A RELATION BETWEEN THE CATEGORIES $\overrightarrow{Set}$, $Set_{\mathbb{T}}$, Set_* AND $Set^{\mathbb{T}}$

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Abstract. In this article, we have shown, for the add-point monad $\mathbb{T}$, the partial morphism category $\overrightarrow{Set}$ is isomorphic to the Kleisli category $Set_{\mathbb{T}}$. Also we have proved that the category, $Set^{\mathbb{T}}$, of $\mathbb{T}$-algebras is isomorphic to the category Set_* of pointed sets. Finally we have established commutative squares involving these categories.

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Keywords: monad, partial morphism category, category of pointed sets, Kleisli category, category of $\mathbb{T}$-algebras

1. Introduction

The partial morphism categories [1, 4, 8, 9, 10, 11, 12, 13], the Kleisli categories [1, 6, 7, 14], the categories of algebras [1, 2, 3, 7] and the pointed categories [1, 5], are all useful categories with a wide range of applications.

In this article we have established a relation between the above mentioned categories, when the base category is the category Set of sets and functions, and the monad is what we have called the add-point monad.

In Section 2, we have defined the add-point monad and we have given functors between the category, $\overrightarrow{Set}$, of partial functions and the Kleisli category $Set_{\mathbb{T}}$. We

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have then shown that these functors are inverses of each other, proving the two categories are isomorphic.

In Section 3, we have given functors between the category, $Set^{\mathbb{T}}$, of $\mathbb{T}$-algebras, and the category Set_* of pointed sets. We have then shown that these functors are inverses of each other, proving the two categories are isomorphic.

Finally in Section 4, we have given functors from $Set_{\mathbb{T}}$ to $Set^{\mathbb{T}}$ and from $\overrightarrow{Set}$ to Set_* and we have established commutative squares involving these categories.

2. $\overrightarrow{Set}$ and $Set_{\mathbb{T}}$ are Isomorphic

Definition 2.1. The partial morphism category, $\overrightarrow{Set}$, associated to the category Set of sets and functions has the same objects as Set, with morphisms $\overrightarrow{f} = [(i_f, f)] : X \to Y$ equivalence classes of pairs $(i_f : D_f \to X, f : D_f \to Y)$ where f is a function and i_f is a mono. Equivalence of (i_f, f) and (i_g, g) means that there is an isomorphism φ for which $i_f = i_g \circ \varphi$ and $f = g \circ \varphi$.

The composition of morphisms $X \xrightarrow{\overrightarrow{f}} Y \xrightarrow{\overrightarrow{g}} Z$ is defined by $\overrightarrow{g} \circ \overrightarrow{f} = [(i_g, g)] \circ [(i_f, f)] = [(i_f(f^{-1}(i_g)), g(i_g^{-1}(f)))]$, where $f^{-1}(i_g)$ is the pullback of i_g along f, etc; and the identity morphism on X is defined to be $[(1_X, 1_X)]$.

Definition 2.2. The add-point monad $\mathbb{T} = (T, \eta, \mu)$, consists of the endofunctor $T : Set \to Set$, where $T(X) = X \sqcup 1$ and $T(f) = f \sqcup 1$; the natural transformation $\eta : I \to T$, where $\eta_X = \nu_1 : X \to X \sqcup 1$ is the first injection of the coproduct, and the natural transformation $\mu : T^2 \to T$, where $\mu_X = 1 \oplus \nu_2 : (X \sqcup 1) \sqcup 1 \to X \sqcup 1$, with ν_2 the second injection of the coproduct.

Definition 2.3. Let $\mathbb{T}$ be the add-point monad. The Kleisli category $Set_{\mathbb{T}}$ has sets as objects, and a morphism $\hat{f} : X \to Y$ corresponds to a morphism $f : X \to Y \sqcup 1$ in Set. The identity morphism on X is $1_X = \widehat{\eta_X} : X \to X$, and the composition of morphisms $X \xrightarrow{\hat{f}} Y \xrightarrow{\hat{g}} Z$ is defined by $\hat{g} \circ \hat{f} = \mu_Z \circ \widehat{(g \sqcup 1)} \circ f$.

