THE e-HYPERSTRUCTURES

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ABSTRACT. The largest class of hyperstructures is the one which satisfy the weak properties. These are called H_v-structures introduced in 1990 and they proved to have a lot of applications on several applied sciences. In this paper we present an introduction on the special classes of hyperstructures used in the Lie-Santilli theory called e-hyperstructures. Several extensions of these hyperstructures are also presented

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1. Introduction

We deal with hyperstructures called H_v-*structures* introduced in 1990 [15], which satisfy the *weak axioms* where the non-empty intersection replaces the equality. Some basic definitions are the following:

In a set H equipped with a hyperoperation (abbreviation *hyperoperation=hope*) $\cdot : H \times H \rightarrow \wp(H) - \{\emptyset\}$, we abbreviate by

WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \emptyset$, $\forall x, y, z \in H$ and by

COW the *weak commutativity*: $xy \cap yx \neq \emptyset$, $\forall x, y \in H$.

The hyperstructure $H, \cdot$ is called H_v-*semigroup* if it is *WASS*, it called H_v-*group* if it is reproductive H_v-*semigroup*, i.e. $xH = Hx = H, \forall x \in H$. The hyperstructure

$(R, +, \cdot)$ is called H_v-*ring* if both $(+)$ and $(\cdot)$ are $WASS$, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y + z) \cap (xy + yz) \neq \emptyset, (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R.$$

Motivation. The motivation for H_v-*structures* is the following: We know that the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v-*group*. This is the motivation to introduce the H_v-*structures* [16].

Specifying this motivation we remark: Let $(G, \cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R, \cdot)$ is an H_v-*group*, therefore we have the quotient $(G/R, \cdot)/\beta^*$ which is a group, the *fundamental* one. Remark that the classes of the fundamental group $(G/R, \cdot)/\beta^*$ are a union of some of the R-classes. Otherwise, the $(G/R, \cdot)/\beta^*$ has elements classes of G where they form a partition which classes are larger than the classes of the original partition R.

In an H_v-*semigroup* the *powers* of an element $h \in H$ are defined as follows

$$h^1 = \{h\}, h^2 = h \cdot h, \ldots, h^n = h^\circ h^\circ \ldots^\circ h,$$

where $(\circ)$ denotes the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-*semigroup* $(H, \cdot)$ is called *cyclic of period s*, if there exists an element g, called *generator*, and a natural number n, the minimum one, such that

$$H = h^1 \cup h^2 \cup \cdots \cup h^s.$$

Analogously the cyclicity for the infinite period is defined [14]. If there is an element h and a natural number s, the minimum one, such that $H = h^s$, then $(H, \cdot)$ is called *single-power cyclic of period s*.

For more definitions and applications on H_v-*structures*, see the books [4, 5, 7] and papers as [1, 3, 6]

The main tool to study hyperstructures are the *fundamental relations* β^*, γ^* and ε^*, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector space,

resp. These relations were introduced by T.Vougiouklis [15, 16]. A way to find the fundamental classes is given by theorems as the following [19, 20, 21]:

Theorem 1. Let $(H, \cdot)$ be an H_v-group and denote by U the set of all finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then β^* is the transitive closure of β.

Analogous theorems for the relations γ^* in H_v-rings, ε^* in H_v-modules and H_v-vector spaces, are also proved. Generalizations of the hyperstructure theory can be found in books as [5, 7, 16]. An element is called single if its fundamental class is singleton [16].

Fundamental relations are used for general definitions. Thus, an H_v-ring $(R, +, \cdot)$ is called H_v-field if R/γ^* is a field. Let $(H, \cdot)$, $(H, *)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an $f \in Aut(H, *)$ such that $xy \subset f(x * y), \forall x, y \in H$.
Then we write $\cdot \leq *$ and we say that $(H, *)$ contains $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called *basic structure* and $(H, *)$ is called H_b-structure.

Theorem 2. (The Little Theorem). Greater hopes than the ones which are *WASS* or *COW*, are also *WASS* or *COW*, respectively.

