GENERALIZED HIGHER-RANK NUMERICAL RANGE

HAMID REZA AFSHIN, HADIS IZADI AND MOHAMMAD ALI MEHRJOOFARD
DEPARTMENT OF MATHEMATICAL SCIENCES, VALI-E-ASR UNIVERSITY OF RAFSANJAN, RAFSANJAN, IRAN
E-MAILS: AFSHIN@VRU.AC.IR, H_IZADI2005@YAHOO.COM, AAHAAY@GMAIL.COM

(Received: 13 October 2012, Accepted: 24 December 2012)

Abstract. In this note, a generalization of higher rank numerical range is introduced and some of its properties are investigated.

AMS Classification: 15A60.
Keywords: generalized projector, higher rank numerical range, generalized higher rank numerical range.

1. Introduction

Let M_n and $\mathbb{C}^n$ denote the spaces of the $n \times n$ complex matrices and $n \times 1$ complex vectors, respectively.

For $A \in M_n$, the numerical range of A is the set

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

The numerical range is a useful tool for studying matrices and operators, and there are many generalizations of this concept motivated by theory and applications. The higher rank numerical range is an extension of the classical numerical range which was first introduced by Choi, Kribs and Życzkowski [4].

JOURNAL OF MAHANI MATHEMATICAL RESEARCH CENTER
VOL. 1, NUMBER 2 (2012) 163-168.

For a positive integer k, the rank-k numerical range of $A \in M_n$ is defined as

$$\Lambda_k(A) := \{\lambda \in \mathbb{C} : PAP = \lambda P \ for\ some\ orthogonal\ projector\ P\ of\ rank\ k\}.$$

This new concept is motivated by the study of quantum error correcting codes [3, 4, 5] and has been further studied recently in [2, 7, 8, 9, 12].

In this note, we extend the notion of rank-k numerical range to $k'-$generalized rank-k numerical range by using $k'-$generalized projectors.

A square matrix A is called an orthogonal projector if $A^2 = A^* = A$, therefore, one of the natural generalizations of this concept, is $k-$generalized projector that is a square matrix A such that $A^k = A^*$, where $k > 1$ [1, 6, 10, 11]. Notice that when $k = 1$ this is the definition of Hermitian matrix, which, in genral, is not a projector, so the assumption $k > 1$ is reasonable. Also notice that Theorem1.1 will show that if we add assuption $A = A^*$ in the definition of $k-$generalized projector, then we will encounter with very special matrices, even for different choices of k.

We will use the following notations: for $k \in \mathbb{N}$ and $k > 1$, the set of complex kth roots of 1 shall be denoted by Ω_k. The symbol $\sigma(A)$ will stand for the spectrum of the matrix A.

For a subset α of $\{1, ..., n\}$, the principal submatrix $A\{\alpha\}$ is obtained from A by deleting all rows and columns not in α.

Let $\alpha \in \mathbb{C}$ and $S \subset \mathbb{C}$, then we denote the set $\{\alpha s : s \in S\}$ by αS.

We end this section with a characterization of k -generalized projectors.

Theorem 1.1. [1] *Let $A \in M_n, k' \in \mathbb{N}$, and $k' > 1$. Then the following statements are equivalent:*

1) *A is a $k'-$generalized projector.*
2) *A is a normal matrix such that $\sigma(A) \subset \{0\} \cup \Omega_{k'+1}$.*
3) *A is a normal matrix such that $A^{k'+2} = A$.*

2. Main results

By replacing an orthogonal projector occurring in the definition of the higher rank numerical range with a so called $k'-$generalized projector, we obtain the following generalization:

Definition 2.1. Let $A \in M_n, k, k' \in \mathbb{N}$, and $k' > 1$. We call the following set as k'-generalized rank-k numerical range of A.

$$G\Lambda_{k',k}(A) := \{\lambda \in \mathbb{C} : PAP = \lambda P \ for\ some\ k' - generalized\ projector\ P\ of\ rank\ k\}$$

Proposition 2.2. *Let k, k_1, k_2, k', n, m be positive integers, $k' > 1$ and $A, A_1 \in M_n, A_2 \in M_m$. Then*

