SOME CHARACTERIZATIONS OF HYPER MV-ALGEBRAS

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ABSTRACT. In this paper we characterize hyper MV-algebras in which 0 or 1 are scalar elements . We prove that any finite hyper MV-algebra that 0 is a scaler element in it, is an MV-algebra. Finally we characterize hyper MV-algebras of order 2 and order 3.

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1. Introduction

The concept of an MV-algebra was introduced by C.C. Chang in 1958 [2] to prove the completeness theorem of infinite valued Łukasiewicz propositional calculus. The hyper structure theory was introduced by F. Marty at 8th congress of Scandinavian Mathematicians in 1934 [4]. Since then many researchers have worked in these areas[1,6]. Recently in [3], Sh. Ghorbani and et.al applied the hyper structure to MV-algebras and introduced the concept of a hyper MV-algebra which is a generalization of an MV-algebra and investigated some related results. In paper [5] authors defined (weak)hyper MV-ideals in hyper MV-algebras. In this paper we want to find conditions that a hyper MV-algebra becomes an MV-algebra. We

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prove that a hyper MV-algebra that 1 is a scalar element on it is an MV-algebra. Also we characterize hyper MV-algebras in which 0 is a scalar element.
In the next section, some preliminary theorems and definitions are stated from [3] and [5]. In section 3, we characterize hyper MV-algebras in which 0 or 1 are scalar. Also we obtain conditions under which a hyper MV-algebra is an MV-algebra. In section 4, we characterize hyper MV-algebras of order 2 and order 3 and we obtain three non-isomorphic hyper MV-algebras of order 2 and thirty-six non-isomorphic hyper MV-algebras of order 3.

2. **Preliminaries**

An MV-algebra is an algebra $(A, \oplus, ^*, 0)$ of type $(2, 1, 0)$ satisfying the following equations:

$(MV1)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
$(MV2)$ $x \oplus y = y \oplus x$,
$(MV3)$ $x \oplus 0 = x$,
$(MV4)$ $x^{**} = x$,
$(MV5)$ $x \oplus 0^* = 0^*$,
$(MV6)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,

for all $x, y, z \in A$.
Consider the real unit interval $[0, 1]$ and for all $x, y \in [0, 1]$, define $x \oplus y = min\{1, x + y\}$ and $x^* = 1 - x$. Then $([0, 1], \oplus, ^*, 0)$ is an MV-algebra. The rational numbers in $[0, 1]$ and the n-element set $L_n = \{0, 1/(n-1), ..., (n-2)/(n-1), 1\}$, for each integer $n \geq 2$, yield examples of subalgebras of $[0, 1]$.
For any two elements x and y of an MV-algebra $(A, \oplus, ^*, 0)$, define $x \leq y$ if and only if $x^* \oplus y = 1$, where $1 := 0^*$. Then $\leq$ is a partial order, called the natural order of A.

Definition 2.1. [3] A hyper MV-algebra is a nonempty set M endowed with a hyper operation "$\oplus$", a unary operation "$*$" and a constant "0" satisfying the following axioms:

(hMV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(hMV2) $x \oplus y = y \oplus x$,
(hMV3) $(x^*)^* = x$,

(hMV4) $0^* \in x \oplus x^*$,
(hMV5) $0^* \in x \oplus 0^*$,
(hMV6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(hMV7) if $0^* \in x^* \oplus y$ and $0^* \in y^* \oplus x$, then $x = y$.
for all $x, y, z \in M$.

Let M be a hyper MV-algebra, $x \ll y$ is defined by $0^* \in x^* \oplus y$, for all $x, y \in M$ and for every $A, B \subseteq M$, we define $A \ll B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \ll b$. Also, we define $1 := 0^*$, $A^* := \{a^* | a \in A\}$ and $A \oplus B := \bigcup_{\substack{a \in A \\ b \in B}} a \oplus b$, for every $A, B \subseteq M$.
Let $[0, 1]$ be real unit interval. We can see that $([0, 1], \oplus, ^*, 0)$ is a hyper MV-algebra, where $x \oplus y = [0, min\{1, x + y\}]$ and $x^* = 1 - x$, for all $x, y \in [0, 1]$.

