PERRON-FROBENIUS THEORY ON THE NUMERICAL RANGE FOR SOME CLASSES OF REAL MATRICES

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Abstract. We give further results for Perron-Frobenius theory on the numerical range of real matrices and some other results generalized from nonnegative matrices to real matrices. We present an extension of Perron-Frobenius theory to the spectrum and the numerical range of real matrices. We indicate two techniques for establishing the main theorem of Perron and Frobenius on the numerical range. In the first method, we use a corresponding version of Wielandt's lemma. The second technique involves graph theory.

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1. Introduction

By a nonnegative matrix we mean a matrix whose entries are nonnegative real numbers. Let $M_n(M_n(\mathbb{R}))$ be the set of $n \times n$ complex (real) matrices. We call a matrix $A \in M_n$ irreducible if $n = 1$, or $n \geq 2$ and there does not exist a permutation matrix P such that

$$P^t AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

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where B and D are nonempty square submatrices and P^t denotes the transpose of the matrix P.

The famous classical Perron-Frobenius theorem on irreducible nonnegative matrices has two parts. The first part says that if A is an irreducible nonnegative matrix, then its spectral radius $\rho(A)$ is always a simple eigenvalue and there is a corresponding positive eigenvector. Here is its second part:

Suppose A has exactly h distinct eigenvalues with modulus $\rho(A)$. (The quantity h is now usually referred to as the index of imprimitivity of A.) Then each of the following conditions holds.

(1) The set of eigenvalues of A with modulus $\rho(A)$ consists precisely of $\rho(A)$ times all the $h-$th roots of unity.
(2) $\sigma(A)$, the spectrum of A, is invariant under a rotation about the origin of the complex plane through an angle of $2\pi/h$; or, in other words, $e^{2\pi i/h}\sigma(A) = \sigma(A)$.
(3) If $h \geq 2$, then A is an $h-$cyclic matrix, i.e., there is a permutation matrix P such that P^tAP is a matrix of the form

$$\begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 & 0 \\ 0 & 0 & A_{23} & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & \dots & 0 & A_{h-1,h} \\ A_{h,1} & 0 & & \dots & 0 & 0 \end{pmatrix}, \tag{1.1}$$

where the zero blocks along the diagonal are all square. The largest positive integer h for which a matrix A is $h-$cyclic is called the cyclic index of A.

It has numerous applications, not only in many branches of mathematics, such as Markov chains, graph theory, game theory and etc. [1], but in various fields of science and technology, e.g. control theory [8, 14] and the population dynamics [7].

In [11] a new quantity for real matrices, the sign-real spectral radius, is defined, which is a generalization of this theory. For $A \in M_n(\mathbb{R})$, the real spectral radius of A is defined by $\rho_0(A) = \max\{|\lambda| : \lambda \textit{ a real eigenvalue of } A\}$, where $\rho_0(A) := 0$ if A has no real eigenvalues. A signature matrix is a diagonal matrix with diagonal entries $+1$ or -1. Note that there are 2^n signature matrices of dimension n. Let φ denote the set of signature matrices. The sign-real spectral radius of a real matrix A is defined by

$$\rho_0^S(A) = \max_{S\in\varphi} \rho_0(SA).$$

The sign-real spectral radius of a real matrix A has the following properties (see [11, Lemma 2.1, Theorem 2.15]):

Lemma 1.1. *Let $A \in M_n(\mathbb{R})$, signature matrices $S_1, S_2 \in \varphi$ a real diagonal matrix D, and a permutation matrix P be given. Then*

(a) $\rho_0^S(A) = \rho_0^S(S_1 A S_2) = \rho_0^S(A^t) = \rho_0^S(P^t A P) = \rho_0^S\left(D^{-1} A D\right)$.

(b) $\rho_0^S(\alpha A) = |\alpha| \rho_0^S(A)$ *for all* $\alpha \in \mathbb{R}$.

(c) $\rho_0^S(AD) = \rho_0^S(DA)$

(d) $\rho_0^S(A) \leq ||A||_p$ *for all* $1 \leq p \leq \infty$.

(e) *If* $A = A^t$ *then* $\rho_0^S(A) = ||A||_2$.

(f) *If A is a nonnegative matrix, then* $\rho_0^S(A) = \rho(A)$.

(g) *If there exists a matrix $C \in M_n(\mathbb{R})$, rank$(C) = 1$ with $c_{ij} = sign(a_{ij})$ if $a_{ij} \neq 0$, and $c_{ij} \in \{-1, 1\}$ if $a_{ij} = 0$, then there exist signature matrices S_1, S_2 such that $|A| = S_1 A S_2$ and $\rho_0^S(A) = \rho_0^S(|A|)$.*

In the sequel, we add the assumption that $|A| = S_1 A S_2$ for some $S_1, S_2 \in \varphi$. This assumption is necessary for the validity of some of the results (see Example 3.4 and Example 4.9).

The sign-real spectral radius of a real matrix A has similar properties to the spectral radius of a nonnegative matrix (cf. [11]). For example, if A is a real matrix, then its sign-real spectral radius is an eigenvalue of SA for some $S \in \varphi$ [11, Lemma 2.2]. In addition, if A is irreducible, then $\rho_0^S(A)$ is a simple eigenvalue of SA for some $S \in \varphi$ [11, Theorems 2.7 and 3.9]. It has also been applied to engineering problems (see, for example, [12, 10, 13] and the references therein). In Section 3, we give some results on the sign-real spectral radius.