(a) $G\Lambda_{k',k}(A) \supset \Lambda_k(A)$,

(b) $G\Lambda_{k',k}(A^*) = \overline{G\Lambda_{k',k}(A)}$,

(c) $G\Lambda_{k',k}(A_1) \cup G\Lambda_{k',k}(A_2) \subset G\Lambda_{k',k}(A_1 \oplus A_2)$,

(d) $G\Lambda_{k',k_1}(A_1) \cap G\Lambda_{k',k_2}(A_2) \subset G\Lambda_{k',k_1+k_2}(A_1 \oplus A_2)$.

Proof. The first two assertions can be verified readily.

For the part (c) consider that if $\lambda \in G\Lambda_{k',k}(A_1)$, then there exists $k'-$generalized projector P_1 of rank k such that $P_1A_1P_1 = \lambda P_1$. Let $P = P_1 \oplus 0_m$, then $P(A_1 \oplus A_2)P = \lambda P$. Similarly, it can be shown that $G\Lambda_{k',k}(A_2) \subset G\Lambda_{k',k}(A_1 \oplus A_2)$.

In the case (d) assume that $\lambda \in G\Lambda_{k',k_1}(A_1) \cap G\Lambda_{k',k_2}(A_2)$, therefore there exist $k'-$generalized projectors P_1 and P_2 of ranks k_1 and k_2, respectively, such that $P_1A_1P_1 = \lambda P_1$ and $P_2A_2P_2 = \lambda P_2$. It is clear that $P = P_1 \oplus P_2$ is a $k'-$generalized projector of rank $k_1 + k_2$ such that $P(A_1 \oplus A_2)P = \lambda P$. □

Proposition 2.3. *Let $A \in M_n, k, k' \in \mathbb{N}$, and $k' > 1$. Then the following statements are equivalent:*

(a) $\lambda \in G\Lambda_{k',k}(A)$.

(b) *There exist $b_1, \cdots, b_k \in \Omega_{k'+1}$ and unitary matrix $U \in M_n$ such that $(U^*AU)\{1, 2, \cdots, k\} = \lambda diag\left(\{b_i\}_{i=1}^k\right)$.*

(c) *There exist $b_1, \cdots, b_k \in \Omega_{k'+1}$ and $X \in M_{n,k}$ where $X^*X = I_k$ and $X^*AX = \lambda diag\left(\{b_i\}_{i=1}^k\right)$.*

(d) *There exist $b_1, \cdots, b_k \in \Omega_{k'+1}$ and orthonormal vectors $u_1, \cdots, u_k \in \mathbb{C}^n$ such that $\langle Au_i, u_j\rangle = \lambda b_i\delta_{ij}$.*

Proof. It is clear that parts (b), (c) and (d) are equivalent. So we only prove equivalency of (a) and (b).

($a \Rightarrow b$): Assume that $\lambda \in G\Lambda_{k',k}(A)$, then there exist a $k'-$generalized projector P of rank k such that $PAP = \lambda P$. Now by Theorem 1.1 there exist unitary matrix U and diagonal matrix Λ such that $P = U\Lambda U^*$ and $\sigma(\Lambda) \subset \{0\} \cup \Omega_{k'+1}$.

We know that $rank\,(P) = rank\,(\Lambda)\,,$ therefore there exist $b_1, \cdots, b_k \in \Omega_{k'+1}$ such that $\Lambda = diag\left(\{b_i\}_{i=1}^k\right) \oplus 0$ and hence $P = U\left(diag\left(\{b_i\}_{i=1}^k\right) \oplus 0\right)U^*$. Finally, $PAP = \lambda P$ implies that $(U^*AU)\{1, 2, \cdots, k\} = \lambda diag\left(\{b_i\}_{i=1}^k\right)$.

