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COMPARISON OF LASSO AND WHISKER TOPOLOGIES ON FUNDAMENTAL GROUPOIDS AND LOCAL TRIVIALITY

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Abstract. This paper is devoted to comparing the Lasso topology and the Whisker topology on the fundamental groupoid. We prove that for locally path connected and semilocally simply connected spaces, the two topologies coincide. However, the converse does not hold in general, and we provide a partial converse. Furthermore, we observe that the topological fundamental groupoid is not étale, and we show that the topological fundamental groupoids of locally path connected and semilocally simply connected spaces, equipped with these topologies, are locally trivial. Through several examples, we illustrate the necessity of these conditions.

Keywords: Topological fundamental groupoid, Locally trivial groupoid, Lasso topology, Whisker topology.
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1. Introduction and motivation

The concept of the topological fundamental groupoid offers a significant extension to the classical fundamental group by removing the restriction of a single fixed base point. It achieves this by considering homotopy classes of paths connecting multiple points within a topological space, thereby providing a richer and more global perspective on the space's homotopical characteristics. This broader viewpoint allows for a more nuanced understanding of the underlying topology, making the fundamental groupoid a vital instrument in contemporary algebraic topology and its adjacent fields [2].

Unlike the fundamental group, which focuses on loops based at one point, the groupoid structure naturally accommodates morphisms between various points, thus capturing the complexity of spaces that may be disconnected or contain intricate path systems. This makes the fundamental groupoid particularly effective for unraveling subtle symmetries and connections that remain hidden under the traditional group framework [2, 8].

Locally trivial topological groupoids naturally arise as models for fiber bundles, especially principal bundles. The local triviality condition corresponds to the classical idea that a fiber bundle looks locally like a product space. The

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groupoid encodes how these local trivializations glue together globally, providing a unified algebraic-topological framework to study bundles and their automorphisms [11]. Also, locally trivial topological groupoids extend classical covering space theory by encoding how multiple local covers patch together. They provide a general setting to study fundamental groupoids with local triviality, which is essential for analyzing spaces that may be disconnected or have complicated local topology [3].

For the first time, fundamental groupoids were topologized by Brown and Danesh-Naruie [3]. They defined the lifted topology on a quotient of the fundamental groupoid of a given space X, making it a topological groupoid, provided that X is locally nice–that is, locally path connected and semilocally simply connected.

Beyond the category of locally nice spaces, in [12], the Lasso topology is introduced on the fundamental groupoid as a generalization of the Lasso topology on the fundamental group and the universal path space [4]. The Lasso topology turns the fundamental groupoid into a topological groupoid when the space is locally path connected [12]. In [14], it is shown that the Lasso topology on the fundamental groupoid of X coincides with the lifted topology, provided that X is locally path connected and semilocally simply connected.

Pakdaman and Shahini [13] also introduced the Whisker topology on the fundamental groupoid. Under this topology, the fundamental groupoid is not necessarily topological unless the space is a small loop transfer space.

Another attempt to topologize the fundamental groupoid was made by Holkar and Hossain [9]. They introduced two equivalent topologies on the fundamental groupoid of a locally path connected and semilocally simply connected space: the CO topology (compact-open topology) and the UC topology (universal cover topology). The UC topology is the same as the Whisker topology in [13], although the Whisker topology is defined for general spaces, while the UC topology is defined only for locally path connected and semilocally simply connected spaces.

In this article, we present a comparison between the Lasso topology and the Whisker topology. When the given space X is locally path connected and semilocally simply connected, these topologies on the fundamental group and the universal path space are known to be equivalent [18]. Since the Lasso and Whisker topologies on the fundamental groupoid are generalizations of their counterparts on the fundamental group and the universal path space, it is natural to expect that these topologies are comparable, and we investigate this relationship.

Through an example, we show that the Whisker topology is strictly finer than the Lasso topology, and we prove that the two are equivalent for locally path connected and semilocally simply connected spaces. As for the converse, we provide an example of a locally path connected space in which the two topologies are equivalent on its fundamental groupoid, although the space is not semilocally simply connected. Nevertheless, we prove that for a locally path connected space X, if the two topologies are equivalent, then X is a strong small loop transfer space.