This Theorem leads to a partial order on H_v-structures and mainly to a correspondence between hyperstructures and posets. The determination of all H_v-groups and H_v-rings is very interesting but hard. Fortunately, in this direction there are results by R. Bayon and N. Lygeros. One can see, for example, in their paper [2] how many H_v-groups and H_v-rings there exist, up to isomorphism, for small sets, for several classes of hyperstructures, of two, three or four elements.

Definitions 3 [18, 19]. Let $(H, \cdot)$ be hypergroupoid. We *remove* $h \in H$, if we take the restriction of $(\cdot)$ in the set $H - \{h\}$. $h \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$ and h does not appear. $\underline{h} \in H$ *merges* with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both $h, \underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

H_v-structures are used in Representation Theory. Representations of H_v-groups can be considered by generalized permutations or by H_v-matrices [16]. Representations by generalized permutations can be achieved using translations. In this theory the single elements are playing a crucial role. H_v-*matrix* is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices is defined in a usual

manner. In representations of H_v-groups by H_v-matrices, there are two difficulties: To find an H_v-ring and an appropriate set of H_v-matrices.

Hopes on any type of matrices can be defined:

Definition 4 [13, 28, 10]. Let A $= (a_{ij}) \in \mathrm{M}_{m \times n}$ be a matrix and $s, t \in N$, with $1 \le s \le m$, $1 \le t \le n$. Then *helix-projection* is a map $\underline{st}$: $\mathrm{M}_{m \times n} \rightarrow \mathrm{M}_{s \times t}$: A $\rightarrow$ A$\underline{st}$ $= (a_{ij})$, where A$\underline{st}$ has entries

$$a_{ij} = a_{i+\kappa s, j+\lambda t} | 1 \le i \le s, 1 \le j \le t \text{ and } \kappa, \lambda \in \mathrm{N}, i + \kappa s \le m, j + \lambda t \le n$$

Let A $= (a_{ij}) \in \mathrm{M}_{m \times n}$, B $= (b_{ij}) \in \mathrm{M}_{u \times v}$ be matrices and $s = min(m, u)$, $t = min(n, v)$. We define a hyper-addition, called *helix-addition*, by

$$\oplus : \mathrm{M}_{m \times n} \times \mathrm{M}_{u \times v} \rightarrow \wp(\mathrm{M}_{s \times t}) : (\mathrm{A,B}) \rightarrow \mathrm{A} \oplus \mathrm{B} = \mathrm{A}\underline{st} + \mathrm{B}\underline{st} = (a_{ij}) + (b_{ij}) \subset \mathrm{M}_{s \times t}$$

where $(a_{ij}) + (b_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) | a_{ij} \in a_{ij} \text{ and } b_{ij} \in b_{ij}\}$.

Let A $= (a_{ij}) \in \mathrm{M}_{m \times n}$, B $= (b_{ij}) \in \mathrm{M}_{u \times v}$, $s = min(n, u)$, define the *helix-multiplication*, by

$$\oplus : \mathrm{M}_{m \times n} \mathrm{M}_{u \times v} \rightarrow \wp(\mathrm{M}_{m \times v}) : (\mathrm{A,B}) \rightarrow \mathrm{A} \oplus \mathrm{B} = \mathrm{A}\underline{ms} \cdot \mathrm{B}\underline{sv} = (a_{ij}) \cdot (b_{ij}) \subset \mathrm{M}_{m \times v}$$

where $(a_{ij}) + (b_{ij}) = \{(c_{ij}) = (a_{ij} b_{ij}) | a_{ij} \in a_{ij} \text{ and } b_{ij} \in b_{ij}\}$. The helix-addition is commutative, WASS but not associative. The helix-multiplication is WASS, not associative and it is not distributive, not even weak, to the helix-addition. For all matrices of the same type, the inclusion distributivity, is valid.