Remark 2.4. For any pair (i_f, f) where f is a function and i_f is a monomorphism, there is a unique morphism $\bar{f}$ making the following square a pullback in Set.

$$\begin{array}{ccc} D_f & \xrightarrow{f} & Y \\ {\scriptstyle i_f}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ X & \xrightarrow[\bar{f}]{} & Y \sqcup 1 \end{array}$$

$\bar{f}$ is defined by:

$$\bar{f}(x) = \begin{cases} \nu_1 f(x') & \text{if } x = i_f(x') \\ 1 & \text{otherwise} \end{cases}$$

and if $\bar{g} : X \to Y \sqcup 1$ is a morphism such that the following square is a pullback,

$$\begin{array}{ccc} D_f & \xrightarrow{f} & Y \\ {\scriptstyle i_f}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ X & \xrightarrow[\bar{g}]{} & Y \sqcup 1 \end{array}$$

then for $x = i_f(x')$, we have $\bar{g}(x) = \bar{g} i_f(x') = \nu_1 f(x') = \bar{f}(x)$; and for $x \notin i_f(D_f)$, $\bar{g}(x) = 1$, since otherwise there is $y \in Y$ such that $\bar{g}(x) = y$ which implies $x = i_f(x')$ for some $x' \in D_f$ and that is a contradiction. Hence $\bar{g} = \bar{f}$.

Proposition 2.5. *The map $\alpha : Set_{\mathbb{T}} \to \overrightarrow{Set}$ that acts like identity on objects and takes each morphism $\hat{f} : X \to Y$ to a morphism $\overrightarrow{f^*} = [(i_{f^*}, f^*)] : X \to Y$, where (i_{f^*}, f^*) is obtained by the pullback,*

$$\begin{array}{ccc} D_f & \xrightarrow{f^*} & Y \\ {\scriptstyle i_{f^*}}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ X & \xrightarrow[f]{} & Y \sqcup 1 \end{array}$$

in Set, is a functor.

Proof. It is easy to verify that α is well-defined and preserves identities. To show α preserves composition, let $\hat{f}, \hat{g} : X \to Y$ be two morphisms in $Set_{\mathbb{T}}$ and set $\hat{h} = \hat{g} \circ \hat{f} = \mu_Z \widehat{(g \sqcup 1)} f$. Then $\alpha(\hat{h}) = \overrightarrow{h^*}$, where the following square is a pullback in Set.

$$\begin{array}{ccc} D_{h^*} & \xrightarrow{h^*} & Z \\ {\scriptstyle i_{h^*}}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ X & \xrightarrow[h]{} & Z \sqcup 1 \end{array}$$

On the other hand we have the composition $\alpha(\hat{g}) \circ \alpha(\hat{f}) = \overrightarrow{g^*} \circ \overrightarrow{f^*} = [(i_{f^*} \circ (f^*)^{-1}(i_{g^*}), g^* \circ (i_{g^*})^{-1}(f^*))]$ with the following pullback squares.

$$\begin{array}{ccccccc}
E & \xrightarrow{(i_{g^*})^{-1}(f^*)} & D_{g^*} & \xrightarrow{g^*} & Z & \xrightarrow{1} & Z \\
{\scriptstyle (f^*)^{-1}(i_{g^*})}\downarrow & & \downarrow{\scriptstyle i_{g^*}} & & \downarrow{\scriptstyle \nu_1} & & \\
D_{f^*} & \xrightarrow[f^*]{} & Y & \xrightarrow[g]{} & Z \sqcup 1 & & \downarrow{\scriptstyle \nu_1} \\
{\scriptstyle i_{f^*}}\downarrow & & \downarrow{\scriptstyle \nu_1} & & \downarrow{\scriptstyle \nu_1'} & & \\
X & \xrightarrow[f]{} & Y \sqcup 1 & \xrightarrow[g \sqcup 1]{} & (Z \sqcup 1) \sqcup 1 & \xrightarrow[\mu_Z]{} & Z \sqcup 1
\end{array}$$