Another direction of this article concerns the local triviality of topological fundamental groupoids. The study of locally trivial groupoids was initiated by Ehresmann [6], and further results have been obtained in [3] and [9]. However, these results are restricted to locally path connected and semilocally simply connected spaces, and no examples have been given to demonstrate the essentiality of these conditions. We prove that the topological fundamental groupoids are locally trivial when the given space X is locally path connected and semilocally simply connected. By means of examples, we show that these conditions are indeed essential.

Finally, we prove that the topological fundamental groupoid of a path connected space is not étale. It is worth noting that even semilocally simply connectedness is not sufficient for the fundamental groupoid to be étale.

2. Preliminaries

2.1. **The Fundamental Groupoid.** A groupoid G over G_0 consists of a set of arrows G and a set of objects G_0, together with two maps $S,\ T : G \longrightarrow G_0$, called respectively the source and target maps, a map $1 : G_0 \longrightarrow G; x \mapsto 1_x$, called the unit map, a map $i : G \longrightarrow G; a \longmapsto a^{-1}$, called the inverse map and a map $m : G_2 \longrightarrow G; (a; b) \mapsto m(a; b) = ab$, called the composition map, where G_2 denotes the set of composable arrows:

$$G_2 = \{(a; b) \in G \times G |\ S(b) = T(a)\}.$$

These structure maps satisfy the following conditions:
i) $S(ab) = S(a)$ and $T(ab) = T(b)$ for all $(a; b) \in G_2$,
ii) $a(bc) = (ab)c$ for all $a, b, c \in G$ such that $S(b) = T(a)$ and $S(c) = T(b)$,
iii) $S(1_x) = T(1_x) = x$ for all $x \in G_0$,
iv) $a1_{T(a)} = a$ and $1_{S(a)}a = a$ for all $a \in G$,
v) each $a \in G$ has a two-sided inverse a^{-1} such that $S(a^{-1}) = T(a)$, $T(a^{-1}) = S(a)$ and $aa^{-1} = 1_{S(a)};\ a^{-1}a = 1_{T(a)}$.

The set of arrows from x to y is denoted by $G(x, y)$ and, in particular, $G(x) := G(x, x)$ is called the object group (or vertex group) at x. Also, we denote $S^{-1}(x)$ by G_x and $T^{-1}(x)$ by G^x.

Definition 2.1. [10] A *topological groupoid* is a groupoid G together with topologies on G and G_0 such that the structure maps are continuous.

For a given topological space X, the fundamental groupoid πX has the set X as its set of objects and for any $x, y \in X$ the set $\pi X(x, y)$ is the set of endpoint fixing homotopy classes of paths in X from x to y. Composition of morphisms $[\alpha], [\beta]$ is $[\alpha * \beta]$, where $\alpha * \beta$ refers to standard path concatenation and the identity in $\pi X(x, x)$ is the $e_x = [c_x]$ for the constant path c_x. The object group at x, $\pi X(x, x)$ considered to be the well-known fundamental group $\pi_1(X, x)$ [1].

If $\mathcal{U}$ is an open cover of X, the subgroup of $\pi_1(X,x)$ consisting of the homotopy classes of loops that can be represented by a product of the following type:

$$\prod_{j=1}^{n} \alpha_j \beta_j \alpha_j^{-1},$$

where the α_j's are arbitrary paths starting at the base point x and each β_j is a loop inside one of the neighborhoods $U_i \in \mathcal{U}$. This group is called the *Spanier group for* $\mathcal{U}$, denoted by $\pi(\mathcal{U},x)$ [7, 16] and the intersection of all the Spanier groups for open covers is called Spanier group of X and is denoted by $\pi_1^{sp}(X,x)$.

2.2. **The Lasso Topology.** Here, we recall the Lasso topology on the fundamental groupoid from [12]. Let $\mathcal{U}$ be an open cover of a given space X and for $x, y \in X$, let $[\alpha] \in \pi X(x,y)$. If V, $W \in \mathcal{U}$ are open neighborhoods of x, y, respectively. Consider

$$N([\alpha],\mathcal{U},V,W) =$$

$$\{[\beta] \in \pi X \mid \beta \simeq \gamma{*}\mu{*}\alpha{*}\mu'{*}\lambda,\ \gamma : I \to V, \lambda : I \to W, [\mu] \in \pi_1(\mathcal{U},x), [\mu'] \in \pi_1(\mathcal{U},y)\}.$$

Theorem 2.2. [12] The family

$$\big\{N([\alpha],\mathcal{U},V,W) \big| \,\mathcal{U} \text{ is an open cover of } X; V,W \in \mathcal{U},\ [\alpha] \in \pi X(x,y),\ x \in V,\ y \in W\big\}$$

forms a basis for a topology on πX in which makes it a topological groupoid, when X is locally path connected.