$(b \Rightarrow a)$: The proof is similar to that of $(a \Rightarrow b)$. □

Corollary 2.4. *Let $n, k \in \mathbb{N}, k' > 1$ and $A \in M_n$. Then*

1) *If A is non-normal, then $G\Lambda_{k',n}(A)$ is empty.*
2) *If A is normal, then $G\Lambda_{k',n}(A)$ consist of all complex scalars s such that $\sigma(A) \subset s\Omega_{k'+1}$.*

Proposition 2.5. *Let $A \in M_n, k, k' \in \mathbb{N}, k' > 1$ and $\lambda \in G\Lambda_{k',k}(A)$. Then*

(a) *There exist rank-k subspace $S \subset \mathbb{C}^n$ and $k'-$generalized projector P such that $range\,(P) = S$ and $(PA - \lambda I_n)\,S \perp S$.*

(b) *There exist rank-k subspace $S \subset \mathbb{C}^n$ and $k'-$generalized projector P such that $range\,(P) = S$ and $(AP - \lambda I_n)\,S \perp S$.*

Proof. (a) Assume that $\lambda \in G\Lambda_{k',k}(A)\,,$ therefore there exists a $k'-$generalized projector P of rank k such that $PAP = \lambda P$. Hence there exist $b_1, \cdots, b_k \in \Omega_{k'+1}$ and matrix $U = [u_1, \cdots, u_k] \in M_{n,k}$ such that $U^*U = I_k$ and $P = U\left(diag\left(\{b_i\}\right)_{i=1}^k \oplus 0\right)U^*$, i.e., $P = \sum_{i=1}^k b_i u_i u_i^*$. It is clear that for any $1 \leq j \leq k$,

$$Pu_j = \sum_{i=1}^k b_i u_i u_i^* u_j = \sum_{i=1}^k b_i u_i \delta_{ij} = b_j u_j.$$

Let $S = span\{u_1, \cdots, u_k\}\,,$ $a = \sum_{i=1}^k a_i u_i$ and $b = \sum_{i=1}^k d_i u_i$ be arbitrary elements of S. Thus

$$\begin{aligned}
\langle (PA - \lambda I_n)\,a, b\rangle &= \left\langle (PA - \lambda I_n)\sum_{i=1}^k a_i u_i, \sum_{j=1}^k d_j u_j \right\rangle \\
&= \sum_{i=1}^k a_i \sum_{j=1}^k \bar{d}_j \langle (PA - \lambda I_n)\,u_i, u_j\rangle \\
&= \sum_{i=1}^k a_i \sum_{j=1}^k \bar{d}_j \left\langle (PA - \lambda I_n)\,\tfrac{1}{b_i}Pu_i, \tfrac{1}{b_j}Pu_j \right\rangle \\
&= \sum_{i=1}^k \tfrac{a_i}{b_i} \sum_{j=1}^k \tfrac{\bar{d}_j}{\bar{b}_j} \langle P^*(PA - \lambda I_n)\,Pu_i, u_j\rangle \\
&= 0.
\end{aligned}$$

(b) Proof is similar to that of (a)

□

Corollary 2.6. *Let k', k, k_1, k_2 be positive integers, $\alpha \in \mathbb{C}, A \in M_n, k_1 > k_2, k' > 1, H = diag\,(a_1, \cdots, a_n)$ be a Hermitian matrix such that $a_1 \leq \cdots \leq a_n$ and $U \in M_n$ be a unitary matrix. Then*

(a) $G\Lambda_{k',k}\,(U^*AU) = G\Lambda_{k',k}\,(A)\,,$

(b) $G\Lambda_{k',k_1}\,(A) \subset G\Lambda_{k',k_2}\,(A)\,,$

(c) $G\Lambda_{k',1}\,(A) = \bigcup\limits_{\omega \in \Omega_{k'+1}} \omega W\,(A) = \bigcup\limits_{\omega \in \Omega_{k'+1}} \omega \Lambda_1\,(A),$

(d) $G\Lambda_{k',k}\,(I_n) = \Omega_{k'+1},$

(e) $G\Lambda_{k',k}\,(\alpha A) = \alpha G\Lambda_{k',k}\,(A)\,,$

(f) $G\Lambda_{k',k}\,(A) = G\Lambda_{k',k}\,(\omega A)$ *for any* $\omega \in \Omega_{k'+1},$