Hence, we have the following pullback squares.

$$\begin{array}{ccc}
E & \xrightarrow{g^* \circ (i_{g^*})^{-1}(f^*)} & Z \\
{\scriptstyle i_{f^*} o (f^*)^{-1}(i_{g^*})}\downarrow & & \downarrow{\scriptstyle \nu_1} \\
X & \xrightarrow[\mu_Z (g \sqcup 1) f]{} & Z \sqcup 1
\end{array}
\qquad\qquad
\begin{array}{ccc}
D_{h^*} & \xrightarrow{h^*} & Z \\
{\scriptstyle i_{h^*}}\downarrow & & \downarrow{\scriptstyle \nu_1} \\
X & \xrightarrow[\mu_Z (g \sqcup 1) f]{} & Z \sqcup 1
\end{array}$$

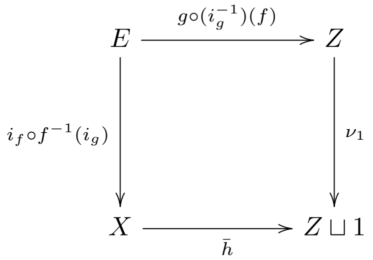
Since pullbacks are unique up to isomorphism, $[(i_{f^*} \circ (f^*)^{-1}(i_{g^*}), g^* \circ (i_{g^*})^{-1}(f^*))]$ $= [(i_{h^*}, h^*)] = \alpha(\hat{g} \circ \hat{f})$. The result then follows. □

Proposition 2.6. *The map $\beta : \overrightarrow{Set} \to Set_{\mathbb{T}}$ that acts like identity on objects and takes each morphism $\overrightarrow{f} : X \to Y$ to a morphism $\hat{\bar{f}} : X \to Y$, where $\bar{f} : X \to Y \sqcup 1$ is the unique function obtained by the pullback*

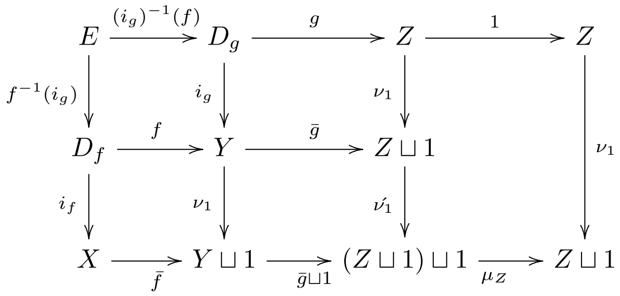
$$\begin{array}{ccc}
D_f & \xrightarrow{f} & Y \\
{\scriptstyle i_f}\downarrow & & \downarrow{\scriptstyle \nu_1} \\
X & \xrightarrow[\bar{f}]{} & Y \sqcup 1
\end{array}$$

in Set, is a functor.

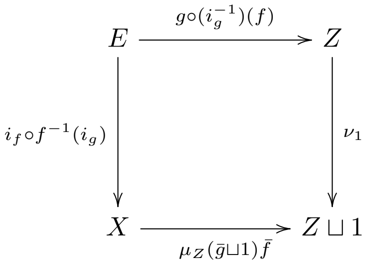
Proof. It is easy to verify that β is well-defined and preserves identities. To show β preserves composition, let $\overrightarrow{f}, \overrightarrow{g} : X \to Y$ be two morphisms in $\overrightarrow{Set}$ and set $\overrightarrow{h} = \overrightarrow{g} \circ \overrightarrow{f} = [(i_f \circ f^{-1}(i_g), g \circ (i_g)^{-1}(f))]$. We have $\beta(\overrightarrow{g} \circ \overrightarrow{f}) = \hat{\bar{h}}$, with the following pullback square.

