MODIFICATION OF THE OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR LANE-EMDEN TYPE EQUATIONS

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Abstract. In this paper, modification of the optimal homotopy asymptotic method (MOHAM) is applied upon singular initial value Lane-Emden type equations and results are compared with the available exact solutions. The modified algorithm give the exact solution for differential equations by using one iteration only.

Keywords: Optimal homotopy asymptotic method; Lane-Emden equations; singular initial value problems.
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1. Introduction

Mathematical modeling of many physical systems and engineering are generally described by differential equations. These equations are often solved by many methods such as, Adomian's decomposition method (ADM) [1–4],Variational iteration method (VIM) [5–8], Homotopy analysis method (HAM) [9–11], and Homotopy perturbation method (HPM) [12–15]. One of these methods is optimal homotopy asymptotic method that was first proposed by Marinca *et al.* [16–23].The present work is motivated to extend the application of MOHAM on the Lane-Emden type equations, that first, this equations were published by Jonathan Homer Lane in 1870 [24], and further explored in detail by Emden [25].

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The Lane-Emden equations have the following form

$$u'' + \frac{m}{x}u' + f(u) = g(x), \quad 0 < x < 1, \ m \geq 1 \tag{1.1}$$

subject to following initial conditions

$$u(0) = \alpha, \quad u'(0) = \beta, \tag{1.2}$$

where α, β and m are constants and $f(u)$ is a real valued continuous function.

This modifications demonstrates a rapid convergence of the series solution if compared with standard OHAM. In addition, the modified algorithm give the exact solution for differential equations by using one iteration only. These results reveal that the MOHAM is very effective, simple and has closed agreement with exact solution.

The rest of the paper is organized as follows. In section 2 we applied the methods of OHAM and MOHAM. The numerical experiments are provided in section 3 and conclusion is in section 4.

2. The methods

2.1 OHAM

Consider the following equation

$$L(u(x)) + g(x) + N(u(x)) = 0, \quad B(u, \frac{\mathrm{d}u}{\mathrm{d}x}) = 0 \tag{2.1}$$

where L is a linear operator, x denotes independent variable, $u(x)$ is an unknown function, $g(x)$ is a known function, N is a nonlinear operator and B is a boundary operator.

According to OHAM we construct a homotopy:
$h(v(x,p),p) : \mathbb{R} \times [0,1] \to \mathbb{R}$ which satisfies

$$\begin{aligned} &(1-p)[L(v(x,p)) + g(x)] = H(p)[L(v(x,p)) + g(x) + N(v(x,p))], \\ &B(v(x,p), \frac{\partial v(x,p)}{\partial x}) = 0, \end{aligned} \tag{2.2}$$

where $x \in \mathbb{R}$ and $p \in [0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$, $H(0) = 0$ and $v(x,p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds that $v(x,0) = u_0(x)$ and $v(x,1) = u(x)$ respectively. Thus, as p varies from $p = 0$ to $p = 1$ the solution $v(x,p)$ approaches from $u_0(x)$ to $u(x)$, where $u_0(x)$ is obtained from Eq.(2.2) for $p = 0$ and we have

$$L(u_0(x)) + g(x) = 0, \quad B(u_0, \frac{\mathrm{d}u_0}{\mathrm{d}x}) = 0. \tag{2.3}$$

Next, we choose auxiliary function $H(p)$ in the form

$$H(p) = pc_1 + p^2 c_2 + \cdots \tag{2.4}$$

where $c_1, c_2, \cdots$ are constants to be determined. $H(p)$ can be expressed in many forms as reported by V. Marinca *et al.* [16]-[18].

To get an approximate solution, we expand $v(x, p, c_i)$ in Taylor's series about p in the following manner

$$v(x, p, c_i) = u_0(x) + \sum_{k=1}^{\infty} u_k(x, c_1, c_2, \cdots, c_k) p^k. \tag{2.5}$$

Substituting Eq.(2.5) into Eq.(2.2) and equating the coefficient of the same power of p, we obtain the following linear equations. The zeroth and the first order are given by Eq.(2.3) and Eq.(2.6) respectively,

$$L(u_1(x)) + g(x) = c_1 N_0(u_0(x)), \quad B(u_1, \frac{\mathrm{d}u_1}{\mathrm{d}x}) = 0. \tag{2.6}$$