The topology that is generated by this basis is called the **Lasso topology**. For a given topological space X, by $\pi^l X$ we mean the fundamental groupoid πX equipped with the Lasso topology on the set of morphisms and original topology on X, as the object set. Hence the operation π^l is a functor from the category of locally path connected topological spaces and continuous maps to the category of topological groupoids [12].

Given a pointed path connected space (X,x), the universal path space $\widetilde{X}$ is the set of homotopy classes of all paths started from x and hence as the sets we have

$$\pi_1(X,x) \subseteq \widetilde{X} = (\pi X)_x \subseteq \pi X.$$

The Lasso topology on the fundamental group [4], has a basis containing the sets in the following form

$$N([\alpha],\mathcal{U}) = \{[\beta] \in \pi_1(X,x) |\ \beta \simeq \alpha * \mu,\ \textit{for some } [\mu] \in \pi_1(\mathcal{U},x)\},$$

where $[\alpha] \in \pi_1(X,x)$ and $\mathcal{U}$ is an open cover of X and is denoted by $\pi_1^l(X,x)$.

The basis of the Lasso topology on $\widetilde{X}$, introduced in [19] is the collection of sets

$$N([\alpha],\mathcal{U},U) = \{[\beta] \in \widetilde{X} |\ \beta \simeq \alpha * \mu * \gamma,\ [\mu] \in \pi_1(\mathcal{U},x), \gamma : I \to U, \gamma(0) = \alpha(1)\},$$

where $[\alpha] \in \widetilde{X}$, $\mathcal{U}$ is an open cover of X and $U \in \mathcal{U}$. The Lasso topology on the universal path space is a generalization of the Lasso topology on the fundamental group [19, Proposition 7] and also the Lasso topology on fundamental groupoid is a generalization of the Lasso topology on the universal path space and hence is a generalization of $\pi_1^l(X,x)$ [12].

2.3. **The Whisker Topology.** For a given topological space X, let $[\alpha] \in \pi X(x,y)$, where $x, y \in X$. If V, W are open neighborhoods of x and y, respectively, one can define [13]

$$N([\alpha], V, W) := \{[\beta] \in \pi X | \beta \simeq \gamma * \alpha * \lambda,\ \gamma(I) \subseteq V,\ \lambda(I) \subseteq W\},$$

where $\gamma(1) = x$ and $\alpha(1) = \lambda(0) = y$. Then the family

$$\{N([\alpha], V, W);\ [\alpha] \in \pi X(x,y),\ x \in V,\ y \in W\},$$

forms a basis for a topology on fundamental groupoid. The topology that is generated by this basis, is called the **Whisker topology** and by $\pi^w X$ we mean the fundamental groupoid πX equipped with the Whisker topology on the set of morphisms and original topology on X, as the objects set.

Definition 2.3. [5] A topological space X is called *small loop transfer* at x (SLT) if for every $x_0, x \in X$, every open neighborhood U of X containing x_0, and every path α in X from x_0 to x, there is an open neighborhood V containing x such that for every loop β in V based at x there is a loop λ in U based at x_0 such that $\alpha * \beta * \alpha^{-1} \simeq \lambda$. Also, X is called SLT if it is SLT at any point x.

If X is a small loop transfer space, then the fundamental groupoid with the Whisker topology is a topological groupoid [13].

The Whisker topology on the fundamental group, which had been used in [16] and named in [4], has a basis containing the sets in the following form

$$N([\alpha], U) = \{[\beta] \in \pi_1(X,x) |\ \beta \simeq \alpha * \mu,\ \textit{for some}\ \mu : I \longrightarrow U,\ \mu(0) = \mu(1) = x\},$$

where $[\alpha] \in \pi_1(X,x)$ and U is an open neighborhood of x.

