiable. If t_0 is left-scattered, then

$$0 \in f^{\nabla}(t_0) = \frac{f(t_0) \ominus_g f(\rho(t_0))}{t_0 - \rho(t_0)}.$$

Consequently, we obtain the following inequalities:

$$\begin{cases} f^L(t_0) - f^L(\rho(t_0)) < 0 \\ f^R(t_0) - f^R(\rho(t_0)) > 0. \end{cases}$$

Therefore, we conclude that

$$f(\rho(t_0)) \subset f(t_0).$$

Let now t_0 be left-dense. Thus,

$$0 \in f^{\nabla}(t_0) = \lim_{t \to t_0} \frac{f(t_0) \ominus_g f(t)}{t_0 - t},$$

which implies that

$$0 \in f(t_0) \ominus_g f(t).$$

Therefore, assume that f is (i)-gH-differentiable. Under this assumption, the following inequalities hold

$$\begin{cases} f^L(t_0) - f^L(t) < 0, \\ f^R(t_0) - f^R(t) > 0. \end{cases}$$

Thus, $f(t) \subset f(t_0)$ for all t in $U \cap \mathbb{T}$ where $t < t_0$. So f achieves its local left-maximum at t_0.
When $0 \in f^\nabla(t_0)$ and f is (ii)-gH-differentiable, it is proven similarly. Thus, $f(t_0) \subset f(t)$ for all $t \in U \cap \mathbb{T}$ where $t < t_0$. Consequently, f achieves its local left-minimum at t_0. □

Corollary 4.6. *Assume $f : \mathbb{T} \longrightarrow I\mathbb{R}$ is differentiable at $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$. If f achieves its local left-maximum (its local left-minimum) at t_0, then f is (i)-gH-differentiable ((ii)-gH-differentiable).*

Theorem 4.7. *Let f be a continuous interval-valued function on interval $[a, b]$ that is differentiable on interval (a, b) with $f(a) = f(b)$. If f has a maximum (minimum) at t_0 then $f^{'}(t_0) = 0$, and there exists a neighborhood $(t - \delta, t_0)$ in which f is (i)-gH-differentiable ((ii)-gH-differentiable) and there also exists a neighborhood $(t_0, t + \delta)$, in which f is (ii)-gH-differentiable ((i)-gH-differentiable).*

Proof. If the function f has a maximum (minimum) at the point t, it is easily seen that $f^L(t)$ is a minimum (maximum) and $f^R(t)$ is a maximum (minimum). Since f is differentiable, we have $(f^L)^{'}(t) = 0$ and $(f^R)^{'}(t) = 0$. So $f^{'}(t) = 0$. Define $h(t) = f^R(t) - f^L(t)$. Since $h(t)$ is continuous and differentiable, it follows that at any point where f attains a maximum (minimum), the function h also attains a maximum (minimum). This is a direct consequence of the definitions of f^L, f^R and h. Given that $f(a) = f(b)$, we have $h(a) = h(b)$, which implies $h^{'}(t_0) = 0$. Thus $h(t)$ is maximum (minimum). Therefore, there exists a neighborhood $(t - \delta, t_0)$ in which $(f^R)^{'} > (f^L)^{'}((f^R)^{'} < (f^L)^{'})$, so f is (i)-gH-differentiable ((ii)-gH-differentiable) on this interval. Similarly, there exists a neighborhood $(t_0, t - \delta)$ in which $(f^R)^{'} < (f^L)^{'}((f^R)^{'} > (f^L)^{'})$, so f is (ii)-gH-differentiable ((i)-gH-differentiable) on this interval. □

Theorem 4.8. *(Mean Value Theorem on Interval-Valued Functions). Consider f, as continuous interval-valued function on $[a, b]$ that is differentiable on $(a, b]$. Then there are points $\xi, \tau \in (a, b]$ so that*

$$f^\nabla(\tau) \subseteq \frac{f(b) \ominus_g f(a)}{b - a} \subseteq f^\nabla(\xi).$$