Proposition 2.2. [3] *Let $(M, \oplus, ^*, 0)$ be a hyper MV-algebra. Then for all $x, y, z \in M$ and for all nonempty subsets A, B and C of M the following statements hold:*
(a_1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
(a_2) $0 \ll x$,
(a_3) $x \ll x$,
(a_4) *if* $x \ll y$, *then* $y^* \ll x^*$ *and* $A \ll B$ *implies* $B^* \ll A^*$,
(a_5) $x \ll 1$,
(a_6) $A \ll A$,
(a_7) $A \subseteq B$ *implies* $A \ll B$,
(a_8) $x \ll x \oplus y$ *and* $A \ll A \oplus B$,
(a_9) $(A^*)^* = A$,
(a_{10}) $0 \oplus 0 = \{0\}$,
(a_{11}) $x \in x \oplus 0$.

A hyper MV-algebra $(M, \oplus, ^*, 0)$ is called nontrivial if $M \neq \{0\}$. It is clear that a hyper MV-algebra is nontrivial if and only if $0 \neq 1$. In this paper, we consider nontrivial hyper MV-algebras. An element $a \in M$ is called a scalar element, if $|a \oplus x| = 1$, for all $x \in M$.
Let $(M_1, \oplus_1, ^{*_1}, 0_1)$ and $(M_2, \oplus_2, ^{*_2}, 0_2)$ be two hyper MV-algebras . A bijection $f : M_1 \to M_2$ is said to be an isomorphism, if for all $x, y \in M$: (i) $f(0) = 0$, (ii) $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$, (iii) $f(x^{*_1}) = (f(x))^{*_2}$.

3. Hyper MV-algebras in which 0 or 1 are scalar elements

In the rest of this paper, we denote a hyper MV-algebra $(M, \oplus, ^*, 0)$ by M.

Theorem 3.1. *If $x \ll y$ and $y \in 0 \oplus a$, for $a, x, y \in M$, then $x \ll a$.*

Proof. By $x \ll y$ we have:

$$1 \in x^* \oplus y \subseteq x^* \oplus (0 \oplus a) = 0 \oplus (x^* \oplus a).$$

Thus there exists $t \in x^* \oplus a$ such that $1 \in 0 \oplus t = 1^* \oplus t$, so we get that $1 \ll t$. Since $t \ll 1$, then $t = 1$. Thus $1 \in x^* \oplus a$, that is $x \ll a$. $\square$

By the above theorem and $y \ll y$ we have:

Corollary 3.2. *If $y \in 0 \oplus a$, for $a, y \in M$, then $y \ll a$.*

Theorem 3.3. *If $x \in 0 \oplus a$ and $x^* \ll x$, for $x, a \in M$, then $a^* \ll a$.*

Proof. By hypothesis we have:

$$1 \in x \oplus x \subseteq x \oplus (0 \oplus a) = 0 \oplus (a \oplus x).$$

Then there is $s \in a \oplus x$ such that $1 \in 0 \oplus s = 1^* \oplus s$ and we have $1 \ll s$, by $s \ll 1$, we get that $s = 1$. Thus

$$1 = s \in x \oplus a \subseteq (0 \oplus a) \oplus a = 0 \oplus (a \oplus a).$$

Hence there exists $l \in a \oplus a$ such that $1 \in 0 \oplus l = 1^* \oplus l$, that is $l = 1$. Therefore $1 \in a \oplus a = (a^*)^* \oplus a$, i.e. $a^* \ll a$. $\square$

Example 3.4. Let $(M, \oplus, ^*, 0)$ be an MV-algebra. Define a hyper operation $\oplus^{'}$ on M by $x \oplus^{'} y = \{x \oplus y\}$, for all $x, y \in M$. Then $(M, \oplus^{'}, ^*, 0)$ is a hyper MV-algebra.

The condition (hMV7) is necessary in any hyper MV-algebra. Since hyper MV-algebras are a generalization of MV-algebras, then hyper MV-algebras whose all elements are scalar should be MV-algebras. By the following example, we show that if the condition (hMV7) is deleted from the definition of a hyper MV-algebra,we obtain a hyper MV-algebra whose all elements are scalar, while it is not an MV-algebra.

Example 3.5. Consider the following tables on $M = \{0, a, 1\}$:

$\oplus$	0	a	1
0	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{1\}$	$\{1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

$*$	0	a	1
	1	a	0

Then $(M, \oplus, ^*)$ satisfies in the axioms (hMV1)-(hMV6) while it does not satisfy in the axiom (hMV7), since $1 \in 0^* \oplus 1$ and $1 \in 1^* \oplus 0$, but $0 \neq 1$. Also M is not an MV-algebra, since $a \oplus 0 = 1 \neq a$.

Note that in a hyper MV-algebra M, if $|x \oplus y| = 1$, for $x, y \in M$, i.e. $x \oplus y = \{a\}$, for $a \in M$, we denote by $x \oplus y = a$.

