THE MODEL UPDATING OF MASS, DAMPING AND STIFFNESS MATRICES

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Abstract. Model updating for quadratic eigenvalue problems ($QEPs$) is proposed by Friswell, Inman and Pilkey (1998), to incorporate the measured model data into the model which only produces the mass and stiffness matrices, that closely match the experimental model data. In this paper, we consider a numerical model for updating $QEPs$ which produces not only the mass and stiffness matrices, but also the damping matrix will be updated. To this end the complete set of eigenpairs will be employed.

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1. Introduction

The problem of finding scalars $\lambda \in \mathcal{C}$ and nontrivial vectors $\mathbf{x} \in \mathcal{C}^n$ such that

$$\mathbf{Q}(\lambda)\mathbf{x} = (\lambda^{\mathbf{2}}\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})\mathbf{x} = \mathbf{0}, \tag{1.1}$$

where $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are given $n \times n$ real matrices, is known as the quadratic eigenvalue problem QEP. The nonzero vectors $\mathbf{x}$ and the corresponding scalars λ are called eigenvectors and eigenvalues of the QEP, respectively. It is known that if

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the leading coefficient matrix $\mathbf{M}$ is nonsingular, then the quadratic pencil will have $2n$ eigenvalues over $\mathcal{C}$. Recently the QEP has received much attention because its information has repeatedly arisen in many different disciplines, including applied mechanics, electrical oscilation, vibro acoustics, fluid mechanics and signal processing. A nice survey paper for the QEP can be found in [15] by Tisseur and Meerbergen. However, due to lack of reliable computational methods to handle distributed parameter systems a finite element method is generally used to discretize such systems to an analytical model(finite element model), namely,

$$\mathbf{Q_a}(\lambda) = \lambda^{\mathbf{2}}\mathbf{M_a} + \lambda\mathbf{C_a} + \mathbf{K_a}, \tag{1.2}$$

where $\mathbf{M_a}$, $\mathbf{C_a}$ and $\mathbf{K_a}$ represent the mass, damping and stiffness matrices, respectively, that all are real $n \times n$ symmetric matrices. See [10] by Friswell and Mottershead for details.

Finite element model updating has emerged in the 1990's as a significant subject to the design, construction, and maintenance of mechanical systems [10, 14]. In the past decades, Baruch, Bar-Itzack [1, 2], Bermann, Nagy [3, 4] and Wei [16, 17, 18] and recently Mohseni and Tajaddini [13] have considered variant aspects of finite element model updating by using measured data for undamped structured systems(i.e. $\mathbf{C} = \mathbf{C_a} = \mathbf{0}$). In the works by Datta, Elhay, Ram, Sarkissian [6, 7, 8], studies are undertaken toward a feedback design problem for second-order system. Recently, Friswell, Inman and Pilkey [9] and Kuo, Lin and Xu [11] proposed to incorporate the measured model data into the finite element model to produce an adjusted finite element model on the damping and stiffness with modal properties that closely math the experimental modal data. The purpose of this paper is to develop an algorithm for the computation of the solutions $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$, that the penalty function

$$\mathbf{J} = \|\mathbf{M} - \mathbf{M_a}\|^{\mathbf{2}}_{\mathbf{F}} + \mu\|\mathbf{C} - \mathbf{C_a}\|^{\mathbf{2}}_{\mathbf{F}} + \|\mathbf{K} - \mathbf{K_a}\|^{\mathbf{2}}_{\mathbf{F}} \tag{1.3}$$

is minimized, subject to

$$\mathbf{M\Phi\Lambda^2} + \mathbf{C\Phi\Lambda} + \mathbf{K\Phi} = \mathbf{0}, \tag{1.4}$$

$$\mathbf{M^T} = \mathbf{M}, \quad \mathbf{C^T} = \mathbf{C}, \quad \mathbf{K^T} = \mathbf{K}, \tag{1.5}$$

where, μ is a weighting parameter, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the updated mass, damping and stiffness matrices, respectively. More over $\mathbf{\Phi} \in \mathbf{R}^{n\times\mathbf{2}n}$ and $\mathbf{\Lambda} \in \mathbf{R}^{\mathbf{2}n\times\mathbf{2}n}$ are newly measured eigenvector and eigenvalue matrices, respectively. The fundamental of the approach are to utilize the parametric representation of **(M,C,K)** that Mohseni and Tajaddini developed earlier [12], and to rewrite the objective function as an unconstrained optimization in terms of the free parameters.