For $A \in M_n$, the numerical range of A is defined as follows (see, for instance, [11, 4]):

$$W(A) = \{x^* A x \,:\, x \in \mathbb{C}^n,\, x^* x = 1\},$$

which is a useful concept in studying matrices and operators. The numerical radius of A is $\omega(A) = \max\{|z| : z \in W(A)\}$. The Perron-Frobenius theory has been extended to the numerical range of a nonnegative matrix by Issos in his unpublished Ph.D. thesis [5] and then generalized in [17]. In the course of proving the main result of Issos for real matrices, a new type of numerical radius is defined in [17] as follows:

Definition 1.2. For $A \in M_n(\mathbb{R})$ the sign-real numerical radius is defined and denoted by $\omega_0^S(A) = \max_{S \in \varphi} \omega_0(SA)$, where $\omega_0(A) = \max\{\, |z| \,:\, z \in W(A) \cap \mathbb{R}\,\}$.

Note that the sign-real numerical radius is well defined, because if A is a real matrix, then its numerical range $W(A)$ is symmetric with respect to the real axis. The following properties of sign-real numerical radius will be used in the proofs of the main results of this paper (see [17, Lemma 2.1, Theorem 2.1 and Lemma 2.2]).

Lemma 1.3. *Let A be a real matrix.*

(1) $\omega_0^S(A) = \omega_0^S(A^t)$.

(2) *For S_1 and S_2 being signature matrices,* $\omega_0^S(A) = \omega_0^S\left(S_1AS_2\right)$.

(3) *For a permutation matrix P,* $\omega_0^S(A) = \omega_0^S(P^tAP)$.

(4) $\rho_0^S(A) \leq \omega_0^S(A) \leq \|A\|$, *where $\|\cdot\|$ is the operator norm.*

(5) *There exists some $S \in \varphi$ such that* $\omega_0^S(A) \in W(SA)$.

(6) *There exist $S_1, S_2 \in \varphi$, and a nonnegative unit vector x such that* $\omega_0^S(A) = x^tS_1AS_2x$.

(7) $\omega_0^S\left(A\right) \leq \omega_0^S\left(|A|\right)$, *where $|A|$ is the matrix whose entries are the absolute values of the entries of A. Suppose, in addition, that A is a nonnegative matrix, then* $\omega(A) = \omega_0^S\left(A\right) = \max\left\{\ z^tA\ z : z \in \mathbb{R}_+^n,\ \|z\| = 1\right\}$.

These show that the sign-real numerical radius of a real matrix has similar properties to the numerical radius of a nonnegative matrix. In Section 4, we show that under some conditions on the matrix A, the set of points of $W(SA)$, with modulus $\omega_0^S(A)$ for some $S \in \varphi$, consists precisely of $\omega_0^S(A)$ times all the h-th roots of unity, i.e., the main theorem of Issos on numerical range of a real irreducible matrix A.

2. Preliminaries

We always use $A = (a_{rs})$ to denote an $n \times n$ complex matrix. The following notation will be adopted:

A^*	the conjugate transpose of A;
$H(A)$	the Hermitian part of A, i.e., $(A + A^*)/2$;
$\lambda_{\max}(H(A))$	the largest eigenvalue of $H(A)$;
$\|x\|$	the vector $(\|x_1\|, \|x_2\|, \cdots, \|x_n\|)^t$;
$\|A\| \leq \|B\|$	$\|a_{rs}\| \leq \|b_{rs}\|$ for all r, s;
$G(A)$	the digraph of A;
$\langle n\rangle$	the set of $\{1, 2, \cdots, n\}$;
$arg(z)$	the argument of the complex number z.

For a vector $x \in \mathbb{C}^n$, we denote by $\|x\|$ the Euclidean norm of x, i.e., $\|x\| = (x^*x)^{1/2}$. For a matrix $A \in M_n$, we denote by $\|A\|$ the operator norm of A, i.e., $\|A\| = \max_{\|x\|=1} \|Ax\|$, where $\|\cdot\|$ is the vector norm. Also, we denote by $||A||_2$, the spectral norm of A, i.e., $||A||_2^2 = \rho(A^*A)$.

The directed graph, $G(A)$, of $A \in M_n$ consists of the set of vertices $\{1, 2, ..., n\}$ and a set of directed edges (i, j) connecting vertex i to vertex j if and only if $a_{ij} \neq 0$. We say

$G(A)$ is strongly connected if for any two vertices i, j of $G(A)$, there is a path in $G(A)$ from i to j and vice versa. It is well known that A is irreducible if and only if $G(A)$ is strongly connected (see, for instance, [3, Theorem 6.2.24]). A circuit is a sequence of directed edges $(r_1, r_2), (r_2, r_3), ..., (r_t, r_1)$ from $G(A)$. A cycle is a circuit such that $r_1, r_2, \cdots, r_t$ are all distinct. We use $r \to s$ to denote the arc (r, s) traversed from r to s and refer to it as a positive link, and we use $s \leftarrow r$ to denote the arc (r, s) traversed from s to r and refer to it as a negative link. Let $\theta = ((r_1, r_2), (r_2, r_3), \cdots, (r_t, r_1))$ be the cycle of $G(A)$. The number of positive links minus the number of negative links in θ is called the signed length of θ and is denoted by $s(\theta)$.

By a block-shift matrix, we mean a square matrix $A = (A_{rs})_{1 \le r,s \le h}$ in block form with square diagonal blocks such that $A_{rs} = 0$ whenever $s \ne r + 1$, or equivalenty if A is in the form (1.1) and with $A_{h,1} = 0$.

