ENTROPY OF DYNAMICAL SYSTEMS ON WEIGHTS OF A GRAPH

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Abstract. Let G be a finite simple graph whose vertices and edges are weighted by two functions. In this paper we shall define and calculate entropy of a dynamical system on weights of the graph G, by using the weights of vertices and edges of G. We examine the conditions under which entropy of the dynamical system is zero, possitive or $+\infty$. At the end it is shown that, for $r \in [0, +\infty]$, there exists an order preserving transformation with entropy r.

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1. Introduction

The study of concept entropy is very important in the current sciences. Entropy plays an important role in a variety of problem areas, including physics, computer science, general systems theory, information theory, statistics, biology, chemistry, sociology and others. One of the applied branches of mathematics is the entropy of a dynamical system. Shannon in 1940 was concerned with the problems of the transmission of information in the presence of noise. Shannon introduced entropy as a measure of information in a probability

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distribution. If $P = (p_1, ..., p_n) \in \mathbf{R^n}$ is a probability distribution, he defined its entropy to be the quantity

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i.$$

In 1958 Kolmogorov introduced the concept of entropy in ergodic theory. Let G be a finite simple graph, V be the set of vertices and E the set of edges of G. We consider two functions $M : V \longrightarrow [0,1]$ and $S : E \longrightarrow [0,1]$. (M,S) is called a weight of G. In this paper we define the entropy of a weight of G. We assume the reader is familiar with the defininition of discrete dynamical system [6].

The definition of the entropy of a dynamical system T might be in three stages [1-5]. For example if T is a measure preserving transformation of probability space (X, β, m):

i) The entropy of a finite partition, ξ, of $(X, \beta.m)$ is defined in [7] as,

$$H(\xi) = -\sum_{i=1}^{n} m(A_i) logm(A_i),$$

where $\xi = \{A_1, ...A_n\} \subset \beta$.

ii) The entropy of T relative to ξ is defined by,

$$h(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H(\vee_{i=0}^{n-1} T^{-i}\xi).$$

iii) The entropy of T is defined by,

$$h(T) = \sup_{\xi} h(T,\xi),$$

where the supremum is taken over all finite partitions of (X, β, m). In this paper the definition of entropy of an order preserving transformation as a dynamical system including the three stages is given. The paper is organized as follows. In Section 2 weights of a finite simple graph G are introduced as two functions on vertices and edges of G. Also entropy of the weights and entropy of an order preserving transformatin on the weights of G is introduced. Finally, some results of the entropy are considered. In Section 3 we examine the conditions under which entropy of the order preserving transformatin is zero, possitive or $+\infty$, and we present some examples for each of these cases. At the end we show that, for $r \in [0, +\infty]$ there exists an order preserving transformation with entropy r.

2. The entropy of weights

Let G be a finite simple graph, V be the set of vertices and E the set of edges of G. We define $F_V := [0,1]^V$, $F_E := [0,1]^E$ and $F := F_V \times F_E$. At first let us define join of two weights and refinement of a weight.

Definition 2.1. If (M_1, S_1) and (M_2, S_2) are two weights of G, we define their join $(M_1, S_1) \vee (M_2, S_2)$ to be the $(\min(M_1, M_2), \min(S_1, S_2))$.

Definition 2.2. A weghit (M_2, S_2) is a refinement of a weight (M_1, S_1), written by $(M_1, S_1) \prec (M_2, S_2)$, if for any $v_i \in V$, there is $v_t \in V$, such that

$$M_2(v_i) \leq M_1(v_t),$$

and,
for any $e_j \in E$, there is $e_l \in E$, such that

$$s_2(e_j) \leq s_1(e_l).$$

Hence for i=1,2, $(M_i, S_i) \prec (M_1, S_1) \vee (M_2, S_2)$, for any weight (M_1, S_1) , (M_2, S_2) of G. Now we would like to define the entropy of a weight of G. In this definition the entropy of the weight of G increases as the weight decreases.

Definition 2.3. Let (M, S) be a weight of G. We define the entropy of M by $H(M) = -\log \max_i M(v_i)$ and the entropy of S by $H(S) = -\log \max_j S(e_j)$. The entropy of (M, S) is defined by

$$H(M, S) = -\log \max_i M(v_i). \max_j S(e_j) = H(M) + H(S).$$

Theorem 2.4. *If (M, S) is a weight of G, Then*
(i) $H(M, S) \geq 0$.
(ii) $H(M, S) = 0$, iff there exist i_0, j_0 such that $M_{v_{i_0}} = S_{e_{j_0}} = 1$.
(iii) $H(M^k, S^t) = kH(M) + tH(S)$ for any $k, t \in \mathbf{N}$.