2. SOLVING AN IQEP

We first consider the self-adjoint pencil

$$\mathbf{Q_a}(\lambda) = \lambda^{\mathbf{2}}\mathbf{M_a} + \lambda\mathbf{C_a} + \mathbf{K_a}.$$

Let $(\mathbf{\Lambda}, \mathbf{\Phi}) \in \mathbf{R}^{\mathbf{2}n\times\mathbf{2}n} \times \mathbf{R}^{n\times\mathbf{2}n}$ be a given pair of matrices, where

$$\mathbf{\Lambda} = \mathbf{diag}\{\lambda_{\mathbf{1}}^{[\mathbf{2}]}, \cdots, \lambda_{\ell}^{[\mathbf{2}]}, \lambda_{\mathbf{2}\ell+\mathbf{1}}, \cdots, \lambda_{\mathbf{2}n}\} \tag{2.6}$$

with $\lambda_j^{[2]} = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$, $\beta_j \neq 0$, for $\mathbf{j} = \mathbf{1}, \cdots, \ell$, and

$$\mathbf{\Phi} = [\varphi_{\mathbf{1R}}, \varphi_{\mathbf{1I}}, \cdots, \varphi_{\ell\mathbf{R}}, \varphi_{\ell\mathbf{I}}, \varphi_{\mathbf{2}\ell+\mathbf{1}}, \cdots, \varphi_{\mathbf{2n}}]. \tag{2.7}$$

We shall make a practical assumption that all eigenvalues are distinct. Such an assumption can be deemed reasonable, because multiple roots are sensitive to perturbation and, hence, are hardly observable in real applications.

Let $\mathbf{\Lambda}$ has only simple eigenvalues, and $\begin{pmatrix} \mathbf{\Phi} \\ \mathbf{\Phi\Lambda} \end{pmatrix}$ is a matrix of full rank. We try to find a general form of symmetric matrices **M**, **C** and **K** that satisfy in(1.4), (1.5). A general solution of the above problem is given the following theorem that Mohseni and Tajaddini developed [12].

Theorem 2.1. *Let a standard eigenpair* $(\mathbf{\Lambda}, \mathbf{\Phi}) \in \mathbf{R}^{\mathbf{2}n\times\mathbf{2}n} \times \mathbf{R}^{n\times\mathbf{2}n}$ *as in (2.6), (2.7), be given then the general solution of IQEP forms are as:*

$$\mathbf{M} = (\mathbf{\Phi\Gamma^{-1}\Lambda^T\Phi^T})^{-\mathbf{1}}, \tag{2.8}$$

$$\mathbf{C} = -\mathbf{M\Phi\Lambda^2\Gamma^{-1}\Phi^TM}, \tag{2.9}$$

and

$$\mathbf{K} = -\mathbf{M\Phi\Lambda^3\Gamma^{-1}\Phi^TM} + \mathbf{CM^{-1}C}, \tag{2.10}$$

where

$$\mathbf{\Gamma^{-1}} = \mathbf{diag}\{\begin{pmatrix} \xi_{\mathbf{1}} & \eta_{\mathbf{1}} \\ \eta_{\mathbf{1}} & -\xi_{\mathbf{1}} \end{pmatrix}, \cdots, \begin{pmatrix} \xi_\ell & \eta_\ell \\ \eta_\ell & -\xi_\ell \end{pmatrix}, \xi_{\mathbf{2}\ell+\mathbf{1}}, \cdots, \xi_{\mathbf{2}n}\}, \tag{2.11}$$

in which $\xi_{\mathbf{i}}$, $i = 1, 2, ..., 2n$ *and* $\eta_{\mathbf{i}}$, $i = 1, 2, ..., n$ *are arbitrary real numbers and*

$$\mathbf{\Phi\Gamma^{-1}\Phi} = \mathbf{0}, \tag{2.12}$$

$$\mathbf{\Gamma^{-1}\Lambda} = \mathbf{\Lambda^T\Gamma^{-1}}. \tag{2.13}$$

Proof. See in [12].