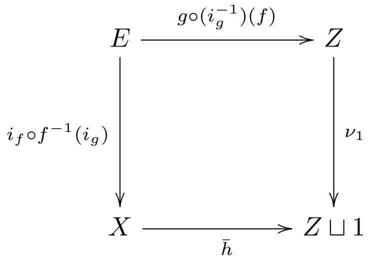


On the other hand we have $\beta(\overrightarrow{g})\circ\beta(\overrightarrow{f}) = \mu_Z\widehat{(\bar{g}\sqcup 1)\bar{f}}$ with the following pullback squares.



So we have the following pullback squares.



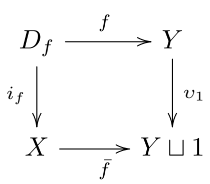


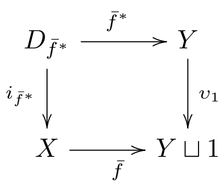
By Remark 2.4, $\bar{h} = \mu_Z(\bar{g}\sqcup 1)\bar{f}$. The result then follows. ☐

Theorem 2.7. *The categories $\overrightarrow{Set}$ and $Set_{\mathbb{T}}$ are isomorphic.*

Proof. We show that the above functors α and β are inverses of each other.

It is Obvious that $\alpha\circ\beta$ is identity on objects. Now let $\overrightarrow{f}$ be a morphism in $\overrightarrow{Set}$. We have $\alpha\circ\beta(\overrightarrow{f}) = \alpha(\hat{\bar{f}}) = \overrightarrow{\bar{f}^*}$ with the following pullback squares in Set.





Since pullbacks are unique up to isomorphism, we have $\overrightarrow{\overline{f}^*} = [(i_{\overline{f}^*}, \overline{f}^*)] = [(i_f, f)] = \overrightarrow{f}$. Hence $\alpha \circ \beta = 1_{\overrightarrow{Set}}$.

Next we show $\beta \circ \alpha : Set_{\mathbb{T}} \to Set_{\mathbb{T}}$ is also the identity functor. Obviously it is on objects. Let $\hat{f} : X \to Y$ be a morphism in $Set_{\mathbb{T}}$. We have $\beta \circ \alpha(\hat{f}) = \beta(\overrightarrow{f^*}) = \hat{\bar{f}}^*$ with the following pullback squares in Set.

$$\begin{array}{ccc} D_f & \xrightarrow{f^*} & Y \\ {\scriptstyle i_{f^*}}\downarrow & & \downarrow{\scriptstyle v_1} \\ X & \xrightarrow[f]{} & Y \sqcup 1 \end{array} \qquad\qquad \begin{array}{ccc} D_{f^*} & \xrightarrow{f^*} & Y \\ {\scriptstyle i_{f^*}}\downarrow & & \downarrow{\scriptstyle v_1} \\ X & \xrightarrow[\bar{f}^*]{} & Y \sqcup 1 \end{array}$$

By Remark 2.4, we have: $\bar{f}^* = f$. Hence $\beta \circ \alpha = 1_{Set_{\mathbb{T}}}$. □

3. Set_* and $Set^{\mathbb{T}}$ are Isomorphic

Definition 3.1. The category Set_* of pointed sets has as objects the pairs (X, x_0), where X is a set and $x_0 \in X$, and as morphisms the point-preserving functions $(X, x_0) \xrightarrow{f} (Y, y_0)$.