Proof. Take the interval-valued function φ, which is defined on $[a,b]$ as:

$$\varphi(t) = f(t) \ominus_g f(a) \ominus_g \frac{f(b) \ominus_g f(a)}{b-a}(t-a).$$

It is apparent that φ is continuous on $[a,b]$ and differentiable on $(a,b]$. Additionally,

$$\varphi(a) = \varphi(b) = 0,$$

and by Theorem 4.7, there are points $\xi, \tau \in (a,b]$ so that $\varphi^{\nabla}(\xi)$ is (i)-gH-differentiable and $\varphi^{\nabla}(\tau)$ is (ii)-gH-differentiable. Therefore, considering that

$$\varphi^{\nabla}(t) = f^{\nabla}(t) \ominus_g \frac{f(b) \ominus_g f(a)}{b-a},$$

we come to the conclusion of the theorem. □

Corollary 4.9. *Let f be a continuous interval-valued function, differentiable on $(a,b]$ except possibly at b. If $f^{\nabla}(t) = 0$ for all $t \in (a,b]$, then f is constant on the interval $[a,b]$.*

Corollary 4.10. *Let f be a continuous interval-valued function, differentiable on $(a,b]$ except possibly at b. Then*
(i) f is increasing on $[a,b]$ if it is (i)-gH-differentiable on $(a,b]$,
(ii) f is decreasing on $[a,b]$ if it is (ii)-gH-differentiable on $(a,b]$,
(iii) f is nondecreasing on $[a,b]$ if $0 \in f^{\nabla}(t)$ and f is (i)-gH-differentiable for all $t \in (a,b]$,
(iv) f is nonincreasing on $[a,b]$ if $0 \in f^{\nabla}(t)$ and f is (ii)-gH-differentiable for all $t \in (a,b]$.

The subsequent theorem extends the findings of Theorem 4.8.

Theorem 4.11. *Assume f, g are continuous interval-valued functions on $[a,b]$ and differentiable on $(a,b]$. Suppose g is (i)-gH-differentiable for all $t \in (a,b]$. Then there exist $\xi, \tau \in (a,b]$ so that*

$$\frac{f^{\nabla}(\tau)}{g^{\nabla}(\tau)} \subseteq \frac{f(b) \ominus_g f(a)}{b-a} \subseteq \frac{f^{\nabla}(\xi)}{g^{\nabla}(\xi)}.$$

Proof. From Theorem 4.8 and since g is (i)-gH-differentiable for all $t \in (a,b]$, we deduce that $g(a) \neq g(b)$. Thus, the auxiliary interval-valued function can be considered

$$\varphi(t) = f(t) \ominus_g f(a) \ominus_g \frac{f(b) \ominus_g f(a)}{g(b) \ominus_g g(a)}[g(t) \ominus_g g(a)].$$

It is evident that φ is continuous on $[a,b]$ and differentiable on $(a,b]$. Additionally,

$$\varphi(a) = \varphi(b) = 0.$$

Applying Theorem 4.7 to the interval-valued function φ and noting that

$$\varphi^{\nabla}(t) = f^{\nabla}(t) \ominus_g \frac{f(b) \ominus_g f(a)}{g(b) \ominus_g g(a)} g^{\nabla}(t),$$

we conclude that the desired result holds. Hence, the proof is complete. □

Theorem 4.12. *Consider F as an interval-valued function and g as a real-valued function defined on $\mathbb{T}$. Suppose F is ∇_{gH}-differentiable and g is ∇-differentiable on $\mathbb{T}_k$. If*

$$\|F^{\nabla}_{gH}(t)\| \leq g^{\nabla}(t)$$

for all $t \in \mathbb{T}_k$, then

$$\|F(t) \ominus_g F(x)\| \geq g(x) - g(t)$$

for all $t \in [x, y]_{\mathbb{T}}$ when x and y are in $\mathbb{T}$ with $x \leq y$.