We try to change the above parametric representation of $(\mathbf{M}, \mathbf{C}, \mathbf{K})$ to a parametric representation that contain n unknown parameters instead of $2n$ unknown parameters.

Let $\mathbf{\Phi}$ have the **SVD**$-$decomposition

$$\mathbf{\Phi} = \mathbf{V}\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \end{pmatrix}\mathbf{U^T} \tag{2.14}$$

where $\mathbf{V} \in \mathbf{R}^{n\times n}$ and $\mathbf{U} \in \mathbf{R}^{\mathbf{2}n\times\mathbf{2}n}$ are orthogonal matrices and $\mathbf{\Sigma} \in \mathbf{R}^{n\times n}$ is a diagonal matrix with positive diagonal entries. Partition $\mathbf{U}$, $\mathbf{\Gamma^{-1}}$ and $\mathbf{\Lambda}$ by

$$\mathbf{U} = \begin{pmatrix} \mathbf{U_{11}} & \mathbf{U_{12}} \\ \mathbf{U_{21}} & \mathbf{U_{22}} \end{pmatrix}, \quad \mathbf{\Gamma^{-1}} = \begin{pmatrix} \mathbf{\Gamma_1^{-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma_2^{-1}} \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda_2} \end{pmatrix}, \tag{2.15}$$

where $\mathbf{U_{ij}} \in \mathbf{R^{n\times n}}$, $\mathbf{\Gamma_j^{-1}} \in \mathbf{R^{n\times n}}$, $\mathbf{\Lambda_j} \in \mathbf{R^{n\times n}}$, $i, j = 1, 2$. Substituting (2.14), (2.15) into (2.12) and simplifying we have

$$\mathbf{\Sigma}(\mathbf{U_{11}^T\Gamma_1^{-1}U_{11}} + \mathbf{U_{21}^T\Gamma_2^{-1}U_{21}})\mathbf{\Sigma} = \mathbf{0}. \tag{2.16}$$

Since $\mathbf{\Sigma}$ is nonsingular, multiplying (2.16) by $\mathbf{\Sigma^{-1}}$ from the right and the left, we get

$$\mathbf{U_{11}^T\Gamma_1^{-1}U_{11}} + \mathbf{U_{21}^T\Gamma_2^{-1}U_{21}} = \mathbf{0}. \tag{2.17}$$

Since $\begin{pmatrix} \mathbf{\Phi} \\ \mathbf{\Phi\Lambda} \end{pmatrix}$ is a matrix full of rank, and substituting (2.14), and (2.15) into $\begin{pmatrix} \mathbf{\Phi} \\ \mathbf{\Phi\Lambda} \end{pmatrix}$, we conclude $\mathbf{U_{11}}, \mathbf{U_{21}}$ are invertible. By (2.17), we have

$$\mathbf{U_{21}^T\Gamma_2^{-1}} = -\mathbf{U_{11}^T\Gamma_1^{-1}U_{11}U_{21}^{-1}}. \tag{2.18}$$

Substituting (2.14), (2.15) and (2.18) into (2.8), (2.9) and (2.10) and simplifying we have

$$\mathbf{V^T M V} = (\mathbf{A\Sigma})^{-1}\mathbf{\Gamma_1}(\mathbf{\Sigma U_{11}^T})^{-1}, \tag{2.19}$$

$$\mathbf{V^T C V} = -(\mathbf{A\Sigma})^{-1}\mathbf{B A^{-1}}\mathbf{\Gamma_1}(\mathbf{\Sigma U_{11}^T})^{-1}, \tag{2.20}$$

and

$$\mathbf{V^T K V} = (\mathbf{A\Sigma})^{-1}\mathbf{E A^{-1}}\mathbf{\Gamma_1}(\mathbf{\Sigma U_{11}^T})^{-1}, \tag{2.21}$$

where

$$\mathbf{A} = \mathbf{\Lambda_1^T U_{11}} - \mathbf{U_{11} U_{21}^{-1} \Lambda_2^T U_{21}}, \tag{2.22}$$

$$\mathbf{B} = \mathbf{\Lambda_1^{2T} U_{11}} - \mathbf{U_{11} U_{21}^{-1} \Lambda_2^{2T} U_{21}}, \tag{2.23}$$

and

$$\mathbf{E} = -\mathbf{\Lambda_1^{3T} U_{11}} + \mathbf{U_{11} U_{21}^{-1} \Lambda_2^{3T} U_{21}} + \mathbf{B A^{-1} B}. \tag{2.24}$$