Lemma 3.6. *Let $(M, \oplus, ^*)$ be a hyper MV-algebra . If $|x \oplus y| = 1$, for all $x, y \in M$, then $(M, \oplus, ^*)$ is an MV-algebra.*

Proof. The proof is easy. □

Theorem 3.7. *Let 1 be a scalar element in a hyper MV-algebra M. Then $|x \oplus y| = 1$, for all $x, y \in M$. Hence M is an MV-algebra.*

Proof. By (hMV6) and Proposition 2.2 for $a \in M$ we have:

$$1 \in a^* \oplus a \subseteq (0 \oplus a)^* \oplus a = (1^* \oplus a)^* \oplus a = (a^* \oplus 1)^* \oplus 1 = 1^* \oplus 1 = 1.$$

Thus $a^* \oplus a = 1$, for all $a \in M$. Let $x, y \in M$ and $e, f \in x \oplus y$. Then by (hMV1) and (hMV6) we have:

$$\begin{aligned}(x \oplus y)^* \oplus (x \oplus y) &= ((x \oplus y)^* \oplus x) \oplus y \\ &= ((x^* \oplus y^*)^* \oplus y^*) \oplus y \\ &= ((x^* \oplus y^*)^* \oplus (y^* \oplus y) \\ &= (x^* \oplus y^*)^* \oplus 1 = 1.\end{aligned}$$

Since $e^* \oplus f$, $f^* \oplus e \subseteq (x \oplus y)^* \oplus (x \oplus y) = 1$, we obtain $e^* \oplus f = f^* \oplus e = 1$, that is $e \ll f$ and $f \ll e$. Hence $e = f$. Thus $|x \oplus y| = 1$, for all $x, y \in M$ and so by Lemma 3.6, M is an MV-algebra. □

Theorem 3.8. *Let 0 be a scalar element in a hyper MV-algebra M. Then $0, x \notin 1 \oplus x$, for all $x \in M - \{1\}$.*

Proof. Let $x \in M - \{1\}$. If $x = 0$, then $0 \notin \{1\} = 1 \oplus 0$. Now let $x \neq 0$. By (hMV6) we have:

$$(1 \oplus x)^* \oplus x = (0^* \oplus x)^* \oplus x = (x^* \oplus 0)^* \oplus 0 = x \oplus 0 = x. \quad (I)$$

If $0 \in 1 \oplus x$, then $0 \in 1 \oplus x = 0^* \oplus x \subseteq (1 \oplus x)^* \oplus x = x$ and so $x = 0$, which is a contradiction. If $x \in 1 \oplus x$, then

$$1 \in x^* \oplus x \subseteq (1 \oplus x)^* \oplus x = x.$$

Hence $x = 1$, which is a contradiction. Therefore $0, x \notin 1 \oplus x$, for all $x \in M - \{1\}$. □

By putting $x = 1$ in (I) of the proof of Theorem 3.8, we can obtain if 0 is a scalar, then $0 \notin 1 \oplus 1$.

Theorem 3.9. *Let 0 be a scalar element in a hyper MV-algebra M and $x, y \in M$. If $y \in 1 \oplus x$, then:*
(i) $x \oplus y^* = x$,
(ii) $(1 \oplus y^*)^* \oplus x = x$,
(iii) $x \oplus y = x \oplus 1$,
(v) $x \neq 1$ *implies* $x \notin 1 \oplus y$,
(iv) $x^* \ll y$ *and* $y^* \ll x$.

Proof. (i) Since 0 is a scalar element, by the proof of Theorem 3.8 we have $(1 \oplus x)^* \oplus x = x$. Thus $y \in 1 \oplus x$ implies $y^* \oplus x = x$.
(ii) By (i) and (hMV6) we have:

$$x = y^* \oplus x = (y \oplus 0)^* \oplus (0 \oplus x) = (0^* \oplus y^*)^* \oplus y^* \oplus x = (1 \oplus y^*)^* \oplus x.$$

(iii) By hypothesis we get :

$$\begin{aligned} x \oplus y &= (y^* \oplus x) \oplus y \\ &= (y^* \oplus y) \oplus x \\ &= ((y \oplus 0)^* \oplus y) \oplus x \\ &= ((y^* \oplus 1)^* \oplus 1) \oplus x \\ &= ((y^* \oplus 1)^* \oplus x) \oplus 1 \\ &= x \oplus 1. \end{aligned}$$

(iv) Let $x \neq 1$ and $x \in 1 \oplus y$. Then by (iii), $x \oplus y = y \oplus 1$ and so $x \oplus 1 = y \oplus 1$. Hence $x \in 1 \oplus y$, implies that $x \in 1 \oplus x$, which is a contradiction by Theorem 3.8.