The general governing equations for $u_k(x)$ are given by

$$\begin{aligned} L(u_k(x)) - L(u_{k-1}(x)) = c_k N_0(u_0(x)) + \sum_{i=1}^{k-1} c_i [L(u_{k-i}(x)) + N_{k-i}(u_0(x), u_1(x), \\ \cdots, u_{k-1}(x))], \quad k = 2, 3, \cdots, \\ B(u_k, \frac{\mathrm{d}u_k}{\mathrm{d}x}) = 0, \end{aligned} \tag{2.7}$$

where $N_m(u_0(x), u_1(x), \cdots, u_m(x))$ is the coefficient of p^m in the expansion of $N(v(x, p))$ about the embedding parameter p.

$$N(v(x, p, c_i)) = N_0(u_0(x)) + \sum_{m=1}^{\infty} N_m(u_0, u_1, u_2, \cdots, u_m) p^m. \tag{2.8}$$

It has been observed that the convergence of the series (2.5) depends upon the auxiliary constants $c_1, c_2, \cdots$. If it is convergent at $p = 1$, one has

$$v(x, c_i) = u_0 + \sum_{k=1}^{\infty} u_k(x, c_1, c_2, \cdots, c_k). \tag{2.9}$$

The result of the m^{th} order approximations are given by

$$\tilde{u}(x, c_1, c_2, \cdots, c_m) = u_0(x) + \sum_{i=1}^{m} u_i(x, c_1, c_2, \cdots, c_i). \tag{2.10}$$

Substituting Eq.(2.10) into Eq.(2.1), it results the following residual

$$R(x, c_1, c_2, \cdots, c_m) = L(\tilde{u}(x, c_1, c_2, \cdots, c_m)) + g(x) + N(\tilde{u}(x, c_1, c_2, \cdots, c_m)). \tag{2.11}$$

If $R = 0$, then $\tilde{u}$ will be the exact solution. Generally it does not happen, especially in nonlinear problems. In order to find the optimal values of $c_i, i = 1, 2, 3, \cdots$, we first construct the functional

$$J(c_1, c_2, \cdots, c_m) = \int_a^b R^2(x, c_1, c_2, \cdots, c_m)\mathrm{d}x \tag{2.12}$$

and then minimizing it, we have

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \cdots = \frac{\partial J}{\partial c_m} = 0, \tag{2.13}$$

where a and b are in the domain of the problem. With these constants known, the approximate solution (of the order m) is well determined.

2.2 MOHAM

The modified form of the OHAM can be established based on the assumption that function $g(x)$ can be divided into two parts namely $g_1(x)$ and $g_2(x)$ [23],

$$g(x) = g_1(x) + g_2(x). \tag{2.14}$$

And to this assumption the Eq.(2.2) becomes

$$\begin{aligned} (1-p)[L(v(x,p) + g_1(x)] =& H(p)[L(v(x,p)) + g_1(x) + g_2(x) + N(v(x,p))], \\ & B(v(x,p), \frac{\partial v(x,p)}{\partial x}) = 0 \end{aligned} \tag{2.15}$$

For to communicate the reliability of MOHAM, we deal with different examples.

3. Examples

In this section, we solve some examples by OHAM and MOHAM.

Example 1. Consider the linear Lane-Emden equation [26]

$$\begin{aligned} & u'' + \frac{2}{x}u' + u - x^5 - 30x^3 = 0, \quad 0 < x \leq 1 \\ & u(0) = 0, \quad u'(0) = 0. \end{aligned} \tag{3.1}$$

The exact solution is $u(x) = x^5$.

a) OHAM

$$(1-p)[u'' + \frac{2}{x}u' + u - x^5 - 30x^3] = H(p)[u'' + \frac{2}{x}u' + u - x^5 - 30x^3] \tag{3.2}$$

The zeroth order problem is

$$\begin{aligned} &u_0'' + \frac{2}{x}u_0' - x^5 - 30x^3 = 0 \\ &u_0(0) = 0, \quad u_0'(0) = 0 \end{aligned} \tag{3.3}$$

$$u_0(x) = \frac{x^7}{56} + x^5. \tag{3.4}$$

The first order problem is

$$\begin{aligned} &u_1'' + \frac{2}{x}u_1' = (1+c_1)u_0'' + (1+c_1)\frac{2}{x}u_0' + c_1u_0 - c_1x^5 - 30c_1x^3 - x^5 - 30x^3, \\ &u_1(0) = 0, \quad u_1'(0) = 0 \end{aligned} \tag{3.5}$$

$$u_1(x) = \frac{c_1x^9}{5050} + \frac{c_1x^7}{56}. \tag{3.6}$$

The second order problem is

$$\begin{aligned} u_2'' + \frac{2}{x}u_2' =& (1+c_1)u_1'' + (1+c_1)\frac{2}{x}u_1' + c_1u_1 + c_2u_0'' + c_2\frac{2}{x}u_0' + c_2u_0 \\ &- c_2u_0 - c_2x^5 - 30c_2x^3, \end{aligned} \tag{3.7}$$