Definition 3.2. Let $\mathbb{T}$ be the add-point monad, the category of $\mathbb{T}$-algebras, $Set^{\mathbb{T}}$, has (X, h) as objects where X is a set and the T-algebra $X \sqcup 1 \xrightarrow{h} X$ is a function such that $h \circ \eta_X = 1_X$ and $h \circ \mu_X = h \circ (h \sqcup 1)$, and a morphism from (X, h) to (Y, h') is a function $f : X \to Y$ such that $f \circ h = h' \circ (f \sqcup 1)$. Composition and identities are as in sets.

Remark 3.3. Let $f : (X, h) \to (Y, h')$ be a morphism in $Set^{\mathbb{T}}$. Then $f \circ h = h' \circ (f \sqcup 1)$ and so $f(h(1)) = h'(f \sqcup 1)(1) = h'(1)$.

Proposition 3.4. *The map $\gamma : Set^{\mathbb{T}} \to Set_*$ which takes the object (X, h) to the object $(X, h(1))$ and the morphism $f : (X, h) \to (Y, h')$ to the morphism $f : (X, h(1)) \to (Y, h'(1))$, is a functor.*

Proof. By Remark 3.3, γ is well-defined on objects; the rest follows easily. □

Remark 3.5. For a morphism $f : (X, x_0) \to (Y, y_0)$ in Set_*, and $\hat{x_0}$ the constant function with value x_0, we have $(1_Y \oplus \hat{y_0}) \circ (f \sqcup 1) = (1_Y \circ f) \oplus (\hat{y_0} \circ 1) = f \oplus \hat{y_0}$ $= (f \circ 1_X) \oplus (f \circ \hat{x_0}) = f \circ (1_X \oplus \hat{x_0})$. Therefore, the following diagram commutes.

$$\begin{array}{ccc} X \sqcup 1 & \xrightarrow{f \sqcup 1} & Y \sqcup 1 \\ {\scriptstyle 1_X \oplus \hat{x_0}} \downarrow & & \downarrow {\scriptstyle 1_Y \oplus \hat{y_0}} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Proposition 3.6. *The map $\delta : Set_* \to Set^{\mathbb{T}}$ that takes the object (X, x_0) to the object (X, h), where $h = 1_X \oplus \hat{x_0} : X \sqcup 1 \to X$; and the morphism $f : (X, x_0) \to (Y, y_0)$ to the morphism $f : (X, h) \to (Y, h')$, is a functor.*

Proof. δ is well-defined by Remark 3.5. The rest follows easily. □

Theorem 3.7. *The categories Set_* and $Set^{\mathbb{T}}$ are isomorphic.*

Proof. We show that the above functors δ and γ are inverses of each other. First for each (X, x_0) in Set_*, we have $\gamma \circ \delta(X, x_0) = \gamma(X, h = 1_X \oplus \hat{x_0}) = (X, h(1)) = (X, x_0)$. So $\gamma \circ \delta$ is identity on objects. It follows easily that it is also identity on morphisms. Hence, $\gamma \circ \delta = 1_{Set_*}$.

Next we show $\delta \circ \gamma : Set^{\mathbb{T}} \to Set^{\mathbb{T}}$ is also the identity functor. For each (X, h) in $Set^{\mathbb{T}}$ we have $\delta \circ \gamma(X, h) = \delta(X, h(1)) = (X, 1_X \oplus \widehat{h(1)}) = (X, h)$. So $\delta \circ \gamma$ acts like identity on objects. It can be easily seen that it acts like identity on morphisms. So $\delta \circ \gamma = 1_{Set^{\mathbb{T}}}$. □

4. A Relation Between $Set_{\mathbb{T}}$, $Set^{\mathbb{T}}$, $\overrightarrow{Set}$ and Set_*

Proposition 4.1. *The map $\varphi : Set_{\mathbb{T}} \to Set^{\mathbb{T}}$ that takes the object X to the object $(X \sqcup 1, \mu_X)$ and the morphism $\hat{f} : X \to Y$ to the morphism $\tilde{f} = f \oplus \nu_2 : (X \sqcup 1, \mu_X) \to (Y \sqcup 1, \mu_Y)$, is a functor.*