Definitions 5 [19]. Let $(F, +, \cdot)$ be an H_v-field, $(V, +)$ be a COW H_v-group and there exists an external hope

$$\cdot : F \times V \rightarrow \wp(V) - \emptyset : (a, x) \rightarrow ax$$

such that, for all a, b in F and x, y in V we have

$$a(x + y) \cap (ax + ay) \ne \emptyset, (a + b)x \cap (ax + bx) \ne \emptyset, (ab)x \cap a(bx) \ne \emptyset,$$

then V is called an H_v-vector space over F. In the case of an H_v-ring instead of an H_v-field then the H_v-modulo is defined. In these cases the fundamental relation ϵ^* is the smallest equivalence relation such that the quotient V/ϵ^* is a vector space over the fundamental field F/γ^*.

The general definition of an H_v-Lie algebra over F was given in [17] as follows:

Definition 6 [23]. Let $(L, +)$ be an H_v-vector space over the H_v-field $(F, +, \cdot)$, take the canonical map $\phi : F \rightarrow F/\gamma^*$ with $\omega_F = x \in F : \phi(x) = 0$, 0 is the zero of

F/γ^*, ω_L the core of $\phi' : L \to L/\epsilon^*$ and denote again 0 the zero of L/ϵ^*. Consider the bracket (commutator) hope:

$$[,] : L \times L \to \wp(L) - \emptyset : (x, y) \to [x, y]$$

then L is an H_v-Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$[\lambda_1 x1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \emptyset$

$[x, \lambda_1 y_1 + \lambda_2 y] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \emptyset, \forall x, x_1, x_2, y, y_1, y_2 \in L$ and $\lambda_1, \lambda_2 \in$ F

(L2)$[x, x] \cap \omega_L \neq \emptyset, \forall x \in L$

(L3)$([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \forall x, y \in L$

In 1989 Corsini and Vougiouklis [16], introduced a method, *the uniting elements method*, to obtain stricter algebraic structures, from given ones, through hyperstructure theory. This method was introduced before the introduction of the H_v-structures, but in fact the H_v-structures appeared in the procedure. This method is the following:

Definition 7. *The uniting elements method* is the following: Let G be a structure and d be a property, which is not valid, and suppose that d is described by a set of equations. Consider the partition in G for which it is put together, in the same class, every pair of elements that causes the non-validity of d. The quotient G/d is an H_v-structure. Then quotient of G/d by the fundamental relation β^*, is a stricter structure $(G/d)\beta^*$ for which d is valid.

An application of the uniting elements is when more than one properties are desired. The reason for this is some of the properties lead straighter to the classes than others. The commutativity and the reproductivity are easily applicable. One can do this because the following Theorem is valid:

Theorem [16]. Let $(G, \cdot)$ be groupoid, and $F = f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n}$ a system of equations on G consisting of two subsystems $F_m = f_1, \ldots, f_m, F_n = f_{m+1}, \ldots, f_{m+n}$. Let σ and σ_m be the equivalence relations defined by the uniting elements using the F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $G_m = (G/\sigma_m)/\beta^*$. Then we have

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*.$$

During last decades hyperstructures seem to have a variety of applications not only in other branches of mathematics but also in many other sciences including social studies. These applications range from biomathematics -conchology, inheritance- and hadronic physics to automata theory, to mention but a few. The hyperstructures theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too.

In several books and papers, such as [3, 5, 7, 9, 12, 15] one can find numerous applications. An important new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis. The suggestion is the following:

Definition 8. *"In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end: The sub-*

$$0 \ \rule{6cm}{0.4pt} \ 1$$

jects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question".

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing [11, 27].

2. Two classes of H_v-structures

2.1. **The theta (∂) hopes.** In [22] a hope, in a groupoid with a map f on it, denoted ∂_f, is introduced. Since there is no confusion, we write simply *theta* ∂. The symbol '∂' appears in Greek papyrus to represent the letter 'theta', usually in middle rather than the beginning of the words.