Proof. It can be deduced from definition 2.3. □

Theorem 2.5. *If (M_1, S_1) and (M_2, S_2) are two weights of G. Then*
(i) If $(M_1, S_1) \prec (M_2, S_2)$, then $H(M_1, S_1) \leq H(M_2, S_2)$.
(ii) For i=1,2, $H(M_i, S_i) \leq H(M_1, S_1) \vee (M_2, S_2)) \leq H(M_1, S_1) + H(M_2, S_2)$.

Proof. (i) By using the definition 2.2, we have

$$\max_i M_2(v_i) \leq \max_i M_1(v_i),$$
$$\max_j S_2(e_j) \leq \max_j S_1(e_j).$$

(ii) By (i), we have the first inquality.
Since for i=1,2, $0 < M_i \leq 1$, $0 < S_i \leq 1$, we have

$$\max_i(\min(M_1, M_2)(v_i) \geq \max_i(M_1 M_2)(v_i),$$
$$\max_j(\min(S_1, S_2)(e_j)) \geq \max_j(S_1 S_2)(e_j).$$

□

Definition 2.6. Let (M_1, S_1) and (M_2, S_2) be two weights of G. We define the conditional entropy of (M_1, S_1) given (M_2, S_2) by

$$H((M_1,S_1)|(M_2,S_2)) = -\log \frac{\max_i(\min(M_1,M_2))(v_i).\max_j(\min(S_1,S_2))(e_j)}{\max_i M_2(v_i).\max_j S_2(e_j)}.$$

Theorem 2.7. *If (M_1, S_1), (M_2, S_2) and (M_3, S_3) are weights of G. Then*
(i) $H((M_1, S_1)|(M_2, S_2)) \geq 0$.
(ii) $H(((M_1, S_1) \vee (M_2, S_2)|(M_3, S_3)) = H((M_1, S_1)|(M_3, S_3)) + H((M_2, S_2)|((M_1, S_1) \vee (M_2, S_2))$.
(iii) $H((M_1, S_1) \vee (M_2, S_2)) = H(M_1, S_1) + H((M_2, S_2)|(M_1, S_1))$.
(iv) If $(M_1, S_1) \prec (M_2, S_2)$, then $H((M_1, S_1)|(M_3, S_3)) \leq H((M_2, S_2)|(M_3, S_3))$.
(v) If $(M_2, S_2) \prec (M_3, S_3)$, then $H((M_1, S_1)|(M_3, S_3)) \leq H((M_1, S_1)|(M_2, S_2))$.
(vi) If $(M_2, S_2) \prec (M_3, S_3)$, then $H((M_1, S_1)|(M_2, S_2)) \leq H((M_1, S_1) \vee (M_3, S_3))$.
(vii) $H(M_1, S_1) \geq H((M_1, S_1)|(M_2, S_2))$.
(viii) $H((M_1, S_1) \vee (M_2, S_2)|(M_3, S_3)) \leq H((M_1, S_1)|(M_3, S_3)) + H((M_2, S_2)|(M_3, S_3))$.

Proof. (i) By the definition 2.6, $H((M_1, S_1)|(M_2, S_2)) \geq 0$.
(ii)

$$\begin{aligned}\frac{\max_i(\min(M_1,M_2,M_3))(v_i).\max_j(\min(S_1,S_2,S_3))(e_j)}{\max_i M_3(v_i).\max_j S_3(e_j)} &= \\ \frac{\max_i(\min(M_1,M_3))(v_i).\max_j(\min(S_1,S_3))(e_j)}{\max_i M_3(v_i).\max_j S_3(e_j)} &\times \\ \frac{\max_i(\min(M_2,M_1,M_3))(v_i).\max_j(\min(S_2,S_1,S_3))(e_j)}{\max_i(\min(M_1,M_3))(v_i).\max_j(\min(S_1,S_3))(e_j)}&.\end{aligned}$$