We now solve the optimization problem (1.3). Since Frobenius norm is invariant respect to orthogonal transformation, we can rewrite the optimization problem (1.3) in the following form

$$\begin{aligned} &\|\mathbf{M}-\mathbf{M_a}\|_\mathbf{F}^\mathbf{2} + \mu\|\mathbf{C}-\mathbf{C_a}\|_\mathbf{F}^\mathbf{2} + \|\mathbf{K}-\mathbf{K_a}\|_\mathbf{F}^\mathbf{2} \\ &= \|\mathbf{V^T}(\mathbf{M}-\mathbf{M_a})\|_\mathbf{F}^\mathbf{2} + \mu\|\mathbf{V^T}(\mathbf{C}-\mathbf{C_a})\mathbf{V}\|_\mathbf{F}^\mathbf{2} + \|\mathbf{V^T}(\mathbf{K}-\mathbf{K_a})\|_\mathbf{F}^\mathbf{2}. \end{aligned} \tag{2.25}$$

Redefine

$$\mathbf{M_a} := \mathbf{V^T M_a V}, \mathbf{C_a} := \mathbf{V^T C_a V}, \mathbf{K_a} := \mathbf{V^T K_a V}. \tag{2.26}$$

We substitute parametric representation of $(\mathbf{V^T M V}, \mathbf{V^T C V}, \mathbf{V^T K V})$ in (2.19), (2.20), (2.21), and (2.26) into (2.25), and obtain an unconstrained optimization problem. our optimization problem is of the following form: Minimize

$$\begin{aligned} f(\mathbf{x}) &= \|\mathbf{M}-\mathbf{M_a}\|_\mathbf{F}^\mathbf{2} + \mu\|\mathbf{C}-\mathbf{C_a}\|_\mathbf{F}^\mathbf{2} + \|\mathbf{K}-\mathbf{K_a}\|_\mathbf{F}^\mathbf{2} \\ &= \|\mathbf{F\Gamma_1 H}-\mathbf{M_a}\|_\mathbf{F}^\mathbf{2} + \mu\|\mathbf{G\Gamma_1 H}-\mathbf{C_a}\|_\mathbf{F}^\mathbf{2} + \|\mathbf{L\Gamma_1 H}-\mathbf{K_a}\|_\mathbf{F}^\mathbf{2} \end{aligned} \tag{2.27}$$

for $\mathbf{x}$, with

$$\mathbf{F} = (\mathbf{A\Sigma})^{-1}, \mathbf{H} = (\mathbf{\Sigma U_{11}^T})^{-1}, \mathbf{G} = -(\mathbf{A\Sigma})^{-1}\mathbf{B A^{-1}}, \mathbf{L} = (\mathbf{A\Sigma})^{-1}\mathbf{E A^{-1}}.$$

In section 3, we solve unconstrained optimization problem.

3. Optimization Method

In this section we shall develop an algorithm for solving the **optimization problem** described in (2.27). We will first solve our optimization problem. Let $(\mathbf{\Lambda}, \mathbf{\Phi}) \in \mathbf{R}^{\mathbf{2}n\times\mathbf{2}n} \times \mathbf{R}^{n\times\mathbf{2}n}$ be given in (2.6), (2.7). Let

$$\mathbf{H} = \begin{pmatrix} h_{\mathbf{11}} & h_{\mathbf{12}} & . & . & . & h_{\mathbf{1}n} \\ h_{\mathbf{21}} & h_{\mathbf{22}} & . & . & . & h_{\mathbf{2}n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ h_{n\mathbf{1}} & h_{n\mathbf{2}} & . & . & . & h_{nn} \end{pmatrix}. \tag{3.28}$$