Definitions 9. Let H be a set equipped with n operations (or hopes) $\otimes_1, \otimes_2, \ldots, \otimes_n$ and a map (or multivalued map) $f : H \to H$ (or $f : H \to \wp(H) - \{\emptyset\}$, respectively), then n hopes $\partial_1, \partial_2, \ldots, \partial_n$ on H can be defined, called *theta-operations* (we rename here *theta-hopes* and we write *∂-hope*) by putting

$$x\partial_i y = \{f(x) \otimes_i y, x \otimes_i f(y)\}, \forall x, y \in H \text{ and } i \in \{1, 2, \ldots, n\}$$

or, in case where $\otimes_i$ is hope or f is multivalued map, we have

$$x\partial_i y = (f(x) \otimes_i y) \cup (x \otimes_i f(y)), \forall x, y \in H \text{ and } i \in \{1, 2, \ldots, n\}.$$

If $\otimes_i$ is associative then ∂_i is WASS.

Analogously one can use several maps f, instead than only one.

Let $(G, \cdot)$ be a groupoid and $f_i : G \rightarrow G, i \in \mathrm{I}$, be a set of maps on G. Take the map $f_\cup : G \rightarrow \wp(G)$ such that $f_\cup(x) = \{f_i(x) | i \in I\}$ and we call it *the union* of the $f_i(x)$. We call *union ∂-hopes*, on G if we consider the map $f \cup (x)$. A special case is to take the union of f with the identity, i.e. $\underline{f} = f_\cup(id)$, so $f(x) = \{x, f(x)\}, \forall x \in G$, which is called *b-$\partial$-hope*. We denote the *b-∂-hope* by $(\underline{\partial})$, so

$$x\underline{\partial}y = xy, f(x) \cdot y, x \cdot f(y), \forall x, y \in G.$$

This hope contains the operation $(\cdot)$ so it is a *b*-hope. If $f : G \rightarrow \wp(G) - \emptyset$, then the *b-$\partial$-hope* is defined by using the map $\underline{f}(x) = x \cup f(x), \forall x \in G$.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for two functions $s(x), t(x)$, we have $s\partial t = s't, st'$ where $(')$ denotes the derivative.

For several results one can see [22, 23, 25, 26].

Examples. (a) Taking the application on the derivative, consider all polynomials of up to first degree $g_i(x) = a_i x + b_i$. We have

$$g_1 \partial g_2 = \{a_1 a_2 x + a_1 b_2, a_1 a_2 x + b_1 a_2\},$$

so this is a hope in the first degree polynomials. Remark that all polynomials $x + c$, where c be a constant, are units.

(b) The *constant map*. Let $(G, \cdot)$ be group and $f(x) = a$, thus $x\partial y = \{ay, xa\}, \forall \, x, y \in G$. If $f(x) = e$, then we obtain $x\partial y = \{x, y\}$, the smallest incidence hope.

Properties. If $(G, \cdot)$ is a semigroup then:

a. For every f, the ∂-hope is WASS.

b. For every f, the *b-∂-hope* $(\underline{\partial})$ is WASS.

c. If f is homomorphism and projection, then (∂) is associative.

Properties. *Reproductivity.* For the reproductivity we must have

$$x\partial G = \cup_{g \in G}\{f(x) \cdot g, x \cdot f(g)\} = G \text{ and } G\partial x = \cup_{g \in G}\{f(g) \cdot x, g \cdot f(x)\} = G.$$

Thus, if $(\cdot)$ is reproductive then $(\cdot)$ is also reproductive.

Commutativity. If $(\cdot)$ is commutative then (∂) is commutative. If f is into the centre of G, then (∂) is commutative. If $(\cdot)$ is *COW* then, (∂) is *COW*.

Unit elements. u is a right unit element if $x\partial u = f(x) \cdot u, x \cdot f(u) \ni x$. So $f(u) = e$, where e be a unit in $(G, \cdot)$. The elements of the kernel of f, are the units of $(G, \cdot)$.