(v) By (iii) we obtain $1 \in x \oplus y = (x^*)^* \oplus y = (y^*)^* \oplus x$, that is $x^* \ll y$ and $y^* \ll x$. $\square$

Theorem 3.10. *Let 0 be a scalar element in a hyper MV-algebra M. Then $1 \oplus 1 = 1$.*

Proof. We have $1 \in 1 \oplus 1$. If $1 \neq b \in 1 \oplus 1$, then by Theorem 3.9 part (iii), $b \oplus 1 = 1 \oplus 1$ and so $b \in b \oplus 1$, which is not true by Theorem 3.8. Therefore $1 \oplus 1 = 1$. $\square$

Theorem 3.11. *Let 0 be a scalar element in a hyper MV-algebra M and $x^* = x$, for all $x \in M$. Then $|x \oplus y| = 1$, for all $x, y \in M$. Moreover $M = L_3 = \{0, 1/2, 1\}$.*

Proof. We show that 1 is a scalar element in M and so by Theorem 3.7 $|x \oplus y| = 1$, for all $x, y \in M$. By Theorem 3.10 we have $1 \oplus 1 = 1$. Now let $1 \neq y \in 1 \oplus x$ for $x \in M - \{1\}$. Then by Theorem 3.9 part (iv) we obtain $x = x^* \ll y$ and $y = y^* \ll x$ and so $x = y$. Hence $x \in 1 \oplus x$ for $x \neq 1$, it is not true by Theorem 3.8. Therefore $1 \oplus x = 1$, that is 1 is a scalar element. $\square$

Theorem 3.12. *Let M be a hyper MV-algebra and $x \oplus x^* = 1$, for all $x \in M$. Then M is an MV-algebra.*

Proof. First we show that 0 is a scalar element in M. Let $a \in M$. We have $a \in 0 \oplus a$. If $x \in 0 \oplus a$, then

$$x \oplus a^* \subseteq (0 \oplus a) \oplus a^* = 0 \oplus (a \oplus a^*) = 0 \oplus 1 = 0 \oplus 0^* = 1.$$

So $x \oplus a^* = 1$, that is $a \ll x$. Since $x \in 0 \oplus a$, by Corollary 3.2 we have $x \ll a$ and so $x = a$. Therefore $0 \oplus a = a$, i.e. 0 is a scalar element, so by Theorem 3.10, $1 \oplus 1 = 1$. Now we show that 1 is a scalar element in M. Let $x \in M - \{1\}$. Then $1 \in 1 \oplus x^*$ implies that $1 \oplus x \subseteq (1 \oplus x^*) \oplus x = 1 \oplus (x^* \oplus x) = 1 \oplus 1 = 1$. Hence $1 \oplus x = 1$, for $x \neq 1$, that is 1 is a scalar element. Therefore by Theorem 3.7, M is an MV-algebra. $\square$

Theorem 3.13. *Let 0 be a scalar element in a hyper MV-algebra M. Then $|1 \oplus x| = 1$ or $x \oplus 1$ is an infinite subset of M, for all $x \in M$.*

Proof. Let $x \in M$. If $x = 1$, then $1 \oplus 1 = 1$ and so $|1 \oplus x| = 1$. Now let $x \neq 1$. Since $1 \in 1 \oplus x$, then $|1 \oplus x| \geq 1$. Let $|1 \oplus x| \neq 1$. We show that $|1 \oplus x| \neq n$, for all $n \in N$.

By induction, for $n = 1$ we have $|1 \oplus x| \neq 1$. If $|1 \oplus x| = 2$, then $1 \oplus x = \{1, y\}$, for $y \in M - \{1\}$. So by (hMV1) we have

$$\{1, y\} = 1 \oplus x = (1 \oplus 1) \oplus x = 1 \oplus (1 \oplus x) = (1 \oplus 1) \cup (1 \oplus y) = \{1\} \cup (1 \oplus y).$$

Hence $y \in 1 \oplus y$ which is a contradiction by Theorem 3.8. Now let induction is true for n. If $|1 \oplus x| = n + 1$, then $1 \oplus x = \{1, a_1, a_2, ..., a_n\}$. So

$$\{1, a_1, a_2, ..., a_n\} = 1 \oplus x = 1 \oplus (1 \oplus x) = (1 \oplus 1) \cup (1 \oplus a_1) \cup ... \cup (1 \oplus a_n).$$