Optimization Problem. Given $\mathbf{F} = [f_{\mathbf{1}}, \cdots, f_n]$, $\mathbf{G} = [g_{\mathbf{1}}, \cdots, g_n]$, $\mathbf{L} = [\ell_{\mathbf{1}}, \cdots, \ell_n] \in \mathbf{R}^{n\times n}$ and let

$$\mathbf{x} = [x_{\mathbf{1}}, x_{\mathbf{2}}, \cdots, x_{\mathbf{2}\ell}, x_{\mathbf{2}\ell+\mathbf{1}}, \cdots, x_n]^{\mathbf{T}} \tag{3.29}$$

be the vector corresponding to the matrix $\mathbf{\Gamma_1}$ where:
i)If $\mathbf{2}\ell < n$,

$$\mathbf{\Gamma_1^{-1}} = \mathbf{diag}\{\begin{pmatrix} \xi_{\mathbf{1}} & \eta_{\mathbf{1}} \\ \eta_{\mathbf{1}} & -\xi_{\mathbf{1}} \end{pmatrix}, \cdots, \begin{pmatrix} \xi_\ell & \eta_\ell \\ \eta_\ell & -\xi_\ell \end{pmatrix}, \xi_{\mathbf{2}\ell+\mathbf{1}}, \ldots, \xi_n\} \in \mathbf{R}^{n\times n}.$$

ii)If $n = \mathbf{2s}$, $\quad \mathbf{s} \leq \ell$,

$$\mathbf{\Gamma_1^{-1}} = \mathbf{diag}\{\begin{pmatrix} \xi_{\mathbf{1}} & \eta_{\mathbf{1}} \\ \eta_{\mathbf{1}} & -\xi_{\mathbf{1}} \end{pmatrix}, \cdots, \begin{pmatrix} \xi_{\mathbf{s}} & \eta_{\mathbf{s}} \\ \eta_{\mathbf{s}} & -\xi_{\mathbf{s}} \end{pmatrix}\} \in \mathbf{R}^{n\times n}.$$

iii)If $n = \mathbf{2s+1}$, $\quad \mathbf{s} < \ell$,

$$\mathbf{\Gamma_1^{-1}} = \mathbf{diag}\{\begin{pmatrix} \xi_{\mathbf{1}} & \eta_{\mathbf{1}} \\ \eta_{\mathbf{1}} & -\xi_{\mathbf{1}} \end{pmatrix}, \cdots, \begin{pmatrix} \xi_{\mathbf{s}} & \eta_{\mathbf{s}} \\ \eta_{\mathbf{s}} & -\xi_{\mathbf{s}} \end{pmatrix}, \xi_{\mathbf{2s+1}}\} \in \mathbf{R}^{n\times n}.$$

Minimize

$$f(\mathbf{x}) = \|\mathbf{F\Gamma_1H} - \mathbf{M_a}\|_{\mathbf{F}}^{\mathbf{2}} + \mu\|\mathbf{G\Gamma_1H} - \mathbf{C_a}\|_{\mathbf{F}}^{\mathbf{2}} + \|\mathbf{L\Gamma_1H} - \mathbf{K_a}\|_{\mathbf{F}}^{\mathbf{2}} = \sum_{\mathbf{j=1}}^{\mathbf{n}} f_{\mathbf{j}}(\mathbf{x}) \tag{3.30}$$

for $\mathbf{x}$,where

$$f_{\mathbf{j}}(\mathbf{x}) = \|\mathbf{F\Gamma_1h_j} - (\mathbf{M_a})_{\mathbf{j}}\|_{\mathbf{2}}^{\mathbf{2}} + \mu\|\mathbf{G\Gamma_1h_j} - (\mathbf{C_a})_{\mathbf{j}}\|_{\mathbf{2}}^{\mathbf{2}} + \|\mathbf{L\Gamma_1h_j} - (\mathbf{K_a})_{\mathbf{j}}\|_{\mathbf{2}}^{\mathbf{2}}, \tag{3.31}$$