Given $A, B \in M_n$, A is said to be diagonally similar to B if there exists a nonsingular diagonal matrix D such that $A = D^{-1}BD$; if, in addition, D can be chosen to be unitary, then we say A is unitarily diagonally similar to B. If there exists a signature matrix S such that $A = SBS$ we say that A and B are sign similar. In 1988, Hershkowitz and Schneider [2] listed equivalent conditions for a matrix A to be unitarily diagonally similar to $|A|$ as follows:
Let a matrix $A = (a_{ij}) \in M_n$ be irreducible. Then the following assertions are equivalent.
(i) $A = DAD^{-1}$, where D is a unitary diagonal matrix;
(ii) $\left|(I - A)^{-1}\right| = (I - |A|)^{-1}$;
(iii) for every circuit $(i_1, \cdots, i_{k+1}),\ i_{k+1} = i_1,\ 1 \le i_j \le n,\ j = 1, \cdots, k,\ k \ge 1$, in the graph of the matrix A the relation

$$\prod_{j=1}^{k} a_{i_j\ i_{j+1}} = \left|\prod_{j=1}^{k} a_{i_j\ i_{j+1}}\right|$$

is valid, i.e., all circuit products of A are positive.

Note that in the above statements, for real matrix A, the unitary diagonal matrix D is a signature matrix, i.e., A and $|A|$ are sign similar. This means that for real matrix A, the relations (ii) and (iii) are equivalent with $|A| = SAS$ for some $S \in \varphi$.

Remark 2.1. In this paper we give results of Perron-Frobenius theory for real and irreducible matrix A such that $|A| = S_1AS_2$ for some $S_1, S_2 \in \varphi$. For the restricted case $S_1 = S_2 := S$, we can obtain the same results with the summation $|A| = SAS$ for some $S \in \varphi$, where is equivalent with the relations (ii) and (iii).

3. Perron-Frobenius theory for real matrices

Rump [11] has offered a way to generalize the Perron-Frobenius theory to arbitrary real matrices. In this section, we generalize Wielandt's lemma and Perron- Frobenius theorem from nonnegative matrices to real matrices.

To prove our main results in this section, we need the following Wielandt's lemma for nonnegative matrices [9, Chapter II, Theorem 2.1]:

Wielandt's lemma. Let $A \geq 0$ be irreducible, and $C \in M_n(\mathbb{C})$ be such that $|C| \leq A$, then for every eigenvalue t of C, we have $|t| \leq \rho(A)$. Furthermore $|t| = \rho(A)$ if and only if $C = e^{i\varphi} EAE^{-1}$, where $t = e^{i\varphi}\rho(A)$ and $|E| = I$.

The following lemma is an extension of the Wielandt's lemma for real matrices.

Lemma 3.1. *Let $A \in M_n(\mathbb{R})$ be irreducible, and $|A| = S_1 A S_2$ for some S_1, $S_2 \in \varphi$. Let $C \in M_n(\mathbb{C})$ be such that $|C| \leq |A|$. Then for every eigenvalue t of C we have $|t| \leq \rho_0^S(A)$. Furthermore $|t| = \rho_0^S(A)$ if and only if $C = e^{i\varphi} E|A|E^{-1}$, where $t = e^{i\varphi}\rho_0^S(A)$, and $|E| = I$.*

Proof. By assumption there exist some $S_1, S_2 \in \varphi$ such that $|A| = S_1 A S_2$. Thus by Lemma 1.1(a) and Lemma 1.1(f) we have $\rho_0^S(A) = \rho_0^S(|A|) = \rho(|A|)$. So the conclusion that the moduli of the eigenvalues of C are bounded by $\rho_0^S(A)$ is clear by Wielandt's lemma. The last part of the theorem is an immediate consequence of the Wielandt's lemma. □

The Perron-Frobenius theorem for nonnegative matrices can now be stated for real matrices.

Theorem 3.2. *Let $A \in M_n(\mathbb{R})$ be irreducible, and $|A| = S_1 A S_2$ for some S_1, $S_2 \in \varphi$. Let $S \in \varphi$ such that $\rho_0^S(A) \in \sigma(SA)$ and suppose SA has $h \geq 1$ distinct eigenvalues $\lambda_1, \ldots, \lambda_h$ of modulus $\rho_0^S(A)$. Then $\lambda_1, \cdots, \lambda_h$ are the distinct h-th roots of $(\rho_0^S(A))^h$.*

Proof. $\rho_0^S(A)$ is an eigenvalue of SA for some $S \in \varphi$, so by applying Lemma 3.1 with $C = SA$, we have

$$SA = E|A|E^{-1}, \;\; |E| = I. \tag{3.1}$$

Let $\lambda_t = e^{i\varphi_t}\rho_0^S(A), \; t = 1, 2, \cdots, h$. Since $|\lambda_t| = \rho_0^S(A)$, again by applying Lemma 3.1 with $C = SA$, we have

$$SA = e^{i\varphi_t} E_t |A| E_t^{-1}, \;\; |E_t| = I, \;\; t = 1, 2, \cdots, h. \tag{3.2}$$

Hence, by (3.1) and (3.2), SA and $e^{i\varphi_t}SA$ are similar. Since $\rho_0^S(A)$ is a simple eigenvalue of SA for some $S \in \varphi$ [11, Theorems 2.7 and 3.9], it follows that, for each t, $\lambda_t = e^{i\varphi_t}\rho_0^S(A)$ is a simple eigenvalue of $e^{i\varphi_t}SA$, and thus of SA. So, again by Lemma 3.1 and (3.2), we have

$$SA = e^{i\varphi_t}E_t\left(e^{i\varphi_r}E_rSAE_r^{-1}\right)E_t^{-1} = e^{i(\varphi_t+\varphi_r)}(E_tE_r)SA(E_tE_r)^{-1}.$$