Proof. Consider x and y as elements of $\mathbb{T}$ with $x \leq y$. For all positive $\varepsilon > 0$, the principle of induction is applicable, as demonstrated in Theorem 2.7 of [10], to verify that

$$S(t) : \|F(t) \ominus_g F(x)\| \geq g(t) - g(x) - \varepsilon(t - x)$$

is valid for all t in the interval $[x, y]_{\mathbb{T}}$. The proof is structured in four parts:
(I) $S(x)$ is clearly true when $t = x$.
(II) Suppose t is left-scattered and $S(t)$ holds true. Based on Theorem 2.6 of [10], it follows that

$$\begin{aligned}
\|F(x) \ominus_g F(\rho(t))\| &= \|F(x) \ominus_g F(t) + F(t) \ominus_g F(\rho(t))\| \\
&\geq \|F(x) \ominus_g F(t)\| - \|F(\rho(t)) \ominus_g F(t)\| \\
&= \|F(x) \ominus_g F(t)\| - \| - \nu(t) F^{\nabla}_{gH}(t)\| \\
&= \|F(t) \ominus_g F(x)\| - \nu(t)\|F^{\nabla}_{gH}(t)\| \\
&\geq g(t) - g(x) - \varepsilon(t - x) - \nu(t) g^{\nabla}(t) \\
&= g(t) - g(x) - \varepsilon(t - x) - g(t) + g(\rho(t)) \\
&= g(\rho(t)) - g(x) - \varepsilon(t - x).
\end{aligned}$$

Therefore, the statement $S(\rho(t))$ holds.
(III) Assume $S(t)$ is true and $t \neq y$ is left-dense. Then, $\rho(t) = t$. Because of the ∇_{gH}-differentiability of F and the ∇-differentiability of g at t, there is a neighborhood $U_{\mathbb{T}}$ of t such that

$$d_H(F(t) \ominus_g F(y), F^{\nabla}_{gH}(t)(t - y)) \leq \frac{\varepsilon}{2}|t - y|$$

for each $y \in U_{\mathbb{T}}$ and

$$|g(t) - g(y) - g^{\nabla}(t)(t - y)| \leq \frac{\varepsilon}{2}|t - y|$$

for each $y \in U_{\mathbb{T}}$. Consequently, it follows that

$$\begin{aligned}
d_H(F(t), F(y)) &= d_H(F(t) \ominus_g F(y), \{0\}) \\
&\leq d_H(F(t) \ominus_g F(y), F^{\nabla}_{gH}(t)(t - y)) + d_H(\{0\}, F^{\nabla}_{gH}(t)(t - y)) \\
&\leq (\|F^{\nabla}_{gH}(t)\| + \frac{\varepsilon}{2})|t - y|
\end{aligned}$$

and

$$g(t) - g(y) - g^{\nabla}(t)(t-y) \geq -\frac{\varepsilon}{2}|t-y|$$

for each $y \in U_{\mathbb{T}}$. Therefore, for each y in $U_{\mathbb{T}} \cap (t, \infty)$, we obtain

$$\begin{aligned}
\|F(x) \ominus_g F(y)\| &= \|(F(x) \ominus_g F(t)) \ominus_g (F(y) \ominus_g F(t))\| \\
&\geq \|F(x) \ominus_g F(t)\| - \|F(y) \ominus_g F(t)\| \\
&\geq g(t) - g(x) - \varepsilon(t-x) - \|F^{\nabla}_{gH}(t)(t-y)\| - \frac{\varepsilon}{2}|t-y| \\
&\geq g(t) - g(x) - \varepsilon(t-x) - g^{\nabla}(t)(t-y) - \frac{\varepsilon}{2}|t-y| \\
&\geq g(t) - g(x) - \varepsilon(t-x) - \frac{\varepsilon}{2}(t-y) - g(t) + g(y) - \frac{\varepsilon}{2}|t-y| \\
&\geq g(y) - g(x) - \varepsilon(y-x).
\end{aligned}$$

This indicates that $S(y)$ holds for all y in $U_{\mathbb{T}} \cap (t, \infty)$.
(IV) Assume t is right-dense and $S(\tau)$ is true for all $\tau < t$. Since F and g are continuous, we then obtain

$$\begin{aligned}
\|F(t) \ominus_g F(x)\| &= \lim_{\tau \to t^-} \|F(x) \ominus_g F(\tau)\| \\
&\geq \lim_{\tau \to t^-} g(\tau) - g(x) + \varepsilon(\tau - x) \\
&= g(t) - g(x) + \varepsilon(t-x).
\end{aligned}$$