(iii)

$$\begin{aligned}&\max_i(\min(M_1,M_2))(v_i).\max_j(\min(S_1,S_2))(e_j)\\ &= \max_i M_1(v_i).\max_j S_1(e_j).\frac{\max_i(\min(M_2,M_1))(v_i).\max_j(\min(S_2,S_1))(e_j)}{\max_i M_1(v_i).\max_j S_1(e_j)}.\end{aligned}$$

(iv) Since $(M_2, S_2) \prec (M_3, S_3)$, for any $v_i \in V$, $e_j \in E$ there exist $v_t \in V$, $e_l \in E$ such that

$$M_1(v_i) \leq M_2(v_t) \leq \max_i M_2(v_i),$$

$$S_1(e_j) \leq S_2(e_l) \leq \max_j S_2(e_j),$$

then

$$\frac{\max_i(\min(M_1,M_3))(v_i).\max_j(\min(S_1,S_3))(e_j)}{\max_i M_1(v_i).\max_j S_1(e_j)} \geq \frac{\max_j(\min(M_2,M_3))(v_i).\max_j(\min(S_1,S_2))(e_j)}{\max_i M_2(v_i).\max_j S_2(e_j)}$$

(v) It is similar to (iv).

(vi) Since $(M_2, S_2) \prec (M_3, S_3)$, for any $v_i \in V$, $e_j \in E$ there exist $v_t \in V$, $e_l \in E$, such that

$$M_2(v_i) \leq M_3(v_t) \leq \max_i M_3(v_i),$$

$$S_2(e_j) \leq S_3(e_l) \leq \max S_3(e_j),$$

therefore

$$\frac{\max_i(\min(M_1, M_2))(v_i).\max_j(\min(S_1, S_2))(e_j)}{\max_i M_2(v_i).\max_j S_2(e_j)} \geq \max_i(\min(M_1, M_3))(v_i).\max_j(\min(S_1, S_3))(e_j).$$

(vii) Since for i=1,2, $0 < M_i \leq 1$, $0 < S_i \leq 1$, we can write

$$\max_i M_1(v_i).\max_i M_2(v_i).\max_j S_1(e_j).\max_j S_2(e_j) \leq \max_i(\min(M_1, M_2))(v_i).\max_j(\min(S_1, S_2))(e_j).$$

So

$$\max_i M_1(v_i).\max_j S_1(e_j) \leq \frac{\max_i(\min(M_1, M_2))(v_i).\max_j(\min(S_1, S_2))(e_j)}{\max_i M_2(v_i).\max_j S_2(e_j)}.$$

(viii) We have $(M_3, S_3) \prec (M_1, S_1) \vee (M_3, S_3)$. The proof is compleet by (ii) and (v). $\square$

Definition 2.8. Let (M_1, S_1) and (M_2, S_2) be two weights of G. We define relation $\sim$ as follows:

$$(M_1, S_1) \sim (M_2, S_2) \Longleftrightarrow \max_i M_1(v_i).\max_j S_1(e_j) = \max_j M_2(v_i).\max_j S_2(e_j).$$

The relation $\sim$ is an equivalence relation on F.

Definition 2.9. Let $T : F \to F$ be a dynamical system. T is said to be an order preserving transformation if:

$$M_i(v_k) \leq M_j(v_l) \Longrightarrow \acute{M}_i(v_k) \leq \acute{M}_j(v_l),$$

$$S_i(e_k) \leq S_j(e_l) \Longrightarrow \acute{S}_i(e_k) \leq \acute{S}_j(e_l)$$

where $T(M_i, S_i) = (\acute{M}_i, \acute{S}_i)$, $i, j \in \{1, 2\}$, $k, l \in \{1, 2, ..., n\}$.

Lemma 2.10. *Let $T : F \to F$ be an order preserving transformation, then*

$$T(M_1, S_1) \vee T(M_2, S_2) = T((M_1, S_1) \vee (M_2, S_2)).$$

Proof. We may assume that for any $v \in V$, $e \in E$,

$$M_1(v) \leq M_2(v), S_2(e) \leq S_1(e).$$

Let $T(M_i, S_i) = (\acute{M}_i, \acute{S}_i)$, $i = 1, 2$, we have

$$\begin{aligned} T((M_1, S_1) \vee (M_2, S_2))(v, e) &= T(\min(M_1, M_2), \min(S_1, S_2))(v, e) \\ &= T(M_1, S_2)(v, e) \\ &= (\acute{M}_1, \acute{S}_2)(v, e). \end{aligned}$$