This shows that SA and $e^{i(\varphi_t+\varphi_r)}SA$ are similar for any t and r. We conclude that $e^{i(\varphi_t+\varphi_r)}\rho_0^S(A)$ is an eigenvalue of SA, and therefore $e^{i(\varphi_t+\varphi_r)}$ must be one of the numbers $e^{i\varphi_1}, \cdots, e^{i\varphi_h}$. Hence the h distinct numbers $e^{i\varphi_1}, \cdots, e^{i\varphi_h}$ are closed under multiplication, and therefore they are the h-th roots of unity. $\square$

Notice that, in the above result, we are assuming that $\lambda_1, \cdots, \lambda_h$ be the eigenvalues of SA of modulus $\rho_0^S(A)$ for some $S \in \varphi$, and also, $\rho_0^S(A) \in \sigma(SA)$. This is the right setting for results. The following example will show it.

Example 3.3. Consider the matrix $A \in M_8(\mathbb{R})$ given by

$$A = \begin{pmatrix} 0 & 0 & -2 & 0 & 0 & -6 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 & 4 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

such that $|A| = S_1AS_2$ for signature matrices $S_1 = diag(-1,-1,1,1,-1,1,1,1)$, and $S_2 = diag(1,-1,1,1,1,1,1,1)$. Then by above result, there exist signature matrices

$$\begin{aligned} T_1 &= diag(1,1,1,-1,1,1,-1,-1), \qquad T_2 = diag(-1,1,-1,-1,1,-1,1,-1), \\ T_3 &= diag(-1,-1,1,-1,1,1,1,-1)\,, \qquad T_4 = diag(-1,1,1,1,-1,1,1,1), \end{aligned}$$

such that $\lambda_1 = -5.9685$, $\lambda_2 = -5.9685i$, $\lambda_3 = 5.9685i$, $\lambda_4 = 5.9685$ are the eigenvalues of T_jA of modulus $\rho_0^S(A) = 5.9685$ for $j = 1, ..., 4$. Also, for the signature matrix $S = diag(-1,1,1,-1,1,1,1,-1)$ with $\sigma(SA) = \{-4.2204+4.2204i,\ -4.2204-4.2204i,\ 4.2204-4.2204i,\ 4.2204+4.2204i,\ 0, ..., 0\}$, we observe that the nonzero eigenvalues of SA have absolute value equal to $\rho_0^S(A) = 5.9685$, but they are not equal to $4-$th roots of $(\rho_0^S(A))^4$.

Our next example shows that Theorem 3.2 fails if the assumption $|A| = S_1AS_2$ is dropped.

Example 3.4. Consider the matrix $A \in M_3(\mathbb{R})$ given by

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 1 & -3 \end{pmatrix},$$

such that $|A| \neq S_1 A S_2$ for all signature matrices $S_1, S_2 \in \varphi$. But $\lambda_1 = -4,\ \lambda_2 = -4,\ \lambda_3 = 4$ are the eigenvalues of A of modulus $\rho_0^S(A) = 4$; this means that they are not equal to $3-$th roots of $(\rho_0^S(A))^3$.

Corollary 3.5. *Let $A \in M_n(\mathbb{R})$ be irreducible, and $|A| = S_1 A S_2$ for some S_1, $S_2 \in \varphi$. Let $S \in \varphi$ such that $\rho_0^S(A) \in \sigma(SA)$ and suppose SA has $h \geq 1$ distinct eigenvalues $\lambda_1, \ldots, \lambda_h$ of modulus $\rho_0^S(A)$. Then the spectrum of SA is invariant under a rotation through $2\pi/h$, but not through a positive angle smaller than $2\pi/h$.*

Proof. Let the spectrum of $e^{i2\pi/h}SA$ be $\beta = (e^{i2\pi/h}\lambda_1,\, e^{i2\pi/h}\lambda_2,\, \cdots,\, e^{i2\pi/h}\lambda_n)$. It was shown in the proof of the preceding theorem that the matrices SA and $e^{2\pi i/h}SA$ are similar, and therefore β is the spectrum of SA as well. $\square$

4. Issos' results for real matrices

We indicate two techniques for establishing the main theorem of Issos on the numerical range of real matrices. In the first method, we use a corresponding version of Wielandt's lemma. The second technique involves graph theory. These two techniques are applied to establish the main theorem (Theorem 4.7) and to derive some results for the sign-real numerical radius.

A relation between the sign-real numerical radius of A and the sign-real spectral radius of $H(SA)$ for some $S \in \varphi$, is observed in the following theorem.

Theorem 4.1. *If $A \in M_n(R)$, then there exists some $S \in \varphi$ such that $\omega_0^S(A) = \rho_0^S(H(SA)) = \lambda_{\max}(H(SA))$.*

Proof. By Lemma 1.3(5), $\omega_0^S(A) \in W(SA)$ for some $S \in \varphi$. Then we can find a nonzero vector u such that $u^*(SA)u = \omega_0^S(A)\|u\|^2$, and so $u^*(SA)^*u = \omega_0^S(A)\|u\|^2$. Adding the two equations, we obtain

$$u^* \left(\omega_0^S(A)\, I - H(SA)\right) u = 0. \tag{4.1}$$

Since $H(SA)$ is Hermitian matrix, therefore by Lemma 1.1 and Lemma 1.3

$$\|H(SA)\|_2 = \rho_0(H(SA)) = \omega_0(H(SA)) \leq \omega_0^S(H(SA)) \leq \|H(SA)\|_2,$$