Thus there is a_k, $1 \leq k \leq n$ such that $|1 \oplus a_k| \neq 1$. Also if $|1 \oplus a_j| = 1$, for all $1 \leq j \leq n$, $j \neq k$, then $1 \oplus a_k = 1 \oplus x$ and so $a_k \in 1 \oplus a_k$, that is not true. Hence there exist $a_k, a_t \in 1 \oplus x$, $t \neq k$, such that $|1 \oplus a_k| \neq 1$ and $|1 \oplus a_t| \neq 1$. Since $a_k \notin 1 \oplus a_k$ and $a_t \notin 1 \oplus a_t$. Then $|1 \oplus a_k| \leq n$, which is a contradiction by hypothesis of induction. Therefore $|1 \oplus x| \neq n + 1$. □

Corollary 3.14. *Let 0 be a scalar element in a finite hyper MV-algebra M. Then 1 is a scalar element.*

By the above corollary and Theorem 3.7 we have:

Corollary 3.15. *Let 0 be a scalar element in a finite hyper MV-algebra M. Then M is an MV-algebra.*

Open problem. If 0 is a scalar element in an infinite hyper MV-algebra. Is M an MV-algebra?

4. Hyper MV-algebras of order 2 and order 3

Theorem 4.1. *There are three non-isomorphic hyper MV-algebras of order 2.*

Proof. Consider $M = \{0, 1\}$. Since M must be a hyper MV-algebra, we have $0 \oplus 0 = 0$. Also $1 \in 0 \oplus 1$, hence $0 \oplus 1 = 1$ or $0 \oplus 1 = \{0, 1\}$. If $0 \oplus 1 = 1$, then 0 is an scalar and so $1 \oplus 1 = 1$, by Corollary 3.14. If $0 \oplus 1 = \{0, 1\}$, by $1 \in 1 \oplus 1$, we have $1 \oplus 1 = 1$ or $1 \oplus 1 = \{0, 1\}$. By considering each case for $1 \oplus 1$, we can obtain a hyper MV-algebra. Therefore we have the following hyper MV-algebras and we can check that they are non-isomorphic.

$\oplus$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{1\}$

$\oplus$	0	1
0	$\{0\}$	$\{0,1\}$
1	$\{0,1\}$	$\{1\}$

$\oplus$	0	1
0	$\{0\}$	$\{0,1\}$
1	$\{0,1\}$	$\{0,1\}$

*	0	1
	1	0

□

Theorem 4.2. *There are thirty-six non-isomorphic hyper MV-algebras of order 3.*

Proof. Let $M = \{0, b, 1\}$. Then the following tables show a probable hyper MV-algebra structure on M:

$\oplus$	0	b	1
0	0	a_{12}	a_{13}
b	a_{21}	a_{22}	a_{23}
1	a_{31}	a_{32}	a_{33}

*	0	b	1
	1	b	0

Since $1 \not\ll b$, then $1 \notin 1^* \oplus b = 0 \oplus b = a_{12}$ and so $1 \notin a_{12}$. By (hMV5) and (hMV4) we have $1 \in a_{13} \cap a_{23} \cap a_{33} \cap a_{22}$ and by (hMV2) $a_{13} = a_{31}, a_{21} = a_{12}$ and $a_{23} = a_{32}$. Also by Proposition 2.2 we have $b \in b \oplus 0$, thus we consider two cases:

case 1: $0 \oplus b = b$, case 2: $0 \oplus b = \{0, b\}$.

In case 1, if $|0 \oplus 1| = 1$, then 0 is a scalar and so by Corollary 3.14, $a_{22} = a_{23} = a_{31} = a_{32} = a_{33} = 1$.

If $|0 \oplus 1| > 1$. Then by (hMV6) we have :

$$0 \oplus 1 \subseteq (b \oplus 1)^* \oplus 1 = (0 \oplus b)^* \oplus b = b \oplus b, \quad (1)$$

$$(1 \oplus b)^* \oplus b = (0^* \oplus b)^* \oplus b = (b \oplus 0)^* \oplus 0 = b \oplus 0 = b. \quad (2)$$

By (1) and (2) we get that $0, b \notin 1 \oplus b$ and thus $a_{32} = 1$. Hence

$$(b \oplus b) \oplus 1 = b \oplus (b \oplus 1) = b \oplus 1 = 1, \quad (3)$$

it implies that $0 \notin b \oplus b$, so by (1), $0 \notin 0 \oplus 1$. Thus $a_{13} = \{1, b\}$ and so by (1), $a_{22} = \{1, b\}$, also by (3), we get that $a_{33} = 1$. Hence by hypothesis we have :

$(0 \oplus 1) \oplus 1 = \{1, b\} \oplus 1 = 1$ and $0 \oplus (1 \oplus 1) = 0 \oplus 1 = \{1, b\}$

that is $(0 \oplus 1) \oplus 1 \neq 0 \oplus (1 \oplus 1)$, i.e. (hMV1) does not hold. Thus in case 1, we have only one the following hyper MV-algebra.