On the other hand since T is an order preserving transformation,

$$\begin{aligned} (T(M_1, S_1) \vee T(M_2, S_2))(v, e) &= ((\acute{M}_1, \acute{S}_1) \vee (\acute{M}_2, \acute{S}_2))(v, e) \\ &= (\acute{M}_1(v), \acute{S}_2(e)). \end{aligned}$$

□

Theorem 2.11. *If $T : F \to F$ is an order preserving transformation and (M, S) is a weight of G and $T(M, S) \sim (M, S)$, then $\lim_{n\to\infty} \frac{1}{n} H(\vee_{i=0}^{n-1} T^i(M, S))$ exists.*

Proof. Let $a_n = H(\vee_{i=0}^{n-1} T^i(M, S))$. We show that for $p \in \mathbf{N}$, $a_{n+p} \leq a_n + a_p$ and then by theorem 4.9 in [7] $\lim_{n\to\infty} \frac{1}{n} a_n$ exists and equals $\inf_n \frac{a_n}{n}$. Since for any $i \in \mathbf{N}$ $T^i(M, S) \sim (M, S)$, by lemma 2.10, we have

$$\begin{aligned} a_{n+p} &= H(\vee_{i=0}^{n+p-1} T^i(M, S)) \\ &\leq H(\vee_{i=0}^{n-1} T^i(M, S)) + H(\vee_{i=n}^{n+p-1} T^i(M, S)) \\ &= a_n + H(\vee_{i=0}^{p-1} T^{n+i}(M, S)) \\ &= a_n + H(T^n(\vee_{i=0}^{p-1} T^i(M, S)) \\ &= a_n + H(\vee_{i=0}^{p-1} T^i(M, S)) \\ &= a_n + a_p. \end{aligned}$$

□

Definition 2.12. Let (M, S) be a weight of G and $T : F \to F$ be an order preserving transformation and $T(M, S) \sim (M, S)$. The entropy of T relative to (M, S) is defined by

$$h(T, (M, S)) = \lim_{n\to\infty} \frac{1}{n} H(\vee_{i=0}^{n-1} T^i(M, S)).$$

Definition 2.13. Let $T : F \to F$ be an order preserving transformation. Entropy of T is defined by

$$h(T) = \sup_{(M,S)} h(T, (M, S)),$$

where (M, S) ranges over all weights of G.

Theorem 2.14. *Let $T : F \to F$ be an order preserving transformation, then*
(i) $h(T,(M,S)) \leq H(M,S)$.
(ii) If $(M_1,S_1) \prec (M_2,S_2)$, *then* $h(T,(M_1,S_1)) \leq h(T,(M_2,S_2))$.
(iii) $h(T,(M_1,S_1) \vee (M_2,S_2)) \leq h(T,(M_1,S_1)) + h(T,(M_2,S_2))$.
(iv) $h(T,(M_1,S_1)) \leq h(T,(M_2,S_2)) + H((M_1,S_1)|(M_2,S_2))$.
(v) $h(T,T^{-1}(M,S)) = h(T,(M,S))$.
(vi) If $k \geq 1$, $h(T, \vee_{i=0}^{k} T^i(M,S)) = h(T,(M,S))$.
(vii) If T is invertible and $k \geq 1$, *then* $h(T, \vee_{i=-k}^{k} T^i(M,S)) = h(T,(M,S))$.
(viii) For $k \geq 1$, $h(T^k) = kh(T)$.
(ix) If T is invertible, then $h(T^k) = |k|h(T)$, $\forall k \in \mathbf{Z}$.

Proof. It is similar to the proof of theorems 4.12 and 4.13 in [7]. □

Corollary 2.15. *If $T : F \to F$ is an order preserving transformation with $T^k = id$ for some $k \in \mathbf{N}$, then* $h(T) = 0$.

Proof. $h(T^k)$=0, Since $T^k = id$. So by theorem 2.14 (ix), we have $h(T) = \frac{1}{k}h(T^k) = 0$. □

3. Main Resultes

In this section we would like to calculate entropy of an order preserving transformation on weights of the graph G. We examine the conditions under which entropy of the dynamical system is zero, possitive or $+\infty$ and we give some examples about these cases. Finally we show that for $r \in [0, +\infty]$, there exists an order preserving transformation with entropy r.