$j = 1, 2, \cdots, n$, where $\mathbf{h_j}$, $(\mathbf{M_a})_\mathbf{j}$, $(\mathbf{C_a})_\mathbf{j}$ and $(\mathbf{K_a})_\mathbf{j}$ are jth column of $\mathbf{H}$, $\mathbf{M_a}$, $\mathbf{C_a}$ and $\mathbf{K_a}$, respectively. In (3.31), the vector $\mathbf{\Gamma_1 h_j}$ can be written by

$$\mathbf{\Gamma_1 h_j} = \mathbf{D_j x}, \quad \mathbf{j} = \mathbf{1}, \mathbf{2}, \cdots, n \tag{3.32}$$

where:

i)If $\mathbf{2}\ell < n$,

$$\mathbf{D_j} = \mathbf{diag}\{\begin{pmatrix} h_{\mathbf{1j}} & h_{\mathbf{2j}} \\ -h_{\mathbf{2j}} & h_{\mathbf{1j}} \end{pmatrix}, \cdots, \begin{pmatrix} h_{\mathbf{2}\ell-\mathbf{1},\mathbf{j}} & h_{\mathbf{2}\ell,\mathbf{j}} \\ -h_{\mathbf{2}\ell,\mathbf{j}} & h_{\mathbf{2}\ell-\mathbf{1},\mathbf{j}} \end{pmatrix}, h_{\mathbf{2}\ell+\mathbf{1},\mathbf{j}}, \cdots, h_{n,\mathbf{j}}\} \in \mathbf{R}^{n\times n}.$$

ii)If $n = \mathbf{2s}$, $\mathbf{s} \le \ell$,

$$\mathbf{D_j} = \mathbf{diag}\{\begin{pmatrix} h_{\mathbf{1j}} & h_{\mathbf{2j}} \\ -h_{\mathbf{2j}} & h_{\mathbf{1j}} \end{pmatrix}, \cdots, \begin{pmatrix} h_{\mathbf{2s}-\mathbf{1},\mathbf{j}} & h_{\mathbf{2s},\mathbf{j}} \\ -h_{\mathbf{2s},\mathbf{j}} & h_{\mathbf{2s}-\mathbf{1},\mathbf{j}} \end{pmatrix}\} \in \mathbf{R}^{n\times n}.$$

iii)If $n = \mathbf{2s} + \mathbf{1}$, $\mathbf{s} < \ell$,

$$\mathbf{D_j} = \mathbf{diag}\{\begin{pmatrix} h_{\mathbf{1j}} & h_{\mathbf{2j}} \\ -h_{\mathbf{2j}} & h_{\mathbf{1j}} \end{pmatrix}, \cdots, \begin{pmatrix} h_{\mathbf{2s}-\mathbf{1},\mathbf{j}} & h_{\mathbf{2s},\mathbf{j}} \\ -h_{\mathbf{2s},\mathbf{j}} & h_{\mathbf{2s}-\mathbf{1},\mathbf{j}} \end{pmatrix}, h_{\mathbf{2s}+\mathbf{1},\mathbf{j}}\} \in \mathbf{R}^{n\times n}.$$

Substituting (3.32) into (3.31) we compute

$$\nabla f_j(x) = (\frac{\partial f_j}{\partial x_1}, \cdots, \frac{\partial f_j}{\partial x_n})^T \tag{3.33}$$

$$= \mathbf{2}(\mathbf{FD_j})^\mathbf{T}(\mathbf{FD_j x} - (\mathbf{M_a})_\mathbf{j}) + \mathbf{2}\mu(\mathbf{GD_j})^\mathbf{T}(\mathbf{GD_j x} - (\mathbf{C_a})_\mathbf{j}) + \mathbf{2}(\mathbf{LD_j})^\mathbf{T}(\mathbf{LD_j x}$$

$$-(\mathbf{K_a})_\mathbf{j}).$$

Consequently,

$$\nabla f(\mathbf{x}) = \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \nabla f_\mathbf{j}(\mathbf{x}) \tag{3.34}$$