$u_2(0) = 0, \quad u_2'(0) = 0.$

$$u_2(x) = \frac{c_1^2x^{11}}{665280} + \frac{c_1^2x^9}{2520} + \frac{c_1^2x^7}{56} + \frac{c_1x^9}{5040} + \frac{c_1x^7}{56} + \frac{c_2x^9}{5040} + \frac{c_2x^7}{56}. \tag{3.8}$$

Now, $u(x)$ can be obtained by adding zeroth-order, first-order and second-order solutions, and other higher order solutions if necessary as:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \tag{3.9}$$

By using the procedure mentioned in section 2, we can calculate the constant c_1 and c_2, as follows: $c_1 = -0.9915643704541924$ and $c_2 = 0.000037809003255378 55$.
By using these values of c_1 and c_2, the approximate solution becomes

$$\begin{aligned} u(x) \quad \approx \quad & x^5 + 1.9458723051687943 \times 10^{-6}x^7 - 3.3117322215657837 \times 10^{-6}x^9 \\ & +1.477873828695014 \times 10^{-6}x^{11}. \end{aligned}$$

The errors of OHAM are shown in Table 1 and Figure 1.

b) MOHAM

For the modification OHAM, we construct homotopy in the following form

$$(1-p)[u'' + \frac{2}{x}u' - 30x^3] = H(p)[u'' + \frac{2}{x}u' + u - x^5 - 30x^3], \quad u(0) = 0, u'(0) = 0. \tag{3.10}$$

We find

$$u_0(x) = x^5, \quad u_1(x) = 0, \quad u_2(x) = 0, \ldots$$

Then

$$u_1 = u_2 = u_3 = \cdots = 0. \tag{3.11}$$

Consequently, the exact solution $u(x) = x^5$ follows immediately.

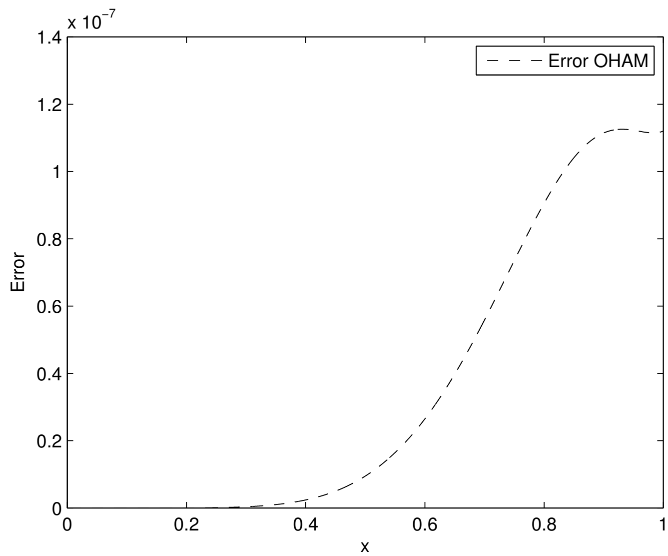

FIGURE 1. The Error between Exact solution and OHAM of order 2.

Example 2. Consider the linear Lane-Emden equation [26]

$$u'' + \frac{8}{x}u' + xu - x^5 + x^4 - 44x^2 + 30x = 0, \quad 0 < x \leq 1, \quad u(0) = 0, \quad u'(0) = 0. \tag{3.12}$$

TABLE 1. The exact and OHAM of order 2 solutions

x	Exact solution	OHAM solution	Error
0.0	0.0000000	0.0000000	0.000000e+0
0.1	0.0000100	0.0000100	1.912903e-13
0.2	0.0003200	0.0003200	2.324183e-11
0.3	0.0024300	0.0024300	3.629955e-10
0.4	0.0102400	0.0102400	2.381953e-9
0.5	0.0312500	0.0312500	9.455518e-9
0.6	0.0777600	0.0777600	2.645902e-8
0.7	0.1680700	0.1680701	5.583301e-8
0.8	0.3276800	0.3276801	9.053422e-8
0.9	0.5904900	0.5904901	1.114442e-7
1.0	1.0000000	1.0000001	1.120139e-7

The exact solution is

$$u(x) = x^4 - x^3.$$

a) OHAM

Using OHAM, the homotopy formula for above equation is

$$\begin{aligned} &(1-p)[u'' + \frac{8}{x}u' - x^5 + x^4 - 44x^2 + 30x] = \\ &H(p)[u'' + \frac{8}{x}u' + xu - x^5 + x^4 - 44x^2 + 30x]. \end{aligned} \tag{3.13}$$