Theorem 3.1. *If $T : F \to F$ is an order preserving transformation with $T \geq id$, then for any $n \in \mathbf{N}$,* $h(T^n)$=$h(T)$=0.

Proof. Since T is order preserving transformation and $T \geq I$, we have $T_V \geq I_{F_V}$,$T_E \geq I_{F_E}$ and for any $n \in \mathbf{N}$, weight (M,S),

$$M \leq T_V M \leq \ldots \leq {T_V}^{n-1} M,$$

$$S \leq T_E S \leq \ldots \leq {T_E}^{n-1} S.$$

So

$$\vee_{i=0}^{n-1} T^i(M,S)) = \vee_{i=0}^{n-1} ({T_V}^i M, {T_E}^i S) = (M,S).$$

Hence

$$h(T,(M,S)) = \lim_{n\to\infty} \frac{1}{n} H(M,S) = 0.$$

Therefore $h(T)$=0. Now by corollary 2.15, $h(T^n) = nh(T)$, for any $n \in \mathbf{N}$. So $h(T^n) = 0$ for any $n \in \mathbf{N}$. □

Example 3.2. Let $T : F \to F$ be defined by $T(M, S) = (\sqrt{M}, \sqrt[3]{S})$. Then $h(T) = 0$ because T is an order preserving transformation and $T \geq I$.

Corollary 3.3. *If $T : F \to F$ is an order preserving transformation and (M_0, S_0) is a fixed point for T. Then there exists a proper subset A of F, such that $h(T) = h(T|_A)$.*

Proof. Let $B = \{(M, S) \in F; T(M, S) \geq (M, S)\}$. Define $A = F - B$. Since $T(M_0, S_0) = (M_0, S_0)$, $B \neq \phi$, therefore $A \subsetneqq F$. Since $T|_B \geq I$ and T is an order preserving transformation, then by theorem 3.1, $h(T|_B) = 0$. So we have

$$\begin{aligned} h(T) &= h(T|_{A\cup B}) \\ &\leq h(T|_A) + h(T|_B) \\ &= h(T|_A) \end{aligned}$$

On the other hand $h(T|_A) \leq h(T)$. Therefore $h(T) = h(T|_A)$. □

Theorem 3.4. *If $T : F \to F$ is defined by $T(M, S) = (PM, QS)$ where $P \in F_V$, $Q \in F_E$. Then $h(T) = H(P, Q)$.*

Proof. We have $T_V M = PM$, $T_E S = QS$. So for $i \in \mathbf{N}$, ${T_V}^i M = P^i M$, ${T_E}^i S = Q^i S$. Since $0 < P \leq 1$, $0 < Q \leq 1$, then for $n \in \mathbf{N}$ we have

$$M \geq T_V M \geq \ldots \geq {T_V}^{n-1} M,$$

$$S \leq T_E S \leq \ldots \leq {T_E}^{n-1} S,$$

so

$$\vee_{i=0}^{n-1} T^i(M, S)) = \vee_{i=0}^{n-1} ({T_V}^i M, {T_E}^i S) = ({T_V}^{n-1} M, {T_E}^{n-1} S).$$

Hence

$$\begin{aligned} h(T,(M,S)) &= \lim_{n\to\infty}\frac{1}{n}H({T_V}^{n-1}M,{T_E}^{n-1}S)\\ &= \lim_{n\to\infty}\frac{1}{n}H(P^{n-1}M,Q^{n-1}S)\\ &= \lim_{n\to\infty}-\frac{1}{n}\log\max_i(P^{n-1}M)(v_i).\max_j(Q^{n-1}S)(e_j)\\ &= \lim_{n\to\infty}-\frac{1}{n}\log\max_i(P^{n-1})(v_i).\max_j(Q^{n-1})(e_j)\\ &= \lim_{n\to\infty}-\frac{n-1}{n}\log\max_i P(v_i).\max_j Q(e_j)\\ &= -\log\max_i P(v_i).\max_j Q(e_j)\\ &= H(P,Q). \end{aligned}$$

□

Corollary 3.5. *Let $T : F \to F$ be an order preserving transformation with $T(M,S) = (PM,QS)$ where $P \in F_V$, $Q \in F_E$, $P < 1$, $Q < 1$. Then $h(T) > 0$.*

Example 3.6. Let $T : F \to F$ is defined by $T(M,S) = (aM,bS)$ where $a,b \in (0,1]$, then by theorem 3.4, $h(T) = \log\frac{1}{ab}$.