The basis of the Whisker topology on $\widetilde{X}$, introduced in [19] is the collection of sets

$$N([\alpha], U) = \{[\beta] \in \widetilde{X} |\ \beta \simeq \alpha * \mu,\ \textit{for some}\ \mu : I \longrightarrow U,\ \mu(0) = \alpha(1)\},$$

where $[\alpha] \in \widetilde{X}$ and U is an open neighborhood of $\alpha(1)$. Notably, the Whisker topology on the fundamental groupoid is a generalization of the Whisker topology on the $\pi_1^w(X,x)$ [12] when X is a small loop transfer space. This property makes the small loops added from the left side of paths homotopically transferable to the right side. We recall that a loop α in X with the base point x is small if there exists a representative of the homotopy class $[\alpha]$ in every open neighborhood U of x.

3. The comparison of Lasso and Whisker topologies on the fundamental groupoid

Virk and Zastrow [18] reviewed the existing topologies on the fundamental group and studied their generalizations to the universal path space. They observed that the Whisker topology is, in general, strictly finer than the Lasso topology on the universal path space, and proved that for locally path connected and semilocally simply connected spaces, all these topologies are equivalent.

In this section, we compare two topologies on πX: the Lasso topology and the Whisker topology.

Although the proof of the following proposition is straightforward and can be easily deduced from the definitions, we present it in full detail to help novice readers become more familiar with the concepts involved.

Proposition 3.1. $\pi^w X$ is finer than $\pi^l X$.

Proof. Let $N([\alpha],\mathcal{U},U,V)$ be a basic open neighborhood of $[\alpha]$ in $\pi^l X$, where α is a path from x to y, $\mathcal{U}$ is an open cover of X and $U, V \in \mathcal{U}$. We show that $N([\alpha],U,V) \subseteq N([\alpha],\mathcal{U},U,V)$, where $N([\alpha],U,V)$ is an open neighborhood of $[\alpha]$ in $\pi^w X$. Let $[\beta] \in N([\alpha],U,V)$, then we have $\beta \simeq \lambda * \alpha * \mu$ in which λ and μ are paths in U and V, respectively. Now we can write $\beta \simeq \lambda * c_x * \alpha * c_y * \mu$ where $c_x \in \pi_1(\mathcal{U}, x)$ and $c_y \in \pi_1(\mathcal{U}, y)$, hence $[\beta] \in N([\alpha],\mathcal{U},U,V)$. □

We recall that a given space X is semilocally simply connected if every point $x \in X$ has an open neighborhood U such that $i_*\pi_1(U, x) = \{e_x\}$, where $i : U \longrightarrow X$ is the inclusion map. It is also emphasized that a semilocally simply connected space does not have to be locally path connected [7].

Proposition 3.2. If X is semilocally simply connected, then $\pi^l X$ is finer than $\pi^w X$ and hence $\pi^l X$ and $\pi^w X$ are equivalent.

Proof. Let $N([\alpha],U,V)$ be a basic open neighborhood of $[\alpha]$ in $\pi^w X$, where α is a path from x to y and U, V are open neighborhoods of x, y, respectively. Since X is semilocally simply connected, there are simply connected neighborhoods W and W' containing x and y, respectively. Let $U' = U \cap W$, $V' = V \cap W'$ and $\mathcal{U}' = \mathcal{U} \cup \{U', V'\}$ where $\mathcal{U}$ is the open cover of X consisting of simply connected neighborhoods. We show that $N([\alpha],\mathcal{U}',U',V') \subseteq N([\alpha],U,V)$. Let $[\beta] \in N([\alpha],\mathcal{U}',U',V')$, then we have $\beta \simeq \lambda * \gamma * \alpha * \gamma' * \mu$, where λ and μ are paths in U' and V', respectively, $\gamma \in \pi_1(\mathcal{U}', x)$ and $\gamma' \in \pi_1(\mathcal{U}', y)$. Since $\gamma \simeq c_x$ and $\gamma' \simeq c_y$, we have $\beta \simeq \lambda * \alpha * \mu$ and therefore $[\beta] \in N([\alpha],U,V)$. □

In the following example, we show that if X is not semilocally simply connected, then $\pi^l X$ is not necessarily finer than $\pi^w X$. This implies that semilocally simply connectedness of X is a necessary condition in Proposition 3.2.