Applying OHAM, we find

$$\begin{aligned} u_0(x) &= \frac{x^7}{98} - \frac{x^6}{78} + x^4 - x^3, \\ u_1(x) &= \frac{c_1 x^{10}}{16660} - \frac{c_1 x^9}{11232} + \frac{c_1 x^7}{98} - \frac{c_1 x^6}{78}, \\ u_2(x) &= \frac{c_1^2 x^{13}}{4331600} - \frac{c_1^2 x^{12}}{2560896} + \frac{c_1^2 x^{10}}{8330} - \frac{c_1^2 x^9}{5616} + \frac{c_1^2 x^7}{98} - \frac{c_1^2 x^6}{78} + \frac{c_1 x^{10}}{16660} \\ &\quad - \frac{c_1 x^9}{11232} + \frac{c_1 x^7}{98} - \frac{c_1 x^6}{78} + \frac{c_2 x^{10}}{16660} - \frac{c_2 x^9}{11232} + \frac{c_2 x^7}{98} - \frac{c_2 x^6}{78}. \end{aligned}$$

Now, $u(x)$ can be obtained by adding the zeroth-order, the first-order and the second-order solutions, and other higher order solution if necessary as:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots . \tag{3.14}$$

By using the procedure mentioned in section 2, we calculate the constants c_1 and c_2, that

$$c_1 = -0.9951916227117169, \; c_2 = 0.00002639270897467315$$

and using this values of $c_1, \; c_2$, the approximate solution becomes:

$$\begin{aligned} u(x) \approx\ & 2.286467739208561 \times 10^{-7} x^{13} - 3.8674212694134408 \times 10^{-7} x^{12} \\ & -5.7287640355933747 \times 10^{-7} x^{10} + 8.4972586211703724 \times 10^{-7} x^{9} \\ & +5.0523674613498198 \times 10^{-7} x^{7} - 6.347846297583312 \times 10^{-7} x^{6} \\ & +x^4 - x^3. \end{aligned}$$

The errors of OHAM are shown in Table 2 and Figure 2.

b) MOHAM

For the modification OHAM, we construct homotopy in the following form

$$\begin{aligned} & (1-p)[u'' + \frac{8}{x}u' - 44x^2 + 30x] = \\ & H(p)[u'' + \frac{8}{x}u' + xu - x^5 + x^4 - 44x^2 + 30x], \\ & u(0) = 0, \quad u'(0) = 0. \end{aligned} \tag{3.15}$$

Consequently, with computing the first few components of the above equation, we obtain $u_0(x) = x^4 - x^3$ and $u_k(x) = 0, \quad k \geq 1$. Thus the exact solution $u(x) = x^4 - x^3$ follows immediately.

Example 3. Consider the nonlinear Lane-Emden equation [26]

$$\begin{aligned} & u'' + \frac{2}{x}u' + u^3 - x^6 - 6 = 0, \quad 0 < x \leq 1 \\ & u(0) = 0; \quad u'(0) = 0. \end{aligned} \tag{3.16}$$

The exact solution was found to be:

$$u(x) = x^2.$$

a) OHAM

TABLE 2. The exact and OHAM of order 2 solutions

x	Exact solution	OHAM solution	Error
0.0	0.00000000	0.00000000	0.000000e+0
0.1	-0.00090000	-0.00090000	5.834689e-13
0.2	-0.00640000	-0.00640000	3.378419e-11
0.3	-0.01890000	-0.01890000	3.390894e-10
0.4	-0.03840000	-0.03840000	1.614572e-9
0.5	-0.06250000	-0.06250000	4.937685e-9
0.6	-0.08640000	-0.08640001	1.091703e-8
0.7	-0.10290000	-0.10290002	1.810387e-8
0.8	-0.10240000	-0.10240002	2.191974e-8
0.9	-0.07290000	-0.07290002	1.735431e-8
1.0	0.00000000	-0.00000001	1.079378e-8

Using OHAM, the homotopy formula for above equation is

$$(1-p)[u'' + \frac{2}{x}u' - x^6 - 6] = H(p)[u'' + \frac{2}{x}u' + u^3 - x^6 - 6] \tag{3.17}$$