Inverse elements. Let $(G, \cdot)$ be a monoid with unit e and u be a unit in $(G, \cdot)$, then $f(u) = e$. For given x, the elements $x' = (f(x))^{-1}u$ and $x' = u(f(x))^{-1}$, are the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$.

From the above properties it is clear that:

Proposition 10. Let $(G, \cdot)$ be a group then, for all maps $f : G \to G$, the hyperstructure (G, ∂) is an H_v-group.

A construction between ∂ and $\underline{\partial}$ is the one which obtained by using special maps.

Definition 10. Let $(G, \cdot)$ be a groupoid and $f : G \to G$ be a map, we call *basic set* of the map f the set $B = B_f = x \in G : f(x) = x$. Then, if $B \in \emptyset$, we have

$$x\partial y = xy, \forall x, y \in B$$
$$x\partial y = \{xy, x \cdot f(y)\}, \forall x \in B, y \in G - B,$$
$$x\partial y = \{f(x) \cdot y, xy\}, \forall x \in G - B, y \in B,$$
$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \forall x, y \in G - B.$$

For $(G, \cdot)$ groups, we obtain the following

(1) If B is a subgroup of $(G, \cdot)$, then (B, ∂) is a sub-H_v-group of (G, ∂).

(2) If $e \in B$, then e is a unit of (B, ∂).

Propositions. (a) Let $g \in G$ is a generator of the group $(G, \cdot)$. If $g \in B$ then, for every f, the element g is a generator in $G, \partial)$.

(b) Let $g \in G$ is a generator of the group $(G, \cdot)$. We suppose that there exists an element w such that $f(w) = g$, then the element w is a generator in (G, ∂).

There is a connection of the ∂-hopes with other classes of hyperstructures:

Example. *Merging and ∂.* If $(H, \cdot)$ is a groupoid and $\underline{h} \in H$ merges with the element $h \in H$, then h does not appeared and we have for the merge hyperstructure $(H, \circ)$,

$$\underline{h} \circ x = \{\underline{h} \cdot x, h \cdot x\}, x \circ \underline{h} = \{x \cdot \underline{h}, x \cdot h\}, \underline{h} \circ \underline{h} = \{\underline{h} \cdot \underline{h}, \underline{h} \cdot h, h \cdot \underline{h}, h \cdot h\}$$

and in the rest cases $(\circ) \equiv (\cdot)$, so we have the hyperstructure $(H - \{h\}, \circ)$.

For a connection of the merge with the ∂-hope, consider the map f such that $f(\underline{h}) = h$ and $f(x) = x$ in the rest cases. Then in $(H - \{h\}, \partial)$ we have,

$\forall\; x, y\; \in H - \{h\}$

$$\underline{h}\partial x = \{h \cdot x, \underline{h} \cdot x\}, x\partial \underline{h} = \{x \cdot \underline{h}, x \cdot h\} \text{ and } \underline{h}\partial \underline{h} = \{h \cdot \underline{h}, \underline{h} \cdot h\}$$

and in the rest cases (∂) coincides with $(\cdot)$. Therefore, $\partial \leq \circ$, more precisely

$$\underline{h}\partial \underline{h} = \{h \cdot \underline{h}, \underline{h} \cdot h\} \subset \{\underline{h} \cdot \underline{h}, \underline{h} \cdot h, h \cdot \underline{h}, h \cdot h\} = \underline{h} \circ \underline{h}$$

and in the remaining cases we have $\partial \equiv \circ$.

Definition 11. Let $(R, +, \cdot)$ be a ring and $f : R \to R$, $g : R \to R$ be two maps. We define two hopes (∂_+) and $(\partial\cdot)$, called both *theta-hopes*, on R as follows

$$x\partial_+ y = \{f(x) + y, x + f(y)\} \text{ and } x\partial \cdot y = \{g(x)y, xg(y)\}, \forall x, y \in G.$$

A hyperstructure $(R, +, \cdot)$, where $(+), (\cdot)$ are hopes which satisfy all H_v-ring axioms, except the weak distributivity, will be called H_v-*near-ring*.