So $\|H(SA)\|_2 = \omega_0^S(H(SA))$. On the other hand, by Lemma 1.1(e) we have $\|H(SA)\|_2 = \rho_0^S(H(SA))$. Thus

$$\omega_0^S(H(SA)) = \rho_0^S(H(SA)). \tag{4.2}$$

It follows from the property of $\omega_0^S(A)$ and [4, Property 1.2.7] that

$$\omega_0^S(A + A^*) \leq \omega_0^S(A) + \omega_0^S(A^*) = 2\omega_0^S(A). \tag{4.3}$$

Thus by (4.2) and (4.3), we have

$$\omega_0^S(A) = \omega_0^S(SA) \geq \omega_0^S(H(SA)) = \rho_0^S(H(SA)) \geq \lambda_{\max}(H(SA)). \tag{4.4}$$

In view of (4.1) and (4.4), the matrix $\left(\omega_0^S(A)\,I - H(SA)\right)$ is positive semidefinite. It follows that $\left(\omega_0^S(A)\,I - H(SA)\right)u = 0$, so u is an eigenvector of $H(SA)$ corresponding to $\omega_0^S(A)$. Then we have $\omega_0^S(A) = \rho_0^S(H(SA)) = \lambda_{\max}(H(SA))$. □

Lemma 4.2. *Let A be a real matrix and $|A| = S_1AS_2$ for some $S_1, S_2 \in \varphi$. Then $\omega_0^S(A) = \max\{\, z^t\,|A|\; z : z \in \mathbb{R}_+^n,\ \|z\| = 1\}$. Moreover, $\omega_0^S(A) \in W(|A|)$.*

Proof. There exists some $S \in \varphi$ such that

$$\begin{aligned}
\omega_0^S(A) = \omega_0(SA) &= \max\ \{\, |x^*SAx| : (x^*SAx) \in \mathbb{R},\ x \in \mathbb{C}^n,\ \|x\| = 1\} \\
&\leq \max\ \left\{\, |x|^t\,|SA|\,|x| : (x^*SAx) \in \mathbb{R},\ x \in \mathbb{C}^n,\ \|x\| = 1\right\} \\
&= \max\ \left\{\, z^t\,|A|\ z :\ z \in \mathbb{R}_+^n,\ \|z\| = 1\right\} \\
&\leq \omega_0(|A|) \leq \omega_0^S(|A|) = \omega_0^S(S_1AS_2) = \omega_0^S(A).
\end{aligned}$$

where the last equality follows from Lemma 1.3(2). Hence, the above inequalities all become equalities, i.e., $\omega_0^S(A) = \max\{\, z^t|A|z : z \in \mathbb{R}_+^n,\ \|z\| = 1\}$. Furthermore, the continuous real-valued map $z \longrightarrow z^t|A|z$ attains its maximum on the intersection of the unit sphere with the nonnegative orthant $\mathbb{R}_+^n$, which is a compact set. Thus $\omega_0^S(A) \in W(|A|)$. □

The next result enables us to show that for real and irreducible matrix A there is a positive unit vector x such that $\omega_0^S(A) = x^*S_1AS_2x$.

Lemma 4.3. *Let A be a real irreducible matrix and $|A| = S_1AS_2$ for some $S_1, S_2 \in \varphi$. Then, any nonnegative unit vector $x \in \mathbb{R}_+^n$ such that $x^t|A|x = \omega_0^S(A)$ is in fact positive.*

Proof. By Lemma 4.2, there exists a unit nonnegative vector x such that $\omega_0^S(A) = x^t|A|x$. Similar to Theorem 4.1, we can conclude $(\omega_0^S(A)I - H(|A|))x = 0$. Therefore $H(|A|)x = \omega_0^S(A)x$, which implies that

$$(H(|A|) + I)^{n-1}x = (\omega_0^S(A) + 1)^{n-1}x. \tag{4.5}$$

By assumption A is irreducible, so $(H(|A|)+I)^{n-1}>0$ (see, for instance, [9, Chapter I, Corollary 2.2]), and $(H(|A|)+I)^{n-1}x>0$. Thus by (4.5), $x>0$. $\square$

In the next lemma, we prove a version of Wielandt's lemma for the sign-real numerical radius.

Lemma 4.4. *Let $A, B \in M_n$, and assume that A is a real irreducible matrix. Let $|A| = S_1AS_2$ for some S_1, $S_2 \in \varphi$ and $|B| \leq |A|$. If ε is a unit complex number such that $\varepsilon\omega_0^S(A) \in W(B)$, then $\varepsilon E|A|E^{-1} = B$ for some unitary diagonal matrix E.*

Proof. Let x be a unit vector such that $x^*Bx = \varepsilon\omega_0^S(A)$. Then

$$\omega_0^S(A) = x^*(\bar{\varepsilon}B)x \leq |x|^t|B||x| \leq |x|^t|A||x| \leq \omega_0^S(A),$$

where the last inequality follow from Lemma 4.2. Hence, the above inequalities all become equalities, i.e., $|x|^t|A||x| = \omega_0^S(A)$. Since A is irreducible, by Lemma 4.3, we have $|x|>0$. Now, in view of

$$|x|^t(|A|-|B|)|x| = \omega_0^S(A) - \omega_0^S(A) = 0, \quad |A|-|B| \geq 0 \ \ \textit{and} \ \ |x|>0,$$

we have $|B| = |A|$. Let E denote the unitary diagonal matrix $diag(x_1/|x_1|, ..., x_n/|x_n|)$, where $x = (x_1, \cdots, x_n)^t$. Then we have

$$|x|^tE^*(\bar{\varepsilon}B)E|x| = x^*(\bar{\varepsilon}B)x = \omega_0^S(A), \tag{4.6}$$

where the second equality has already been established above. But we also already have $|x|^t|A||x| = \omega_0^S(A)$, so by (4.6) we have

$$|x|^tE^*(\bar{\varepsilon}B)E|x| = |x|^t|A||x| = \omega_0^S(A). \tag{4.7}$$

Since $|E^*(\bar{\varepsilon}B)E| = |B| = |A|$, $|x|>0$, therefore by (4.7), we obtain $\bar{\varepsilon}E^{-1}BE = |A|$. $\square$