$$= \mathbf{2}\sum_{\mathbf{j}=\mathbf{1}}^{n}[(\mathbf{FD_j})^\mathbf{T}(\mathbf{FD_j x} - (\mathbf{M_a})_\mathbf{j}) + \mu(\mathbf{GD_j})^\mathbf{T}(\mathbf{GD_j x} - (\mathbf{C_a})_\mathbf{j})$$

$$+(\mathbf{LD_j})^\mathbf{T}(\mathbf{LD_j x} - (\mathbf{K_a})_\mathbf{j})].$$

Setting $\nabla f(\mathbf{x}) = \mathbf{0}$ we end up with the following linear system of equations:

$$\mathbf{Px} = \mathbf{b}, \tag{3.35}$$

where

$$\mathbf{P} = \sum_{\mathbf{j=1}}^{n}[(\mathbf{FD_j})^{\mathbf{T}}(\mathbf{FD_j}) + \mu(\mathbf{GD_j})^{\mathbf{T}}(\mathbf{GD_j}) + (\mathbf{LD_j})^{\mathbf{T}}(\mathbf{LD_j})], \tag{3.36}$$

$$\mathbf{b} = \sum_{\mathbf{j=1}}^{n}[(\mathbf{FD_j})^{\mathbf{T}}(\mathbf{M_a})_{\mathbf{j}} + \mu(\mathbf{GD_j})^{\mathbf{T}}(\mathbf{C_a})_{\mathbf{j}} + (\mathbf{LD_j})^{\mathbf{T}}(\mathbf{K_a})_{\mathbf{j}}]. \tag{3.37}$$

Since the function $f(\mathbf{x})$ in (3.30) must have an optimum value, the linear system of equations (3.35) is a consistent system.

The computational steps for solving the optimization problem (1.3), (1.4), (1.5) are summarized in the following algorithm.

Algorithm 3.1 Given $\mathbf{Q_a}(\lambda) = \lambda^{\mathbf{2}}\mathbf{M_a} + \lambda\mathbf{C_a} + \mathbf{K_a}$ and $(\mathbf{\Lambda}, \mathbf{\Phi}) \in \mathbf{R}^{\mathbf{2}n\times\mathbf{2}n} \times \mathbf{R}^{n\times\mathbf{2}n}$ as in (2.6), (2.7). The optimal solutions $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ of (1.3), (1.4) and (1.5) are computed by

Step 1. Compute the SVD-decomposition of $\mathbf{\Phi}$:

$$\mathbf{\Phi} = \mathbf{V}\begin{pmatrix} \mathbf{\Sigma} \\ \mathbf{0} \end{pmatrix}\mathbf{U}^{\mathbf{T}};$$

Step 2. Set $\mathbf{M_a} := \mathbf{V^T M_a V}$, $\mathbf{C_a} := \mathbf{V^T C_a V}$, $\mathbf{K_a} := \mathbf{V^T K_a V}$;

Step 3. Solve $\mathbf{Px} = \mathbf{b}$ for $\mathbf{x} = [x_{\mathbf{1}}, x_{\mathbf{2}}, \cdots, x_{\mathbf{2}\ell}, x_{\mathbf{2}\ell+\mathbf{1}}, \cdots, \ \mathbf{x}_n]^{\mathbf{T}}$, where $\mathbf{P}$ and $\mathbf{b}$ are given by (3.35), (3.36).

Step 4. Set $\mathbf{\Gamma_1}$ of the following form:

i)If $\mathbf{2}\ell < n$,

$$\mathbf{\Gamma_1} = \mathbf{diag}\{\begin{pmatrix} x_{\mathbf{1}} & x_{\mathbf{2}} \\ x_{\mathbf{2}} & -x_{\mathbf{1}} \end{pmatrix}, \cdots, \begin{pmatrix} x_{\mathbf{2}\ell-\mathbf{1}} & x_{\mathbf{2}\ell} \\ x_{\mathbf{2}\ell} & -x_{\mathbf{2}\ell-\mathbf{1}} \end{pmatrix}, x_{\mathbf{2}\ell+\mathbf{1}}, \cdots, x_n\} \in \mathbf{R}^{n\times n}.$$