Propositions 12. (a) Let $(R, +, \cdot)$ be a ring and $f : R \to R$, $g : R \to R$ be maps. The $(R, \partial_+, \partial\cdot)$, called *theta*, is an H_v-near-ring. Moreover (∂_+) is commutative.
(b) Let $(R, +, \cdot)$ be a ring and $f : R \to R$, $g : R \to R$ maps, then $(R, \underline{\partial}_+, \partial\cdot)$, is an H_v-ring.

Properties. *(Special classes).* The theta hyperstructure $(R, \underline{\partial}_+, \partial\cdot)$ takes a new form and has some properties in several cases as the following ones:
(a) If f is a homomorphism and projection, then

$$x\partial \cdot (y\partial_+ z) \cap (x\partial \cdot y)\partial_+(x\partial \cdot z) = \{f(x)f(y) + f(x)z, f(x)y + f(x)f(z)\} \neq \emptyset.$$

Therefore, $(R, \partial_+, \partial\cdot)$ is an H_v-ring.
(b) If $f(x) = x, \forall x \in R$, then $(R, +, \partial\cdot)$ becomes a multiplicative H_v-ring:

$$x\partial \cdot (y + z) \cap (x\partial \cdot y) + (x\partial \cdot z) = \{g(x)y + g(x)z\} \neq \emptyset.$$

If, moreover, f is a homomorphism, then we have a 'more' strong distributivity:

$$x\partial \cdot (y + z) \cap (x\partial \cdot y) + (x\partial \cdot z) = \{g(x)y + g(x)z, xg(y) + xg(z)\} \neq \emptyset.$$

Theorem 13. Consider the group of integers $(Z, +)$ and $n \neq 0$ be a natural number. Take the map f such that $f(0) = n$ and $f(x) = x, \forall x \in Z - 0$. Then

$$(Z, \partial)/\beta^* \cong (Z_n, +).$$

Theorems 14. (a) Take the ring of integers $(Z, +, \cdot)$ and fix $n \neq 0$ a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x, \forall x \in Z - 0$. Then $(Z, \partial_+, \partial \cdot)$ is an H_v-near-ring, with

$$(Z, \partial_+, \partial \cdot)/\gamma^* \cong Z_n.$$

(b) Consider the $(Z, +, \cdot)$ and $n \neq 0$ a natural. Take the map f such that $f(n) = 0$ and $f(x) = x, \forall x \in Z - n$. Then $(Z, \partial_+, \partial \cdot)$ is an H_v-ring, moreover,

$$(Z, \partial_+, \partial \cdot)/\gamma^* \cong Z_n.$$

Special case of the above is for $n = p$, prime, then $(Z, \partial_+, \partial \cdot)$ is an H_v-field.

Now we can see theta hopes in H_v-vector spaces and H_v-Lie algebras:

Theorem 15. Let $(V, +, \cdot)$ be an algebra over the field $(F, +, \cdot)$ and $f : V \to V$ be a map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(V, +, \partial)$ is an H_v-algebra over F, where the related properties are weak. If, moreover f is linear then we have more strong properties.

Proof. It is clear that (∂) is WASS. Moreover

$$x \partial(y + z) = \{f(x)(y + z), x f(y + z)\}$$

and $(x \partial y) + (x \partial z) = \{f(x)y + f(x)z, f(x)y + x f(z), x f(y) + f(x)z, x f(y) + x f(z)\}$. Thus the weak distributivity is valid: $x \partial(y + z) \cap (x \partial y) + (x \partial z) = f(x)(y + z) \neq \emptyset$. If f is linear then the inclusion distributivity is valid. Moreover

$$\lambda(x \partial y) = \{\lambda f(x)y, \lambda x f(y)\}$$

$$(\lambda x) \partial y = \{f(x)y, \lambda x f(y)\} \text{ and } x \partial(\lambda y) = \{\lambda f(x)y, x f(\lambda y)\}$$

so

$$\lambda(x \partial y) \cap (\lambda x) \partial y = \lambda x f(y) \neq \emptyset \text{ and } \lambda(x \partial y) \cap x \partial(\lambda y) = \lambda f(x)y \neq \emptyset$$

But, if f is linear, then we have $\lambda(x \partial y) = (\lambda x) \partial y = x \partial(\lambda y)$.