Example 3.3. Let $\mathbb{H}$ be the Hawaiian earring space, i.e, the shrinking wedge of circles, α_n be the circle of radius $\frac{1}{n}$ with the center $(\frac{1}{n}, 0)$, for every $n \in \mathbb{N}$ and

y be the common point of all circles. Let $\mathcal{U}$ be the open cover of $\mathbb{H}$ consisting of all open balls, and let $N([\alpha_1], U, V)$ be an open neighborhood of $[\alpha_1]$ in $\pi^w\mathbb{H}$ such that $U, V \in \mathcal{U}$ are open balls centered at the origin and with a radius less than 1 which do not contain $[\alpha_1]$. We show that $N([\alpha_1], U, V)$ is not open in $\pi^l\mathbb{H}$.

By contrary assume that $N([\alpha_1], U, V)$ is open in $\pi^l\mathbb{H}$. Hence there exist $\gamma \in \pi\mathbb{H}$ and an open cover $\mathcal{U}'$ such that the basic open neighborhood $N([\gamma], \mathcal{U}', U', V')$ of $[\alpha_1]$ is contained in $N([\alpha_1], U, V)$, where $U', V' \in \mathcal{U}'$ and $y = \alpha(0) = \alpha(1) = \gamma(0) = \gamma(1) = (0,0) \in U' \cap V'$. By [12, Lemma 2.3], $N([\gamma], \mathcal{U}', U', V') = N([\alpha_1], \mathcal{U}', U', V')$. Since $y = (0,0)$ is the common point of all α_n's and their radii converge to zero, we can choose $n > 1$ so that $\alpha_n(I) \subseteq U \cap U'$. Then $[\alpha_1 * \alpha_n * (\alpha_1)^{-1}] \in \pi(\mathcal{U}', y)$ which implies that

$$[(\alpha_1 * \alpha_n * (\alpha_1)^{-1}) * \alpha_1 * (\alpha_1 * \alpha_n * (\alpha_1)^{-1})] \in N([\alpha_1], \mathcal{U}', U', V').$$

But $[(\alpha_1 * \alpha_n * (\alpha_1)^{-1}) * \alpha_1 * (\alpha_1 * \alpha_n * (\alpha_1)^{-1})] = [\alpha_1 * \alpha_n * \alpha_1 * \alpha_n * (\alpha_1)^{-1}] \notin N([\alpha_1], U, V)$ because $\alpha_1(I)$ does not lie in U and V [5, Proposition 4.10]. This is a contradiction.

Now, of course, we have to examine the question whether the converse of Proposition 3.2 is true or not, and if it is not, what conditions on X will guarantee the converse of Proposition 3.2?

In the following example, we show that the converse of Proposition 3.2 does not hold, and in the next corollary, we will present a partial converse for it.

Example 3.4. Let S be the small loop space introduced in [17]. The space S, which is constructed using Harmonic Archipelago, has this property that for every $x \in S$, every nontrivial loop α based at x is small.

We know that $\pi^w S$ is finer than $\pi^l S$, in general. Let $N([\alpha], U, V)$ be an arbitrary basic open neighborhood of $[\alpha] \in \pi^w S$, where U, V are open neighborhood of $x := \alpha(0), y := \alpha(1)$, respectively. For every open cover $\mathcal{U}$ of S, there are $O, W \in \mathcal{U}$ such that $x \in O$ and $y \in W$. We claim that $N([\alpha], \mathcal{U}', U \cap O, V \cap W) \subseteq N([\alpha], U, V)$, where $\mathcal{U}' = \mathcal{U} \cup \{U \cap O, V \cap W\}$. For if, let $\beta \in N([\alpha], \mathcal{U}', U \cap O, V \cap W)$. We have $\beta \simeq \lambda * \mu * \alpha * \mu' * \lambda'$, where λ is a path in $U \cap O$ by $\lambda(1) = x$, $\mu \in \pi(\mathcal{U}', x)$, λ' is a path in $V \cap W$ by $\lambda'(0) = y$ and $\mu' \in \pi(\mathcal{U}', y)$. Since X is small loop space, there are $\eta : I \longrightarrow U \cap O$ and $\eta' : I \longrightarrow V \cap W$ such that $\mu \simeq \eta$ and $\mu' \simeq \eta'$. Hence $\beta \simeq \lambda * \eta * \alpha * \eta' * \lambda'$. Since $\lambda * \eta$ is a path in $U \cap O$ and $\eta' * \lambda'$ is a path in $V \cap W$, we have $[\beta] = [(\lambda * \eta) * \alpha * (\eta' * \lambda')] \in N([\alpha], U, V)$, which implies that $\pi^l S$ is finer than $\pi^w S$, as desired.