Proof. Straightforward. □

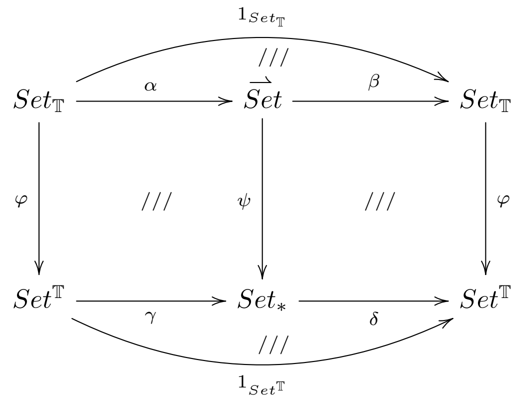
Proposition 4.2. *The map $\psi : \overrightarrow{Set} \to Set_*$ that takes the object X to the object $(X \sqcup 1, 1)$ and the morphism $\overrightarrow{f} : X \to Y$ to the morphism $\bar{f} \oplus \nu_2 : (X \sqcup 1, 1) \to (Y \sqcup 1, 1)$, where $\bar{f}$ is obtained by the pullback*

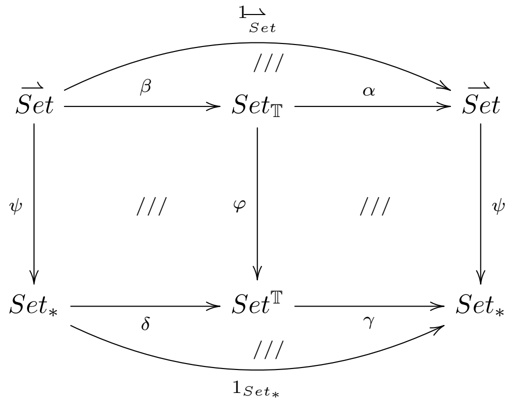
$$\begin{array}{ccc} D_f & \xrightarrow{f} & Y \\ {\scriptstyle i_f} \downarrow & & \downarrow {\scriptstyle \nu_1} \\ X & \xrightarrow[\bar{f}]{} & Y \sqcup 1 \end{array}$$

is a functor.

Proof. Straightforward. □

Theorem 4.3. *We have the following commutative diagrams of functors.*





Proof. We only need to show the commutativity of the squares. For $\hat{f} : X \to Y$ a morphism in $Set_{\mathbb{T}}$, we have $(\psi \circ \alpha)(\hat{f}) = \psi(\overrightarrow{f^*}) = \bar{f}^* \oplus \nu_2$ with the following pullback squares in Set.

By Remark 2.4, $f = \bar{f}^*$. Hence $(\psi \circ \alpha)(\hat{f}) = f \oplus \nu_2$. On the other hand, we have $(\gamma \circ \varphi)(\hat{f}) = \gamma(f \oplus \nu_2) = (f \oplus \nu_2)$, therefore $\psi \circ \alpha = \gamma \circ \varphi$.

Now let $\overrightarrow{f} : X \to Y$ be a morphism in $\overrightarrow{Set}$. We have $(\varphi \circ \beta)(\overrightarrow{f}) = \varphi(\hat{\bar{f}}) = \bar{f} \oplus \nu_2$ and $(\delta \circ \psi)(\overrightarrow{f}) = \delta(\bar{f} \oplus \nu_2) = (\bar{f} \oplus \nu_2)$. Where $\bar{f}$ is the morphism making the following square a pullback in Set.

$$
\begin{array}{ccc}
D_f & \xrightarrow{f} & Y \\
{\scriptstyle i_f}\big\downarrow & & \big\downarrow{\scriptstyle \nu_1} \\
X & \xrightarrow[\bar{f}]{} & Y \sqcup 1
\end{array}
$$

Therefore $\varphi \circ \beta = \delta \circ \psi$. □

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