(g) $G\Lambda_{k',k}\,(A) \supset \bigcup\limits_{\omega \in \Omega_{k'+1}} \Lambda_k\,(\omega A),$

(h) $\bigcup\limits_{\omega \in \Omega_{k'+1}} \omega\,[a_k, a_{n-k+1}] \subset G\Lambda_{k',k}\,(H) \subset \bigcup\limits_{\omega \in \Omega_{k'+1}} \omega[a_1, a_n].$

Proof. Proof of all parts except (h) can be deduced directly from the definitions, so we leave them to the interested reader. The left inclusion in (h) is an immediate result of (g) and [4, Theorem 2.4], and the right inclusion can be obtained by using (b) and (c). □

Theorem 2.7. *Let $A \in M_n$ and $k^{'}, k$ are positive integers such that $k^{'} > 1$. Then $G\Lambda_{k',k}(A)$ is compact.*

Proof. By parts (b) and (c) of Corollary 2.6 we deduce that $G\Lambda_{k',k}(A)$ is bounded. Let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of complex numbers in $G\Lambda_{k',k}(A)$, which converges to complex number λ. For any $i = 1, 2, ...$ there exist unitary matrix $U_i \in M_n$ and $b_{i_1}, ..., b_{i_k} \in \Omega_{k'+1}$ such that

$$(U_i^* AU_i)\,\{1, ..., k\} = \lambda_i diag(\{b_{i_j}\}_{j=1}^{k}).$$

Since the set of unitary matrices is a compact subset of M_n, $\{U_i\}_{i=1}^{\infty}$ has a convergent subsequence $\{U_{m_i}\}_{i=1}^{\infty}$ which converges to some unitary matrix U. Hence there exist $b_1, ..., b_k \in \Omega_{k'+1}$ such that

$$(U^* AU)\,\{1, ..., k\} = \lambda diag(\{b_i\}_{i=1}^{k}).$$

□

References

[1] J. Benítez and N. Thome, Characterizations and linear combinations of k-generalized projectors, Linear Algebra and its Applications, Vol. 410 (2005) 150-159.

[2] M.D. Choi, M. Giesinger, J. A. Holbrook, and D. W. Kribs, Geometry of higher-rank numerical ranges, Linear and Multilinear Algebra, Vol. 56 (2008) 53-64.

[3] M.D. Choi, J.A. Holbrook, D.W. Kribs, and K. Życzkowski, Higher-rank numerical ranges of unitary and normal matrices, Operators and Matrices, Vol 1 (2007) 409-426.

[4] M.D. Choi, D.W. Kribs and K. Życzkowski, Higher-rank numerical ranges and compression problems, Linear Algebra and its Applications, Vol. 418 (2006) 828-839.

[5] M.D. Choi, D.W. Kribs and K. Życzkowski, Quantum error correcting codes from the compression formalism, Reports on Mathematical Physics, Vol. 58 (2006) 77-91.

[6] H.K. Du, W.F. Wang, Y.T. Duan, Path connectivity of k-generalized projectors, Linear Algebra and its Applications, Vol. 422 (2007) 712-720.

[7] C.K. Li, Y.T. Poon, N.S. Sze, Higher rank numerical ranges and low rank perturbations of quantum channels, Journal of Mathematical Analysis and Applications, Vol. 348 (2008) 843-855.

[8] C.K. Li, Y.T. Poon, N.S. Sze, Condition for the higher rank numerical range to be non-empty, Linear and Multilinear Algebra, Vol. 57 (2009) 365-368.

[9] C.K. Li and N.S. Sze, Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations, Proceedings of the American Mathematical Society, Vol. 136 (2008) 3013-3023.

[10] L. Lebtahi, N. Thome, A Note on k-generalized projections, Linear Algebra and its Applications, Vol. 420 (2007) 572-575.

[11] B. Schipp, W. Kräer, G. Trenkler, Statistical Inference, Econometric Analysis and Matrix Algebra, Springer, 2009.

[12] H. Woerdeman, The higher rank numerical range is convex, Linear and Multilinear Algebra, Vol. 56 (2008) 65-67.