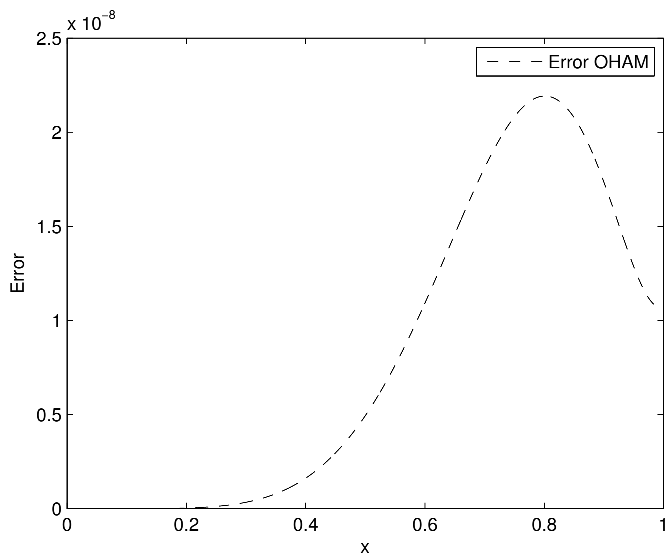

FIGURE 2. The Error between Exact solution and OHAM of order 2.

Applying OHAM, we have the following zero, first and second orders solution:

$$\begin{aligned} u_0(x) &= \frac{x^8}{72} + x^2, \\ u_1(x) &= \frac{c_1 x^{26}}{262020096} + \frac{c_1 x^{20}}{725760} + \frac{c_1 x^{14}}{5040} + \frac{c_1 x^8}{72}, \\ u_2(x) &= \frac{c_1^2 x^{44}}{896486037258240} + \frac{491 c_1^2 x^{38}}{652367154216960} \\ &+ \frac{67 c_1^2 x^{32}}{293462507520} + \frac{41 c_1^2 x^{26}}{91707336} + \frac{47 c_1^2 x^{20}}{8467200} + \frac{c_1^2 x^{14}}{2520} + ... \end{aligned}$$

Now, $u(x)$ can be obtained by adding zeroth-order, first-order and second-order solutions, and other higher order solution if necessary as:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots . \tag{3.18}$$

By using the procedure mentioned in section 2, we calculate the constants c_1 and c_2,
That $c_1 = -0.9828444161739907$ and $c_2 = 0.00007151252608689145$ and using these values of c_1 and c_2 solution becomes:

$$\begin{aligned} u(x) \approx\ & 1.077521686069644 \times 10^{-15} x^{44} + 7.2704108693801268 \times 10^{-13} x^{38} \\ & + 2.2054221289131272 \times 10^{-10} x^{32} + 3.568499285010508 \times 10^{-8} x^{26} \\ & + 2.6536524254083618 \times 10^{-6} x^{20} - 6.676910740296036 \times 10^{-6} x^{14} \\ & + 5.0809247569184018 \times 10^{-6} x^8 + x^2. \end{aligned}$$

The errors of OHAM are shown in Table 3 and Figure 3.

b) MOHAM

For the modification OHAM, we construct homotopy in the following form:

$$\begin{aligned} & (1-p)[u'' + \frac{2}{x}u' - 6] = H(p)[u'' + \frac{2}{x}u' + u^3 - X^6 - 6]. \\ & u(0) = 0, \quad u'(0) = 0. \end{aligned} \tag{3.19}$$

Consequently, with computing the first few components of the equation in above , we obtain:
$u_0(x) = x^2$ and $u_k(x) = 0, \quad k \geq 1.$
Thus the exact solution $u(x) = x^2$ follows immediately.

Example 4. Consider nonlinear Lane-Emden equation

$$\begin{aligned} & u'' + \frac{1}{x}u' + u'u - 2x^3 - 2x - 4 = 0., \quad , 0 < x \leq 1 \\ & u(0) = 1, \quad u'(0) = 0. \end{aligned} \tag{3.20}$$

TABLE 3. The exact and OHAM of order 2 solutions

x	Exact solution	OHAM solution	Error
0.0	0.00000000	0.00000000	0.000000e+0
0.1	0.01000000	0.01000000	5.080918e-14
0.2	0.04000000	0.04000000	1.300607e-11
0.3	0.09000000	0.09000000	3.330402e-10
0.4	0.16000000	0.16000000	3.311941e-9
0.5	0.25000000	0.25000002	1.944237e-8
0.6	0.36000000	0.36000008	8.020490e-8
0.7	0.49000000	0.49000025	2.497424e-7
0.8	0.64000000	0.64000059	5.894919e-7
0.9	0.81000000	0.81000098	9.846713e-7
1.0	1.00000000	1.00000109	1.093692e-6