Example 3.7. Let $T_i : F \to F$, $i = 1,2$, be defined by

$$T_1(M,S) = (\frac{2}{3}M,\frac{3}{4}S),$$

$$T_2(M,S) = (\frac{2}{5}M,\frac{5}{6}S),$$

then by example 3.6, we have $h(T_1) = \log 2$, $h(T_2) = \log 3$.

Theorem 3.8. *If $T : F \to F$ is an order preserving transformation and $T \leq I$, then*

$$h(T,(M,S)) = \lim_{n\to\infty}\frac{1}{n}H(T^n(M,S)).$$

Proof. Since T is an an order preserving transformation and $T \leq I$, we have

$$\vee_{i=0}^{n}T^i(M,S) = T^n(M,S).$$

Therefore

$$h(T,(M,S)) = \lim_{n\to\infty}\frac{1}{n}H(\vee_{i=0}^{n}T^i(M,S)) = \lim_{n\to\infty}\frac{1}{n}H(T^n(M,S)).$$

□

Theorem 3.9. *If $T : F \to F$ is an order preserving transformation with $T_V M = M^k$ or $T_E S = S^k$ for $k \in (1, +\infty)$, then $h(T) = +\infty$.*

Proof. Case (a). Let $T_V M = M^k$. Since ${T_V}^i M = M^{k^i}$, $0 < M \leq 1$, then for $n \in \mathbf{N}$

$$\min(M, T_V M, \ldots, {T_V}^{n-1} M) = M^{k^{n-1}}.$$

So

$$\vee_{i=0}^{n-1}({T_V}^i M, {T_E}^i S) = (M^{k^{n-1}}, \min(S, T_E S, \ldots, {T_E}^{n-1} S)).$$

Therefore

$$\begin{aligned} h(T,(M,S)) &= \lim_{n\to\infty} \frac{1}{n} H(\vee_{i=0}^{n-1}({T_V}^i M, {T_E}^i S)) \\ &= \lim_{n\to\infty} -\frac{1}{n} \log \max_i (M^{k^{n-1}})(v_i). \max_j \min(S, T_E S, \ldots, {T_E}^{n-1} S) \\ &\geq \lim_{n\to\infty} -\frac{k^{n-1}}{n} \log \max(M)(v_i) \\ &= +\infty. \end{aligned}$$

Therefore $h(T) = +\infty$.
Case (b). Let $T_E S = S^k$. It is similar to case (a) to see that $h(T) = +\infty$. $\square$

Example 3.10. Let $T_i : F \to F$, $i = 1, 2, 3, 4$ be defined by

$$T_1(M, S) = (M^2, \frac{3}{5} S),$$

$$T_2(M, S) = (\frac{1}{4} M, S^{\frac{3}{2}}),$$

$$T_3(M, S) = (M^3, \sqrt{S}),$$

$$T_4(M, S) = (M^{\frac{5}{4}}, S^4).$$

Then for $i = 1, 2, 3, 4$, $h(T_i) = +\infty$, because $2, \frac{3}{2}, 3, \frac{5}{4} \in (1, +\infty)$.

The next corollary implies that, for $r \in [0, +\infty]$ there exists an order preserving transformation with entropy r.

Corollary 3.11. *Let U denote the set of order preserving transformations. Then the function $E : U \to [0, +\infty]$ which is defined by $E(T) = h(T)$, is surjective.*

Proof. If $E(T) = 0$, by corollary 2.15 it is sufficient to put $T = I$. Let $E(T) = h(T) < +\infty$. Since log is surjective, example 3.6 implies that the function E is surjective. If $E(T) = h(T) = +\infty$, then example 3.10 implies that there exists an order preserving transformation with entropy $+\infty$. $\square$

4. Conclusion

This paper has introduced and calculated entropy of an order preserving transformation on weights of a finite simple graph G by using the changes of the weights of vertices and edges of G. We presented some conditions as to when the entropy of dynamical system T is zero, possitive or $+\infty$. Also we presented some examples for the cases of zero, possitive and $+\infty$, and we have shown for $r \in [0, +\infty]$ there exists an order preserving transformation with entropy r.

For future research we present some questions:

• If $h(T_1) = h(T_2)$, then what is the relation between T_1 and T_2 ?

• Is there an order preserving transformation T such that $T < I$ and $h(T) = 0$?

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