This demonstrates that $S(t)$ is valid. Since ε is arbitrary, the inequality $\|F(t) \ominus_g F(x)\| \geq g(t) - g(x) + \varepsilon(t-x)$ holds for all ε. Hence, the inequality $\|F(t) \ominus_g F(x)\| \geq g(t) - g(x)$ follows, and the proof is complete. □

Example 4.13. *Let $\mathbb{T} = [0,1]$ and let $F(t) = [t, 2t]$ and $g(t) = 2t - 1$. If $x = \frac{1}{4}$ and $t = \frac{1}{3}$, then $F'_{gH}(t) = F^{\nabla}_{gH}(t) = [1,2]$ and $g^{\nabla}(t) = 2$. Since $\|F^{\nabla}_{gH}(t)\| \leq 2$,*

$$\begin{aligned}
\|F(\frac{1}{3}) \ominus_g F(\frac{1}{4})\| &\geq g(\frac{1}{4}) - g(\frac{1}{3}), \\
\|[\frac{1}{3}, \frac{2}{3}] \ominus_g [\frac{1}{4}, \frac{1}{2}]\| &\geq (\frac{1}{2} - 1) - (\frac{2}{3} - 1), \\
\|[\frac{1}{12}, \frac{1}{6}]\| &\geq -\frac{1}{2} + \frac{1}{3}, \\
\frac{1}{6} &\geq -\frac{1}{6}.
\end{aligned}$$

Example 4.14. *Let $\mathbb{T} = \mathbb{Z}$ and let $F(t) = [\sin^2(t), \cos^2(t)]$ and $g(t) = \sin(t)$. If $x = -\frac{\pi}{4}$ and $t = 0$, then $F'_{gH}(t) = F^{\nabla}_{gH}(t) = [-2\sin(t)\cos(t), 2\sin(t)\cos(t)]$ and $g^{\nabla}(t) = \cos(t)$. Since $\|F^{\nabla}_{gH}(t)\| = \|[0,0]\| \leq \cos(t) = 1$,*

$$\|F(0) \ominus_g F(-\frac{\pi}{4})\| \geq g(-\frac{\pi}{4}) - g(0),$$
$$\|[0,1] \ominus_g [\frac{1}{2}, \frac{1}{2}]\| \geq -\frac{\sqrt{2}}{2} - 0,$$
$$\|[-\frac{1}{2}, \frac{1}{2}]\| \geq -\frac{\sqrt{2}}{2},$$
$$\frac{1}{2} \geq -\frac{\sqrt{2}}{2}.$$

5. Conclusion

This paper investigates the behavior of classical theorems—specifically, the Intermediate Value Theorem, Rolle's Theorem, and the Mean Value Theorem—within the framework of interval-valued functions. We demonstrate that these theorems require modified hypotheses to remain valid in the interval context, and we propose new formulations of the Mean Value Theorem under both (i)-gH-differentiable and (ii)-gH-differentiable. These contributions advance the theoretical foundations of interval analysis and extend core concepts of classical calculus to broader function spaces. The results also provide a groundwork for future studies in areas such as robust control, uncertain dynamical systems, and time scale calculus. Future research directions include the generalization of additional classical results, the study of topological and algebraic properties of interval-valued functions, and the development of efficient numerical methods for solving interval differential equations.

6. Author Contributions

Conceptualization, S.M.S.M.; methodology, T.S. and A.K.; software, T.S.; validation, S.M.S.M.; formal analysis, T.S., A.K., and S.M.S.M.; investigation, T.S. and A.K.; resources, T.S.; data curation, T.S.; writing—original draft preparation, T.S.; writing—review and editing, T.S.; supervision, S.M.S.M.; project administration, T.S. All authors have read and agreed to the published version of the manuscript.

7. Data Availability Statement

Not applicable.

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9. Ethical considerations

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10. Funding

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11. Conflict of interest

The authors declare no conflict of interest.

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