The exact solution for this problem is

$$u(x) = 1 + x^2.$$

a) OHAM

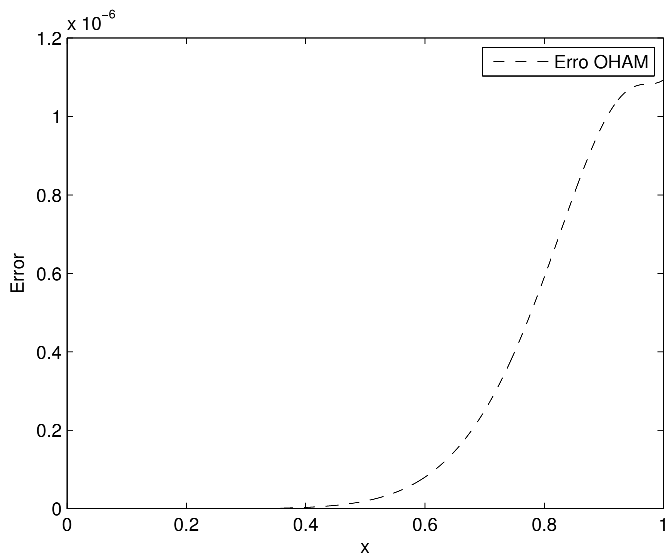

FIGURE 3. The Error between Exact solution and OHAM of order 2.

Using OHAM, the homotopy formula for above equation is

$$(1-p)[u'' + \frac{1}{x}u' - 2x^3 - 2x - 4] = H(p)[u'' + \frac{1}{x}u' + u'u - 2x^3 - 2x - 4]. \tag{3.21}$$

Applying OHAM, we have the following zero, first and second orders solutions

$$\begin{aligned}
u_0(x) &= \frac{2x^5}{25} + \frac{2x^3}{9} + x^2 + 1, \\
u_1(x) &= \frac{4c_1x^3}{27} - \frac{c_1x^2}{2} + \frac{c_1x^4}{24} - \frac{2c_1x^5}{125} + \frac{c_1x^6}{90} - \frac{x^2}{2} + \frac{10x^3}{27} + \frac{x^4}{12} + \frac{8x^5}{125} \\
&\quad + \frac{43x^6}{810} + \frac{4x^7}{1323} + \frac{7x^8}{800} + \frac{32x^9}{18225} - \frac{4x^{11}}{15125}, \\
u_2(x) &= \frac{11c_1x^3}{27} - c_1x^2 + \frac{7c_1x^4}{36} - \frac{7c_1x^5}{375} + \frac{2651c_1x^6}{24300} + \frac{169c_1x^7}{5670} + \frac{13841c_1x^8}{1008000} \\
&\quad + \frac{40003c_1x^9}{3280500} + \frac{41513c_1x^{10}}{1984500} + \frac{21125249c_1x^{11}}{14407470000} + \frac{1448683c_1x^{12}}{2309472000} + \ldots
\end{aligned}$$

Now, $u(x)$ can be obtained by adding zeroth-order, first-order and second-order solutions, and other higher order solution if necessary as:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots . \tag{3.22}$$

By using the procedure mentioned in section 2, we calculate the constants c_1 and c_2;
That $c_1 = -0.72628701832663696695952478829064$ and
$c_2 = -1.5666846630398228174101179583022$ and using these values of c_1 and c_2 solution becomes:

$$\begin{aligned}
u(x) \approx\ & -0.0000008507168140541351 4x^{17} - 0.000009003708114859284 5x^{15} \\
& -0.000046460197145527956x^{14} - 0.000032595707226913386 7x^{13} \\
& -0.0004196708982501607x^{12} - 0.0010061383709219621x^{11} \\
& -0.0012438236844093688x^{10} - 0.0070845567198262452x^9 \\
& -0.0088779323735504041x^8 - 0.014802190408287729x^7 \\
& -0.03949085161037252x^6 + 0.060720483845049083x^5 \\
& -0.033464849107268245x^4 + 0.23085479925375204x^3 \\
& +0.82868411099505713x^2 + 1.0
\end{aligned}$$

The errors of OHAM are shown in Table 4 and Figure 4.

b) MOHAM

For the modification OHAM, we construct homotopy in the following form:

$$(1-p)[u'' + \frac{1}{x}u' - 4] = H(p)[u'' + \frac{1}{x}u + u'u - 2x^3 - 2x - 4]. \tag{3.23}$$
$$u(0) = 1, \quad u'(0) = 0.$$

Consequently, with computing the first few components of the equation in above, we obtain:
$u_0(x) = x^2 + 1$ and $\quad u_k(x) = 0, \quad k \geq 1.$
Thus the exact solution $u(x) = x^2 + 1$ follows immediately.