Definition 16. Let $(A, +, \cdot)$ be an algebra over the field F. Take any map $f : A \to A$, then the ∂-hope on the Lie bracket $[x, y] = xy - yx$, is defined as follows

$$x \partial y = \{f(x)y - f(y)x, f(x)y - y f(x), x f(y) - f(y)x, x f(y) - y f(x)\}.$$

Remark that if we take the identity map $f(x) = x, \forall x \in A$, then $x \partial y = \{xy - yx\}$, thus we have not a hope and remains the same operation.

Proposition 17. Let $(A, +, \cdot)$ be an algebra F and $f : A \to A$ be a linear map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(A, +, \partial)$ is an H_v-algebra over F, with respect to the ∂-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

Proposition. Let $(A, +, \cdot)$ be an algebra and $f : A \to A : f(x) = a$ be a constant map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(A, +, \partial)$ is an H_v-Lie algebra over F.

In the above theorem if one takes $a = e$, the unit element of the multiplication, then the properties become more strong.

2.2. **The theta P-hopes.** A general way to define hopes, which are not always of a constant length, from given operations [14, 16, 24] can be generalized as follows:

Definition 18. Let $(G, \cdot)$ be a groupoid, then for every $P \subset G, P \neq \emptyset$, we define the following hopes called P-hopes: for all $x, y \in G$

$$\underline{P} : xPy \cup (xP)y, \underline{P}_r : xP_ry = (xy)P \cup x(yP), \underline{P}_1 : xP_ly = (Px)y \cup P(xy).$$

The (G, P), (G, P_r) and (G, P_1) are called P-hyperstructures. The most usual case is if $(G, \cdot)$ is semigroup, then

$$xPy = (xP)y \cup x(Py) = xPy$$

and (G, P) is a semihypergroup but we do not know about (G, P_r) and (G, P_l). In some cases, depending on the choice of P, the (G, P_r) and (G, P_l) can be associative or WASS. If in the set G, more operations are defined then obviously for each one operation several P-hopes can be defined.

Definition 19. Let $M = M_{m \times n}$ be a module of $m \times n$ matrices over a ring R and $P = P_i : i \in I \subseteq M$. We define, a kind of, a P-hope P on M as follows

$$P : M \times M \to P(M) : (A, B) \to A\underline{P}B = AP_i^t B : i \in IM$$

where P^t denotes the transpose of the matrix P.

The hope $\underline{P}$, which is a bilinear map, is a generalization of Rees' operation where, instead of one sandwich matrix, a set of sandwich matrices is used. The hope $\underline{P}$ is strong associative and the inclusion distributivity with respect to addition of matrices

$$A\underline{P}(B + C) \subseteq A\underline{P}B + A\underline{P}C \forall A, B, C \in M$$

is valid. Therefore, $(M, +, P)$ defines a multiplicative hyperring on non-square matrices. Multiplicative hyperring means that only the multiplication is a hope.

Definition 20. Let $M = M_{m \times n}$ be module of $m \times n$ matrices over R and let take sets $S = s_k : k \in K \subseteq R, Q = Q_i : j \in J \subseteq M, P = P_i : i \in I \subseteq M$. Define three hopes as follows

$$\underline{S} : R \times M \to P(M) : (r, A) \to r\underline{S}A = (r_{sk})A : k \in K \subseteq M$$

$$\underline{Q}_+ :\to M \times M \to P(M) : (A, B)A\underline{Q}_+B = A + Q_j + B : j \in J \subseteq M$$

$$\underline{P} : M \times M \to P(M) : (A, B) \to A\underline{P}B = AP_i^t B : i \in IM$$

Then (M, S, Q_+, P) is a hyperalgebra over R called general matrix P-*hyperalgebra*.