$\oplus$	0	b	1
0	0	b	1
b	b	1	1
1	1	1	1

In case 2, we consider the following subcases :
(a) $b \oplus b = 1$, (b) $b \oplus b = \{0, 1\}$, (c) $b \oplus b = \{1, b\}$, (d) $b \oplus b = \{0, b, 1\}$.
(a) By (hMV1) and (hMV6) we have:

$$1 \oplus 0 = (b \oplus b) \oplus 0 = b \oplus (b \oplus 0) = b \oplus \{0, b\} = \{0, b, 1\},$$

$$1 \oplus b = (b \oplus b) \cup (1 \oplus b) = (0 \oplus b)^* \oplus b = (b \oplus 1)^* \oplus 1 = \{0, b, 1\},$$

$$1 \oplus 1 = (b \oplus b) \oplus 1 = b \oplus (b \oplus 1) = b \oplus \{0, b, 1\} = \{0, b, 1\}.$$

Therefore in this case we have the following hyper MV-algebra

$\oplus$	0	b	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
b	$\{0, b\}$	$\{1\}$	$\{0, b, 1\}$
1	$\{0, b, 1\}$	$\{0, b, 1\}$	$\{0, b, 1\}$

(b) By (hMV1) and (hMV6) we obtain:

$$\{0\} \cup (0 \oplus 1) = (b \oplus b) \oplus 0 = b \oplus (b \oplus 0) = \{0, b, 1\} \Rightarrow \{1, b\} \subseteq 0 \oplus 1, \quad (4)$$

$$b \in 0 \oplus 1 \subseteq (b \oplus 1)^* \oplus 1 = (0 \oplus b)^* \oplus b = \{1, b\} \oplus b = (1 \oplus b) \cup \{0, 1\}, \quad (5)$$

$$\{0, 1, b\} = (1 \oplus 0) \oplus b = 1 \oplus (0 \oplus b) = 1 \oplus \{0, b\} = (1 \oplus 0) \cup (1 \oplus b) \quad (6)$$

$$\{0, b\} = b \oplus 0 \subseteq (0 \oplus 1) \oplus 0 = (0 \oplus 0) \oplus 1 = 0 \oplus 1 \quad (7)$$

By (4) and (7) we get that $0 \oplus 1 = \{0, b, 1\}$ and also by (5) we have $\{1, b\} \subseteq 1 \oplus b$. If $1 \oplus b = \{1, b\}$, since $(1 \oplus 1) \oplus b = 1 \oplus (1 \oplus b) = 1 \oplus \{1, b\} = (1 \oplus 1) \cup \{1, b\}$, we get that $1 \oplus 1 \neq \{1, b\}$. While we can check that $1 \oplus 1 = \{1\}, \{0, 1\}$ or $\{0, b, 1\}$ Also for $1 \oplus b = \{0, b, 1\}$, we can obtain 4 cases for $1 \oplus 1$, that is $1 \oplus 1 = \{1\}$, $1 \oplus 1 = \{0, 1\}$, $1 \oplus 1 = \{1, b\}$ or $1 \oplus 1 = \{0, b, 1\}$.
Therefore in this case we have 7 non-isomorphic hyper MV-algebras:

$\oplus$	0	b	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
b	$\{0, b\}$	$\{0, 1\}$	$\{1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{1\}$

$\oplus$	0	b	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
b	$\{0, b\}$	$\{0, 1\}$	$\{1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{0, 1\}$

$\oplus$	0	b	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
b	$\{0, b\}$	$\{0, 1\}$	$\{1, b\}$
1	$\{0, b, 1\}$	$\{1, b\}$	$\{0, b, 1\}$

$\oplus$	0	b	1
0	0	$\{0, b\}$	$\{0, b, 1\}$
b	$\{0, b\}$	$\{0, 1\}$	$\{0, 1, b\}$
1	$\{0, b, 1\}$	$\{0, 1, b\}$	$\{1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1\}$	$\{0,1,b\}$
1	$\{0,b,1\}$	$\{0,1,b\}$	$\{0,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1\}$	$\{0,1,b\}$
1	$\{0,b,1\}$	$\{0,1,b\}$	$\{b,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1\}$	$\{0,1,b\}$
1	$\{0,b,1\}$	$\{0,1,b\}$	$\{0,b,1\}$