Strong small loop transfer (SSLT for brevity) spaces were introduced for the first time by Brodskiy et al. to determine the condition for coincidence of the Lasso topology and the Whisker topology on $\widetilde{X}$ [5, Theorem 4.11]. We recall that a space X is called an SSLT space if for every $x_0, x \in X$ and for every open neighborhood U of X containing x_0, there is an open neighborhood V

containing x such that for every loop β in V based at x and for every path α in X from x_0 to x, there is a loop γ in U based at x_0 such that $\alpha * \beta * \alpha^{-1} \simeq \gamma$.

If the two topologies on the fundamental groupoid are equivalent, then so are they on the fundamental group. Now Theorem 4.11 in [5] implies the following corollary.

Corollary 3.5. For a connected and locally path connected space X, if topological groupoids $\pi^l X$ and $\pi^w X$ are equivalent, then X is an SSLT space.

Note that every locally path connected and semilocally simply connected space is SSLT, but the converse is not necessarily true [15], and so the Corollary 3.5 is a partial converse of Proposition 3.2.

4. **Locally triviality of topological fundamental groupoid**

In this section, we prove that for locally nice spaces, the topological fundamental groupoid is locally trivial. Moreover, through examples, we demonstrate that these conditions are essential. It should be noted that in [9, Corollary 2.9 and Remark 2.12], the authors proved the local triviality of the fundamental groupoid for the topologies considered there—namely, the compact-open topology and the universal cover topology.

Unlike the Lasso and Whisker topologies, when the space is not locally nice, the fundamental groupoid equipped with the compact-open or universal cover topology is not necessarily a topological groupoid. Perhaps for this reason, no example of a space with a non-locally trivial topological fundamental groupoid is provided in [9].

A surjective map $p : X \longrightarrow Y$ is called a local homeomorphism if for a given $x \in X$ there exists an open neighborhood $U \subseteq X$ such that $p(U) \subseteq Y$ is open and $p|_U : U \longrightarrow p(U)$ is a homeomorphism.

Definition 4.1. [6] A topological groupoid G is called locally trivial if for each object x, the restriction of the target (source) map $T : G_x \longrightarrow G_0$ ($S : G^x \longrightarrow G_0$) is a local homeomorphism.

Theorem 4.2. If X is locally path connected and semilocally simply connected, then $\pi^w X$ ($\pi^l X$) is locally trivial i.e., for each $x \in X$, the target map $T : (\pi^w X)_x \to X$ defined by $T([\alpha]) = \alpha(1)$ is a local homeomorphism.

Proof. Since X is semilocally simply connected, $\pi^l X$ and $\pi^w X$ are equivalent. So it suffices to show that $T : (\pi^w X)_x \to X$ is a local homeomorphism.
Let $[\alpha] \in (\pi^w X)_x$ and $N([\alpha], U, V) \cap (\pi^w X)_x$ be an open neighborhood of $[\alpha]$ in $(\pi^w X)_x$, where α is a path from x to y. Since X is semilocally simply connected, there are simply connected neighborhoods W and W' containing x and y, respectively. Let $U' = U \cap W$, $V' = V \cap W'$ and $O = N([\alpha], U', V') \cap (\pi^w X)_x$. Then we have $[\alpha] \in O \subseteq N([\alpha], U, V) \cap (\pi^w X)_x$. Since $T(\alpha) = \alpha(1) = y \in V'$, $\alpha \in O$ and the map T takes each member of O (which is obtained from the concatenation of the path α by a path in V') to its endpoint (which is a member