Lemma 4.5. *Let $A \in M_n(\mathbb{R})$ be irreducible, and $|A| = S_1AS_2$ for some S_1, $S_2 \in \varphi$. Then there exists some $S \in \varphi$ such that $E^{-1}|A|E = SA$, where $|E| = I$.*

Proof. By Lemma 1.3 and [6, Proposition 3.3], we have

$$\omega_0^S(A) = \omega_0^S(S_1AS_2) = \omega_0^S(|A|) = \omega(|A|) = \rho(H(|A|)) = \rho_0^S(H(|A|)). \tag{4.8}$$

where the last equality follows from Lemma 1.1(f). Thus, from (4.8) and Theorem 4.1, we conclude $\rho_0^S(H(|A|)) = \omega_0^S(A) = \rho_0^S(H(SA))$ for some $S \in \varphi$. In view of $|H(SA)| \leq H(|A|)$ and the irreducibility of $H(|A|)$, by Lemma 3.1, it follows that

$$E\left(H\left(SA\right)\right)E^{-1} = H(|A|), \ \ \textit{i.e.,} \ \ E\left(SA+(SA)^t\right)E^{-1} = |A|+|A|^t,$$

where $E := diag(e_1, \cdots, e_n)$, $|E| = I$. By equating the corresponding entries of both sides, we obtain $e_r\left(b_{rt}+b_{tr}\right)e_t^{-1} = |a_{rt}|+|a_{tr}|$, for all $r, t \in \langle n\rangle$, where $b_{rt} = (SA)_{rt}$.

Since $\left|e_r\, b_{rt}\, e_t^{-1}\right| = |a_{rt}|$ and $\left|e_r\, b_{tr}\, e_t^{-1}\right| = |a_{tr}|$ (*as* $|e_r| = |e_t| = 1$), it follows that $e_r\, b_{rt}\, e_t^{-1} = |a_{rt}|$ $\big($*and* $e_r\, b_{tr}\, e_t^{-1} = |a_{tr}|\big)$ for all $r, t \in \langle n\rangle$. Hence, we have $ESAE^{-1} = |A|$. $\square$

Remark 4.6. In view of Lemma 4.5, the matrix E is a signature matrix, i.e., there exists $S \in \varphi$ such that $|A|, SA$ are sign similar. So in the proofs of Theorem 4.1 and Lemma 4.5, we see that the signature matrix S is equal to the signature matrix S_1 that $\omega_0^S(A) \in W(S_1 A)$.

Now we are ready to prove our main result in this section.

Theorem 4.7. *Let $A \in M_n(\mathbb{R})$ be irreducible, and $|A| = S_1 A S_2$ for some S_1, $S_2 \in \varphi$. Let $S \in \varphi$ such that $\rho_0^S(A) \in \sigma(SA)$ and suppose SA has $h \geq 1$ distinct eigenvalues $\lambda_1, \ldots, \lambda_h$ of modulus $\rho_0^S(A)$. Then*

$$\{z \in W(TA) \,:\, |z| = \omega_0^S(A)\} = \{\omega_0^S(A) e^{i(\theta + 2\pi t/h)} \,:\, t = 0, 1, \cdots, h-1\}, \tag{4.9}$$

for some $T \in \varphi$ with $\theta = 0$ or $\theta = \pi/h$.

Proof. By Theorem 3.2, we have $e^{2\pi i/h}\rho_0^S(A) \in \sigma(SA)$ for some $S \in \varphi$, and so by (3.1) and (3.2), $e^{2\pi i/h}A$ is unitarily diagonally similar to A. Hence we have $W(A) = W(e^{2\pi i/h}A)$ [4, Property 1.2.8]. But we also have $\omega_0^S(A) \in W(TA)$ for some $T \in \varphi$ (Lemma 1.3(5)). Let $\theta \in [0, 2\pi)$ be the principal argument such that $\omega_1^S(A) \in W(e^{-i\theta}TA)$. Hence $\omega_0^S(A)e^{i(\theta+2\pi t/h)} \in W(TA)$ for $t = 0, 1, \cdots, h-1$, i.e.,

$$\{\omega_0^S(A)e^{i(\theta+2\pi t/h)} \;:\; t = 0, 1, \cdots, h-1\} \subseteq \{z \in W(TA) \,:\, |z| = \omega_0^S(A)\}.$$

On the other hand, consider any unit complex number ε for which $\varepsilon\omega_0^S(A) \in W(TA)$ for some $T \in \varphi$. By applying Lemma 4.4, with $B = TA$, and Remark 4.6, εA is unitarily diagonally similar to A, equivalently, the matrices $e^{-i\theta}A$ and $e^{-i\theta}\varepsilon A$ are unitarily diagonally similar. But $\rho_0^S(A)$ is an eigenvalue of $\sigma(SA)$ for some $S \in \varphi$, hence so is $\varepsilon\rho_0^S(A)$. By Theorem 3.2, ε must be an h-th root of unity, i.e., $\varepsilon = e^{2\pi t i/h}$ for $t = 0, 1, \cdots, h-1$. This follows that

$$\{z \in W(TA) \,:\, |z| = \omega_0^S(A)\} \subseteq \{\omega_0^S(A)e^{i(\theta+2\pi t/h)} \;:\; t = 0, 1, \cdots, h-1\}.$$