(c) If $0\oplus 1=\{1\}$, then $(1\oplus 1)^*\oplus 1=(0\oplus 0)^*\oplus 0=1\oplus 0=\{1\}$, so $0\not\in 1\oplus 1$. By (hMV6) we have

$$\{1,b\}\cup(1\oplus b)=(1^*\oplus b)^*\oplus b=(b\oplus 1)^*\oplus 1. \quad (8)$$

If $b\in 1\oplus 1$, then $b\oplus 1=1$. So $b\in 0\oplus 1$, which is not true. Thus $b\not\in 1\oplus 1$ and so $1\oplus 1=\{1\}$. By (8) we can obtain $b\oplus 1\neq 1$ and $b\oplus 1\neq\{0,1\}$. Also $1\oplus b=(1\oplus 0)\oplus b=(1\oplus b)\oplus 0$ implies that if $b\in 1\oplus b$, then $1\oplus b=\{0,b,1\}$. By considering $1\oplus b=\{0,b,1\}$ and some manipulations we can obtain the following hyper MV-algebra:

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{1\}$
b	$\{0,b\}$	$\{1,b\}$	$\{0,1,b\}$
1	$\{1\}$	$\{0,1,b\}$	$\{1\}$

If $0\oplus 1=\{0,1\}$, then by (8), $0\oplus 1\subseteq(1\oplus b)\cup\{1,b\}$ and so $0\in 1\oplus b$. Also $\{0,b,1\}=(1\oplus 0)\oplus b=(1\oplus b)\oplus 0$ implies that $1\oplus b=\{0,b,1\}$. Since $(1\oplus 1)^*\oplus 1=1\oplus 0=\{0,1\}$, so $b\not\in 1\oplus 1$. We obtain $1\oplus 1=\{1\}$ or $1\oplus 1=\{0,1\}$, in each case we can check that M is a hyper MV-algebra. So we have 2 the following non-isomorphic hyper MV-algebras.

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,1\}$
b	$\{0,b\}$	$\{1,b\}$	$\{0,b,1\}$
1	$\{0,1\}$	$\{0,b,1\}$	$\{1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,1\}$
b	$\{0,b\}$	$\{1,b\}$	$\{0,b,1\}$
1	$\{0,1\}$	$\{0,b,1\}$	$\{0,1\}$

If $b\in 0\oplus 1$, then by $(0\oplus 0)\oplus 1=(0\oplus 1)\oplus 0$ we get that $0\oplus 1=\{0,b,1\}$. By

$$(1\oplus b)^*\oplus b=(b\oplus 0)^*\oplus 0=\{0,b,1\}, \quad (9)$$

we get that $1 \oplus b \neq 1$. Also by (8) we can obtain $0 \in 1 \oplus b$ and so $\{0,1\} \subseteq 1 \oplus b$. If $1 \oplus b = \{0,1\}$, then by $(1 \oplus 1) \oplus b = (1 \oplus b) \oplus 1 = \{0,b,1\}$, we have $1 \oplus 1 \neq 1$. Thus $1 \oplus 1 = \{0,1\}, \{1,b\}$ or $\{0,b,1\}$. So by some manipulations we can obtain 3 hyper MV-algebras. If $1 \oplus b = \{0,b,1\}$, there are four cases for $1 \oplus 1$. So in this case we have 7 non-isomorphic hyper MV-algebras.

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{1,b\}$	$\{0,1\}$
1	$\{0,b,1\}$	$\{0,1\}$	$\{0,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{1,b\}$	$\{0,b\}$
1	$\{0,b,1\}$	$\{0,1\}$	$\{b,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{1,b\}$	$\{0,1\}$
1	$\{0,b,1\}$	$\{0,1\}$	$\{0,1,b\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{1,b\}$	$\{0,1,b\}$
1	$\{0,b,1\}$	$\{0,1,b\}$	$\{0,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{1,b\}$	$\{0,b,1\}$
1	$\{0,b,1\}$	$\{0,b,1\}$	$\{1,b\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{1,b\}$	$\{0,1,b\}$
1	$\{0,b,1\}$	$\{0,1,b\}$	$\{0,b,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{1,b\}$	$\{0,1,b\}$
1	$\{0,b,1\}$	$\{0,1,b\}$	$\{1\}$