of V' connected to $\alpha(1)$), $T(O)$ is the path component of V' containing $\alpha(1)$ which is open because X is locally path connected.
Now we show that $T|_O : O \to T(O)$ is a homeomorphism. Let $[\beta], [\gamma] \in O$ with $\beta(1) = \gamma(1)$. We have $\beta \simeq \lambda * \alpha * \mu$ and $\gamma \simeq \lambda' * \alpha * \mu'$, where λ, λ' are paths in U' with $\lambda(0) = \lambda'(0) = x$ and μ, μ' are paths in V' with $\mu(1) = \mu'(1)$. λ and λ' are loops in $U' \subseteq W$ which implies $\lambda \simeq \lambda'$. Also $\mu \simeq \mu'$ since $\mu * (\mu')^{-1}$ is a loop in $V' \subseteq W'$. Therefore $[\beta] = [\gamma]$ and hence $T|_O$ is injective. Since the target map is continuous, it suffices to show that $T|_O$ is an open map. Let $N([\beta], L, K)$ be an open basic neighborhood in O, then $T(N([\beta], L, K))$ is open because it is the path component of K containing $\beta(1)$.

□

Remark 4.3. In [3] it is proved that by the lifted topology on πX, the target map $T : (\pi X)_x \longrightarrow X$ is a covering map, for every $x \in X$, and so is a local homeomorphism. When the given space X is locally path connected and semilocally simply connected, the lifted topology and the Lasso topology are equivalent [13], and so we can consider Theorem 4.2 as a result of Proposition 3.2 and [3, Theorem 1]

Let $\mathbb{H}$ be the Hawaiian earring space. We want to prove that $\pi^l\mathbb{H}$ and $\pi^w\mathbb{H}$ are not locally trivial. But we cannot provide a single proof for both at once because in Example 3.3 we have shown that $\pi^l\mathbb{H}$ and $\pi^w\mathbb{H}$ are not equivalent.

Example 4.4. $T : (\pi^l\mathbb{H})_x \to \mathbb{H}$ is not a local homeomorphism.

Proof. For an arbitrary $[\alpha] \in (\pi^l\mathbb{H})_x$, let $O = N([\alpha], \mathcal{U}, U, V) \cap (\pi^l\mathbb{H})_x$ be an open neighborhood of $[\alpha]$. We show that $T|_O : O \to T(O)$ is not one to one. Let y be the common point of all circles and $W \in \mathcal{U}$ be an open neighborhood of y. There exists $0 < N \in \mathbb{N}$ such that for all $n > N$, α_n lies in W, where α_n denotes the circle of radius $\frac{1}{n}$. Let λ be a path from x to y and

For $n \neq m > N$, define $\gamma = \alpha*(\lambda*\alpha_m*\lambda^{-1})$ and $\beta = \alpha*(\lambda*\alpha_n*\lambda^{-1})$. Since $[\lambda*\alpha_m*\lambda^{-1}], [\lambda*\alpha_n*\lambda^{-1}] \in \pi_1(\mathcal{U}, x)$, we have $[\beta], [\gamma] \in N([\alpha], \mathcal{U}, U, V) \cap (\pi^l\mathbb{H})_x$. The contradiction arises from that

$$T([\gamma]) = T([\beta]),$$

while $[\gamma] \neq [\beta]$ because $[\alpha_n] \neq [\alpha_m]$. □

Example 4.5. $T : (\pi^w\mathbb{H})_x \to \mathbb{H}$ is not a local homeomorphism.

Proof. Let $N(\alpha_1, U, V) \cap (\pi^w\mathbb{H})_x$ be an open neighborhood of $[\alpha_1]$. There exists $0 < N \in \mathbb{N}$ such that for all $m, k > N$, α_m and α_k lie in V. For $n \neq m > N$, define $\gamma = \alpha_1 * \alpha_m$ and $\beta = \alpha_1 * \alpha_n$. Since $[\beta], [\gamma] \in N([\alpha], U, V) \cap (\pi^l\mathbb{H})_x$ and

$$T([\gamma]) = T([\beta]),$$

we have a contradiction because $[\gamma] \neq [\beta]$.

□

In the following example, we show that local path connectedness is also essential in Theorem 4.2. Although it is known that when a space is not locally path connected, the fundamental groupoid cannot be endowed with a topological groupoid structure, and locally trivial groupoids are assumed to be inherently topological, it is nevertheless interesting to present an example of a non-locally path connected space whose topologized fundamental groupoid is not locally homeomorphic to the space itself.