Since $A \in M_n(\mathbb{R})$, so $W(A)$ is symmetric with respect to the real axis. If we consider $\theta \neq 0$, we obtain $2\pi - \theta = \theta + 2\pi(h-1)/h$. Hence, $\theta = \pi/h$ and this completes the proof. $\square$

Remark 4.8. If in Theorem 4.7, the assumption $|A| = S_1 A S_2$ is dropped, then (4.9) is not satisfied. Since the above proof depends heavily on the Theorem 3.2 and in

Example 3.4 we observed that Theorem 3.2 fails if the assumption $|A| = S_1AS_2$ is dropped.

We illustrate Theorem 4.7 in the following example.

Example 4.9. In this example, we show that there exists some $S \in \varphi$ such that (4.9) holds with $\theta = 0$, and also there exists some $T \in \varphi$ such that (4.9) holds with $\theta = \pi/h$. Consider the real matrix A shown in Example 3.4. Then there exists some $S \in \varphi$ such that $\omega_0^S(A) = 7.151$. For example, for the signature matrix $S = diag(1,1,1,-1,1,1,-1,-1)$, we obtain $\omega_0^S(A) = \omega_0(SA) = 7.151$, and the relation (4.9) holds with $\theta = 0$. For the signature matrix $T = diag(1,1,-1,-1,1,-1,-1,-1)$, we have also the relation (4.9) with $\theta = \pi/4$.

Figure 1 illustrates in detail the above example.

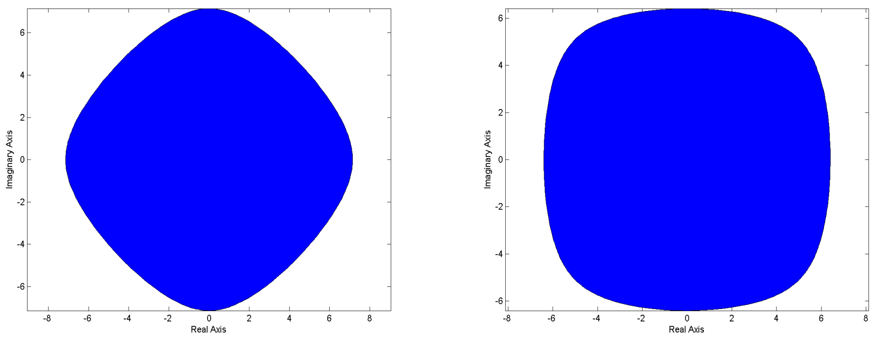

FIGURE 1. The numerical range of SA, TA. The graph on the left shows the numerical range of SA such that the numbers $\omega_0^S(A), e^{i\pi/2}\omega_0^S(A), e^{i\pi}\omega_0^S(A), e^{i3\pi/2}\omega_0^S(A)$ are belong to $W(SA)$, with modulus $\omega_0^S(A)$. The graph on the right shows the numerical range of TA such that the numbers $e^{i\pi/4}\omega_0^S(A), e^{i3\pi/4}\omega_0^S(A)$, $e^{i5\pi/4}\omega_0^S(A), e^{i7\pi/4}\omega_0^S(A)$ are belong to $W(TA)$, with modulus $\omega_0^S(A)$.

In the rest of this section, we will prove Theorem 4.7 in a graph theory manner by using the following results.

Lemma 4.10. *Let $A \in M_n(\mathbb{R})$ be such that $H(A)$ is irreducible, and $|A| = S_1AS_2$ for some S_1, $S_2 \in \varphi$. Let α be a real number such that $e^{i\alpha} \neq 1$, and suppose $e^{i\alpha}\omega_0^S(A) \in W(SA)$ for some $S \in \varphi$.*

(i) *If α is an irrational multiple of π, then A is permutationally similar to a block-shift matrix.*

(ii) *If α is a rational multiple of π, say $\alpha = 2\pi p/q$, where p, q are relatively prime integers, q being positive, then all cycles of $G(A)$ are of signed length an integral multiple of q.*

In any case, A is diagonally similar to $e^{i\alpha}A$.

Proof. Let $z = (\beta_1, \cdots, \beta_n)^t$ be a unit vector of $\mathbb{C}^n$ such that $z^*SAz = e^{i\alpha}\omega_0^S(A)$ for some $S \in \varphi$. We have

$$\begin{aligned}(4.10)\quad \omega_0^S(A) &= |z^*SAz| = \left|\sum_{r,t\in\langle n\rangle} \bar{\beta}_r\, \beta_t\, s_{rr}\, a_{rt}\right| \le \sum_{r,t\in\langle n\rangle} \left|\bar{\beta}_r\right|\, |\beta_t|\, |a_{rt}| \\ &= |z|^*\, |A|\, |z| = |z|^*\, H(|A|)\, |z| \le \lambda_{max}\left(H(|A|)\right) = \rho\left(H(|A|)\right) \\ &= \omega(|A|) = \omega_0^S(|A|) = \omega_0^S(A).\end{aligned}$$