In a similar way a generalization of this hyperalgebra can be defined if one considers an H_v-ring or an H_v-field instead of a ring and using H_v-matrices.

Connection of ∂-hopes with P-hopes: Let $(G, \cdot)$ be commutative semigroup and $P \subset G$. Consider the multivalued map $f : f(x) = P \cdot x, \forall x \in G$. Then we have $x\partial y = x \cdot y \cdot P, \forall x, yG$. So the ∂-hope coincides with the class of P-hopes.

3. THE e-HYPERSTRUCTURES

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1999 [12] and they are called e-hyperfields. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology. We present in the following the main definitions and results restricted in the H_v-structures.

Definition 22. A hyperstructure $(Hm; , \cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we normally assume that for

every element x, there exists an inverse element x^{-1}, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$. Remark that the inverses are not necessarily unique.

Definition 23. A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following axioms are valid:

(1) $(F, +)$ is an abelian group with the additive unit 0,

(2) $(\cdot)$ is WASS,

(3) $(\cdot)$ is weak distributive with respect to $(+)$,

(4) 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$,

(5) there exists a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$,

(6) for every $x \in F$ there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an e-hyperfield are called e-hypernumbers. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield*.

Now we present a general construction which is based on the partial ordering of the H_v-structures and on the Little Theorem.

Definition 24 [12, 8]. *The Main e-Construction.* Given a group $(G, \cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:

$$x \otimes y = xy, g_1, g_2, \dots, \forall x, y \in G - e, \text{ and } g_1, g_2, \dots G - e$$

$g_1, g_2, \dots$ are not necessarily the same for each pair (x, y). Then $(G, \otimes)$ becomes an H_v-group, in fact is H_b-group which contains the $(G, \dots)$. The H_v-group $(G, \otimes)$ is an e-hypergroup. Moreover, if for each x, y such that $xy = e$, so we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong e-hypergroup.

The proof is immediate since for both cases we enlarge the results of the group by putting elements from G and applying the Little Theorem. Moreover one can see that the unit e is a unique scalar element and for each x in G, there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$ and if this condition is valid then we have $1 = x \cdot x^1 = x^{-1} \cdot x$. So the hyperstructure $(G, \otimes)$ is a strong e-hypergroup.

Remark. The above main e-construction gives an extremely large class of e-hopes. These e-hopes can be used in the several more complicate hyperstructures to obtain appropriate e-hyperstructures. However, notice that the most useful are the ones where only few products are enlarged.

Example. Consider the finite-non-commutative quaternion group
$Q = 1, -1, i, -i, j, -j, k, -k$ whose multiplication is given by the table

·	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

Using this operation one can obtain several hopes which define very interesting e-groups. For example, denoting $\underline{i} = i, -i, \underline{j} = j, -j, \underline{k} = k, -k$ we may define the $(*)$ hope by the table:

$*$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$\underline{k}$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	$\underline{i}$	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$\underline{k}$	$\underline{j}$	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

The hyperstructure $(Q, *)$ is strong e-hypergroup because 1 is scalar unit and the elements $-1, i, -i, -j, k$ and $-k$ have unique inverses the elements $-1, -i, i, -j, j, -k$ and k, respectively, which are the inverses in the basic group.

The important thing of this example is that one can have more strict hopes.

The reason we gave the above example is to see that there is a large variety of e-hyperstructures we can construct from given classical structures. One can see in the papers [8, 12] and in the book [7] some classes of e-hyperstructures and their properties and results connected them with the classical theory. The representation theory and the Lie algebras as well as in hypermatrix theory large classes of e-hyperstructures appear and can offer to Lie-Santilli algebraic theory

models to represent their theory. In the Lie admissible algebras the P-hopes can be used as one can see the replacement of the unit matrix in the Lie-bracket (in isoproducts): by any set of matrices P:

$[X, Y]_P = (XP)Y - (YP)X$, for all matrices or hypermatrices X and Y.

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