Let C be the comb space: The subset C of the Euclidean plane formed by the union of the x-axis, the line segment with interval $[0,1]$ of the y-axis, and the sequence of segments with endpoints $(\frac{1}{n},0)$ and $(\frac{1}{n},1)$ for all positive integers n. Concerning the relative topology, C is path connected. It is therefore connected, but not locally path connected at any point of the interval $\{0\}\times(0,1]$ since each open disk centered at one of these points intersects C in a union of parallel segments, forming a disconnected set.

Example 4.6. $T:(\pi^w C)_x \to C$ is not a local homeomorphism.

Proof. Let α be a path in $\{0\}\times[0,1]$ from $y=(0,0)$ to $x=(0,1)$ and let $O=N([\alpha],U,V)\cap(\pi^w C)_x$ be an open neighborhood of $[\alpha]$. We show that $T|O:O\to T(O)$ is not open. Let V' be an open subset of V containing $x=(0,1)$ such that $V'\cap([0,1]\times\{0\})=\emptyset$. $T(N([\alpha],U,V'))$ is the path component of V' containing $\alpha(1)$ i.e. $T(N([\alpha],U,V'))\subseteq\{0\}\times(0,1]$ and hence it is not open in C.

□

Remark 4.7. It is shown in [13] that $\pi^w\mathbb{H}$ is not a topological fundamental groupoid. Also, if X is a locally path connected space, $\pi^l X$ is a topological groupoid [12] and we cannot use the term topological fundamental groupoid for $\pi^l C$.

Corollary 4.8. The fundamental groupoids $\pi^l\mathbb{H}$, $\pi^w\mathbb{H}$, $\pi^l C$ and $\pi^w C$ are not locally trivial.

Definition 4.9. [10] A topological groupoid G is called étale if the target (source) map $T:G\longrightarrow G_0$ $(S:G\longrightarrow G_0)$ is a local homeomorphism.

Note that in the definition of locally trivial groupoid, the target map is restricted to G_x, but in the definition of étale groupoid, the target map is restricted to the entire groupoid G.

Theorem 4.10. For a path connected space X, $\pi^l X$ is not étale.

Proof. Let $\alpha\in\pi X(x,y)$. For every open subset $O\subseteq\pi X$ containing α, we prove that $T|_O$ can not be a homeomorphism. Consider a basic open neighborhood $N([\alpha],\mathcal{U},U,V)\subseteq O$, where $\mathcal{U}$ is an open cover of X and $U,V\in\mathcal{U}$. Choose a point $x\neq z\in U$ that is connected to x by a path, named λ. Obviously, $[\lambda*\alpha]\in N([\alpha],\mathcal{U},U,V)$ while $[\lambda*\alpha]\neq[\alpha]$ and $T([\lambda*\alpha])=T([\alpha])$.

□

In a similar way, we have this result for the Whisker topology.

Corollary 4.11. For a path connected space X, $\pi^w X$ is not étale.

Remark 4.12. It should be noted that in the previous corollary, the path connectedness is not critical, and the existence of only one nonconstant path in the space X is sufficient for the proof process to proceed correctly.

5. Conclusions and future works

In this paper, we compare two recently introduced topologies on the fundamental groupoid: the Lasso topology [12] and the Whisker topology [13]. We show that for locally path connected and semilocally simply connected spaces, these two topologies concide, and the converse is not necessarily true. The equality of these two topologies on the fundamental groupoid of the space X makes X a strong small loop transfer space.

Furthermore, we prove that the fundamental groupoid endowed with either of these topologies is locally trivial when the underlying space is locally path connected and semilocally simply connected. We also provide examples illustrating that these conditions are essential. Finally, we demonstrate that the fundamental groupoid equipped with the Lasso or Whisker topology is not necessarily étale. Since the fundamental groupoid with these topologies forms a topological groupoid for a broader class of spaces than just locally nice spaces, comparing these topologies and analyzing their properties is of greater significance than for other known topologies [3, 9].

Given that the bases for the Lasso and Whisker topologies are explicit, it appears promising to investigate additional topological properties of the fundamental groupoid–such as Hausdorffness, local path connectivity, and even local compactness or compactness–as directions for future research.

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8. Conflict of interest

The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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