Hence, the above inequalities all become equalities. As a consequence, $|z|$ is an eigenvector of the irreducible nonnegative matrix $H(|A|)$ corresponding to its spectral radius, and then, by Perron-Frobenius theorem, $|z| > 0$. By the condition for equality in (4.10) and the assumption that $z^*SAz = e^{i\alpha}|z^*SAz|$, we have

$$\overline{\beta}_r\, \beta_t\, |a_{rt}| = e^{i\alpha}\left|\overline{\beta}_r\, \beta_t\, s_{rr}\, a_{rt}\right| \quad \textit{for all} \quad r, t \in \langle n\rangle.$$

This implies that if (r, t) is an arc in $G(A)$, then the quantity $arg(\beta_t) - arg(\beta_r)$, is equal to α or $-\alpha$ according to whether the link is positive or negative. Hence, if we go around a cycle θ of $G(A)$ once, the total change of argument of the corresponding components of z is $s(\theta)\alpha$, and clearly this latter quantity must be an integral multiple of 2π. Thus, if α is an irrational multiple of π, then necessarily all cycles of $G(A)$ have zero signed length. In this case by [15, Theorem 1] A is permutationally similar to a block-shift matrix, and by [16, Corollary 4.7] A is diagonally similar to γA for any nonzero complex number γ, and in particular A is diagonally similar to $e^{i\alpha}A$. If $\alpha = 2\pi p/q$, where p, q are relatively prime integers, q being positive, then necessarily all cycles of $G(A)$ have signed length an integral multiple of q. In this case [16, Theorem 4.1] the latter is equivalent to the condition that A is diagonally similar to $e^{2\pi i/q}A$. It follows that A is diagonally similar to $e^{2\pi pi/q}A$. The proof is complete. □

Lemma 4.11. *Let $A \in M_n(\mathbb{R})$ be irreducible, and $|A| = S_1AS_2$ for some S_1, $S_2 \in \varphi$. If α is any real number, then the following are equivalent:*

(a) *A is diagonally similar to $e^{i\alpha}A$.*

(b) *$e^{i\alpha}W(SA) = W(SA)$ for some $S \in \varphi$.*

Proof. Suppose A is diagonally similar to $e^{i\alpha}A$. By [6, Remark 2.1], A is unitarily diagonally similar to $e^{i\alpha}A$, therefore the condition (b) holds. When A is real matrix, we have $\omega_0^S(A) \in W(SA)$ for some $S \in \varphi$. In this case, if condition (b) holds, then $e^{i\alpha}\omega_0^S(A) \in W(SA)$. By Lemma 4.10, the condition (a) holds. □

Lemma 4.12. *Let $A \in M_n(\mathbb{R})$ be irreducible, and $|A| = S_1AS_2$ for some S_1, $S_2 \in \varphi$. If the directed graph of A has at least one cycle with nonzero signed length, then the cyclic index of A is equal to the largest positive integer h that satisfies $e^{2\pi i/h}W(SA) = W(SA)$ for some $S \in \varphi$. In addition, this common value is also equal to the number of eigenvalues with absolute value equal to $\rho_0^S(A)$.*

Proof. By Lemma 4.11, there exists some $S \in \varphi$ such that the largest positive integer h for which $e^{2\pi i/h}W(SA) = W(SA)$ is the same as the largest positive integer h for which A is diagonally similar to $e^{2\pi i/h}A$. If $G(A)$ has at least one cycle with nonzero signed length, then by [16, Theorem 4.1] the latter number is equal to the largest positive integer h for which A is $h-$cyclic, i.e., the cyclic index of A. By [16, Theorem 4.1], again this common value is also equal to the largest positive integer h for which A and $e^{2\pi i/h}A$ have the same peripheral spectrum, which by Theorem 3.2 is clearly the number of eigenvalues with absolute value equal to $\rho_0^S(A)$. □

In the following, we prove Theorem 4.7 in a graph theory manner:

Consider an irreducible real matrix A such that $|A| = S_1AS_2$ for some S_1, $S_2 \in \varphi$. Let $S \in \varphi$ such that $\rho_0^S(A) \in \sigma(SA)$ and suppose SA has $h \geq 1$ distinct eigenvalues $\lambda_1, \ldots, \lambda_h$ of modulus $\rho_0^S(A)$. We have $\omega_0^S(A) \in W(TA)$ for some $T \in \varphi$ (Lemma 1.3(5)). Let $\theta \in [0, 2\pi)$ be the principal argument such that $\omega_0^S(A) \in W(e^{-i\theta}TA)$, and hence by Lemma 4.12, we have

$$\{\omega_0^S(A)e^{i(\theta+2\pi t/h)} \ : \ t = 0, 1, \cdots, h-1\} \subseteq \{z \in W(TA) \, : \, |z| = \omega_0^S(A)\}.$$

On the other hand, if there exists a real number α such that $e^{i\alpha}\omega_0^S(A) \in W(TA)$ for some $T \in \varphi$, then A can not be permutationally similar to a block-shift matrix (because A is irreducible). By Lemma 4.10, $\alpha = 2\pi p/q$, where p, q are relatively prime integers, and q is positive and divides the greatest common divisor of the signed lengths of the cycles of $G(A)$. By [16, Theorem 4.1] the latter number is equal to h (cf. the proof of Lemma 4.12). It follows that $e^{i\alpha}\omega_0^S(A)$ is equal to one of the h numbers $e^{2\pi ti/h}\omega_0^S(A)$, $t = 0, 1, \cdots, h-1$. Therefore we have proved that

$$\{z \in W(TA) \, : \, |z| = \omega_0^S(A)\} \subseteq \{\omega_0^S(A)e^{i(\theta+2\pi t/h)} \ : \ t = 0, 1, \cdots, h-1\}.$$

The proof that $\theta = 0$ or $\theta = \pi/h$, is similar to Theorem 4.7.

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