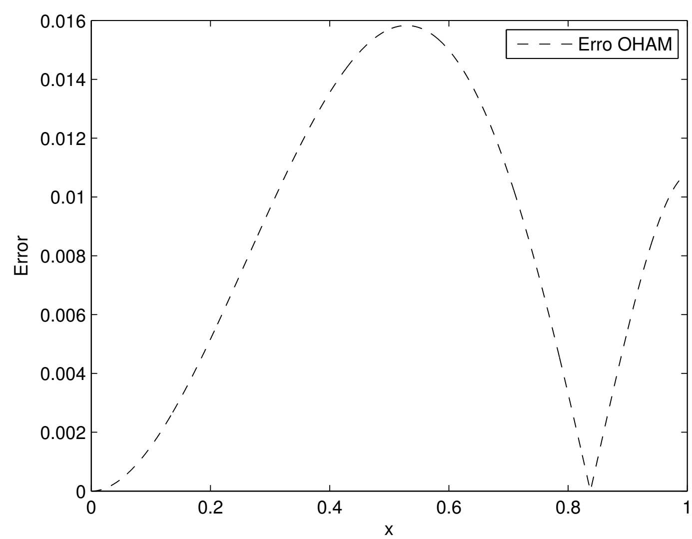

FIGURE 4. The Error between Exact solution and OHAM of order 2.

Example 5. Consider the nonlinear Lane-Emden equation together with non-homogenous initial conditions

$$u'' + \frac{1}{x}u' + u^2 - x^4 - 2x^3 - 7x^2 - 6x - \frac{1}{x} - 13 = 0, \quad 0 < x \leq 1, \tag{3.24}$$
$$u(0) = 3, \quad u'(0) = 1.$$

The exact solution is

$$u(x) = x^2 + x + 3.$$

a) OHAM

TABLE 4. The exact and OHAM of order 2 solutions

x	Exact solution	OHAM solution	Error
0.0	1.0000000	1.0000000	0.000000e+0
0.1	1.0100000	1.0113646	1.515084e-3
0.2	1.0400000	1.0460277	5.162654e-3
0.3	1.0900000	1.1047812	9.611622e-3
0.4	1.1600000	1.1882937	1.354462e-2
0.5	1.2500000	1.2969916	1.569918e-2
0.6	1.3600000	1.4308165	1.499403e-2
0.7	1.4900000	1.5887828	1.078549e-2
0.8	1.6400000	1.7682158	3.324701e-3
0.9	1.8100000	1.9634760	5.473674e-3
1.0	2.0000000	2.0137805	1.378047e-2

Using OHAM, the homotopy formula for above equation is

$$\begin{aligned}&(1-p)[u''+\frac{1}{x}u'-x^4-2x^3-7x^2-6x-\frac{1}{x}-13]\\ =&H(p)[u''+\frac{1}{x}u'+u^2-x^4-2x^3-7x^2-6x-\frac{1}{x}-13].\end{aligned} \tag{3.25}$$

Applying OHAM, we have the following zero, first and second orders solutions

$$
\begin{aligned}
u_0(x) &= \frac{x^6}{36}+\frac{2x^5}{25}+\frac{7x^4}{16}+\frac{2x^3}{3}+\frac{13x^2}{4}+x+3,\\
u_1(x) &= \frac{c_1x^{14}}{254016}+\frac{c_1x^{13}}{38025}+\frac{5527c_1x^{12}}{25920000}+\frac{289c_1x^{11}}{326700}+\frac{27569c_1x^{10}}{5760000}\\
&+\frac{1043c_1x^9}{72900}+\frac{26027c_1x^8}{460800}+\frac{3413c_1x^7}{29400}+\frac{697c_1x^6}{1728}\\
&+\frac{21c_1x^5}{50}+\frac{41c_1x^4}{32}+\frac{2c_1x^3}{3}+\frac{9c_1x^2}{4},\\
u_2(x) &= \frac{c_1^2x^{22}}{2212987392}+\frac{187c_1^2x^{21}}{39440746800}+\frac{10762123c_1^2x^{20}}{220776192000000}\\
&+\frac{18102734527c_1^2x^{19}}{58599022482000000}+\frac{136024795219c_1^2x^{18}}{67319076464640000}\\
&+\frac{5659427479327c_1^2x^{17}}{584591739732000000}+\frac{45839677162501c_1^2x^{16}}{957426865274880000}\\
&+\frac{9518465151599c_1^2x^{15}}{525930284880000000}+\frac{19174941827183c_1^2x^{14}}{26024731729920000}\\
&+\frac{2865630583c_1^2x^{13}}{1284323040000}+\frac{4028274223c_1^2x^{12}}{526727577600}+\frac{672969337c_1^2x^{11}}{36883123200}\\
&+\frac{189132731c_1^2x^{10}}{3386880000}+\frac{8513557c_1^2x^9}{85730400}+\frac{10403c_1^2x^8}{36864}+\frac{869c_1^2x^7}{2352}+\frac{229c_1^2x^6}{216}+\ldots
\end{aligned}
$$