(d) by (9) we have $1 \oplus b \neq 1$. Also

$$\{0,b,1\} = (1^* \oplus b)^* \oplus b = (1 \oplus b)^* \oplus 1. \quad (10)$$

If $1 \oplus b = \{0,1\}$, then $(1 \oplus 1) \cup (0 \oplus 1) = \{0,b,1\}$. If $0 \oplus 1 = 1$, then $1 \oplus 1 = \{0,b,1\}$, it is not true, since the equality $(1 \oplus 1)^* \oplus 1 = (0 \oplus 0)^* \oplus 0$ does not hold. If $0 \oplus 1 = \{0,1\}$, then $\{0,b,1\} = (0 \oplus 1) \oplus b = (0 \oplus b) \oplus 1 = \{0,1\}$, it is not true. Similar to the above argument for (c), if $b \in 0 \oplus 1$, then $0 \oplus 1 = \{0,b,1\}$. We can consider 4 cases for $1 \oplus 1$, i.e. $1 \oplus 1 = 1, \{0,1\}, \{1,b\}$ or $\{0,b,1\}$. Therefore we have

4 the following non-isomorphic hyper MV-algebras.

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1\}$
1	$\{0,b,1\}$	$\{0,1\}$	$\{1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1\}$
1	$\{0,b,1\}$	$\{0,1\}$	$\{0,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1\}$
1	$\{0,b,1\}$	$\{0,1\}$	$\{b,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1\}$
1	$\{0,b,1\}$	$\{0,1\}$	$\{0,1,b\}$

If $1 \oplus b = \{1,b\}$, then by (10), $0 \in 0 \oplus 1$. If $0 \oplus 1 = \{0,1\}$, similar to the above argument $b \notin 1 \oplus 1$, so $1 \oplus 1 = 1$ or $\{0,1\}$. If $0 \oplus 1 = \{0,1,b\}$, we can consider 4 cases for $1 \oplus 1$ and so we have 6 the following non-isomorphic hyper MV-algebras.

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{1,b\}$
1	$\{0,b,1\}$	$\{1,b\}$	$\{1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{1,b\}$
1	$\{0,b,1\}$	$\{1,b\}$	$\{0,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{1,b\}$
1	$\{0,b,1\}$	$\{1,b\}$	$\{1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{1,b\}$
1	$\{0,b,1\}$	$\{1,b\}$	$\{0,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{1,b\}$
1	$\{0,b,1\}$	$\{1,b\}$	$\{1,b\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,b,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{1,b\}$
1	$\{0,b,1\}$	$\{1,b\}$	$\{0,1,b\}$

If $1 \oplus b = \{0,1,b\}$, we have the above cases for $0 \oplus 1$ and additional case for $0 \oplus 1$, i.e. $0 \oplus 1 = 1$, in this case we have $1 \oplus 1 = 1$. So we obtain 7 the following non-isomorphic hyper MV-algebras.

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1,b\}$
1	$\{1\}$	$\{0,1,b\}$	$\{1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1,b\}$
1	$\{0,1\}$	$\{0,1,b\}$	$\{1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,1\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1,b\}$
1	$\{0,1\}$	$\{0,1,b\}$	$\{0,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,1,b\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1,b\}$
1	$\{0,1,b\}$	$\{0,1,b\}$	$\{1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,1,b\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1,b\}$
1	$\{0,1,b\}$	$\{0,1,b\}$	$\{0,1\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,1,b\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1,b\}$
1	$\{0,1,b\}$	$\{0,1,b\}$	$\{1,b\}$

$\oplus$	0	b	1
0	0	$\{0,b\}$	$\{0,1,b\}$
b	$\{0,b\}$	$\{0,1,b\}$	$\{0,1,b\}$
1	$\{0,1,b\}$	$\{0,1,b\}$	$\{0,1,b\}$

By defining of isomorphism, it is easy to see that the above hyper MV-algebras are not isomorphic. □

5. **Conclusion and Future Research**

In this paper, we find some conditions which a hyper MV-algebra is an MV-algebra. For this purpose we consider hyper MV-algebras which 0 or 1 is scalar on them. Then hyper MV-algebras of order 2 and order 3 are determined. Application of this paper is to find more hyper MV-algebras which are not MV-algebras.
Important issues for future work are
(i) Characterization of hyper MV-algebra with at least one scalar which this scalar is not 0 or 1.
(ii) Characterization of hyper MV-algebras of order more than 3.
(iii) Find an algorithm for computing hyper MV-algebra.

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