Now, $u(x)$ can be obtained by adding zeroth-order, first-order and second-order solutions, and other higher order solution if necessary as:

$$u(x)=u_0(x)+u_1(x)+u_2(x)+\cdots . \tag{3.26}$$

By using the procedure mentioned in section 2, we calculate the constants c_1 and c_2; That $c_1 = -0.63298312$ and $c_2 = 0.027003035$ and using these values of c_1 and c_2 solution becomes:

$$\begin{aligned} u(x) \approx\ & 1.81053 \times 10^{(-10)} x^{22} + 1.89968 \times 10^{(-9)} x^{21} \\ & +1.95312 \times 10^{(-8)} x^{20} + 1.23776 \times 10^{(-7)} x^{19} \\ & +8.09588 \times 10^{(-7)} x^{18} + 0.00000387886 x^{17} \\ & +0.0000191832 x^{16} + 0.0000725142 x^{15} \\ & +0.00029033 x^{14} + 0.000861402 x^{13} \\ & +0.00280001 x^{12} + 0.00621459 x^{11} \\ & +0.0164444 x^{10} + 0.0220626 x^{9} \\ & +0.0430888 x^{8} + 0.00420649 x^{7} - 0.0471842 x^{6} \\ & -0.135857 x^{5} - 0.298503 x^{4} + 0.107803 x^{3} + 1.36383 x^{2} + x + 3. \end{aligned}$$

The errors of OHAM are shown in Table 5 and Figure 5.

b) MOHAM

For the modification OHAM, we construct homotopy in the following form:

$$\begin{aligned} & (1-p)[u'' + \frac{1}{x}u' - \frac{1}{x} - 4] = H(p)[u'' + \frac{1}{x}u' + u^2 - x^4 - 2x^3 - 7x^2 - 6x - \frac{1}{x} - 13]. \\ & u(0) = 3, \quad u'(0) = 1. \end{aligned} \tag{3.27}$$

Consequently, with computing the first few components of the equation in above, we obtain:
$u_0(x) = x^2 + x + 3$ and $u_k(x) = 0, \quad k \geq 1$.
Thus the exact solution $u(x) = x^2 + x + 3$ follows immediately.

4. **Conclusion**

In this paper, the modified OHAM is applied to approximate solutions of linear and non-linear Lane-Emden equations. The results show us that this method can obtain the exact solution by only one iteration. So it is concluded that MOHAM is reliable and efficient technique for finding the solutions of Lane-Emden equations.

TABLE 5. The exact and OHAM of order 2 solutions

x	Exact solution	OHAM solution	Error
0.0	3.0000000	3.0000000	0.000000e+0
0.1	3.1100000	3.1137148	-3.714898e-3
0.2	3.2400000	3.2548917	-1.489190e-2
0.3	3.3900000	3.4228773	-3.287772e-2
0.4	3.5600000	3.6159290	-5.592984e-2
0.5	3.7500000	3.8310578	-8.105909e-2
0.6	3.9600000	4.0640059	-1.040077e-1
0.7	4.1900000	4.3095555	-1.195580e-1
0.8	4.4400000	4.5625206	-1.225238e-1
0.9	4.7100000	4.8200323	-1.100364e-1
1.0	5.0000000	5.0861540	-8.615903e-2

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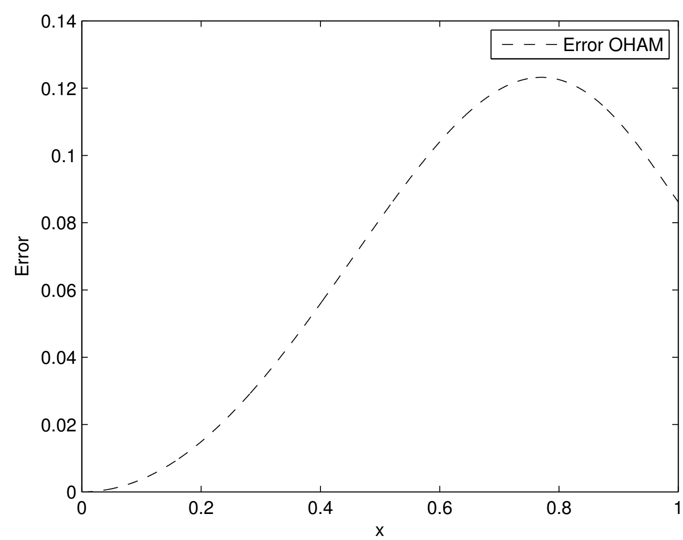

FIGURE 5. The Error between Exact solution and OHAM of order 2.

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