ii)If $n = \mathbf{2s}$, $\mathbf{s} \leq \ell$,

$$\mathbf{\Gamma_1} = \mathbf{diag}\{\begin{pmatrix} x_{\mathbf{1}} & x_{\mathbf{2}} \\ x_{\mathbf{2}} & -x_{\mathbf{1}} \end{pmatrix}, \cdots, \begin{pmatrix} x_{\mathbf{2s-1}} & x_{\mathbf{2s}} \\ x_{\mathbf{2s}} & -x_{\mathbf{2s-1}} \end{pmatrix}\} \in \mathbf{R}^{n\times n}.$$

iii)If $n = \mathbf{2s+1}$, $\mathbf{s} < \ell$,

$$\mathbf{\Gamma_1} = \mathbf{diag}\{\begin{pmatrix} x_{\mathbf{1}} & x_{\mathbf{2}} \\ x_{\mathbf{2}} & -x_{\mathbf{1}} \end{pmatrix}, \cdots, \begin{pmatrix} x_{\mathbf{2s-1}} & x_{\mathbf{2s}} \\ x_{\mathbf{2s}} & -x_{\mathbf{2s-1}} \end{pmatrix}, \mathbf{x_{2s+1}}\} \in \mathbf{R}^{n\times n}.$$

Step 4. Compute $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ with

$$\mathbf{M} = \mathbf{V}(\mathbf{A\Sigma})^{-\mathbf{1}}\mathbf{\Gamma_1}(\mathbf{\Sigma U_{11}^T})^{-\mathbf{1}}\mathbf{V^T},$$

$$\mathbf{C} = -\mathbf{V}(\mathbf{A\Sigma})^{-\mathbf{1}}\mathbf{BA^{-1}}\mathbf{\Gamma_1}(\mathbf{\Sigma U_{11}^T})^{-\mathbf{1}}\mathbf{V^T},$$

$$\mathbf{K} = \mathbf{V}(\mathbf{A\Sigma})^{-\mathbf{1}}\mathbf{EA^{-1}}\mathbf{\Gamma_1}(\mathbf{\Sigma U_{11}^T})^{-\mathbf{1}}\mathbf{V^T},$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{E}$ are given by (2.22), (2.23),(2.24).

4. Numerical results

In this section, we will present a numerical Example to show that our Algorithm is reliable. We will report all numbers in 16 significant digits.

Example 4.1. *To generate test data, we first randomly generate a* 3×3 *real symmetric quadratic pencil* $\mathbf{Q_a}(\lambda) = \lambda^{\mathbf{2}}\mathbf{M_a} + \lambda\mathbf{C_a} + \mathbf{K_a}$*, where*

$$\mathbf{M_a} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 3 \end{pmatrix}, \quad \mathbf{C_a} = \begin{pmatrix} 4 & -3 & 0 \\ -3 & 4 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

$$\mathbf{K_a} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

and compute its "exact" eigenpairs $(\mathbf{\Lambda}, \mathbf{\Phi})$. *We conclude that*

$$\mathbf{\Lambda} = \mathbf{diag}\{\begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix}, \lambda_{\mathbf{3}}, \cdots, \lambda_{\mathbf{6}}\},\ \Phi = [\varphi_{1R}, \varphi_{1I}, \varphi_3, \cdots, \varphi_6],$$

with

$$\lambda_1 = \underbrace{-2.784355624909507e-001}_{\alpha_1} + \underbrace{5.477022711808772e-001}_{\beta_1} i,\ \lambda_2 = \bar{\lambda_1},\ \lambda_3 = 0,$$

$\lambda_4 = -1.830798151417329e+000$, $\lambda_5 = -6.593324492213939e-001$,
$\lambda_6 = -3.376136589947590e-001$,

and the corresponding eigenvectors

$$\varphi_{1R} = \begin{pmatrix} -5.917257037379675e-001 \\ -9.111035949173539e-001 \\ 1 \end{pmatrix}, \varphi_{1I} = \begin{pmatrix} 5.541299377683790e-001 \\ 4.469246200079373e-001 \\ 0 \end{pmatrix},$$

$$\varphi_3 = \begin{pmatrix} 9.999999999999997e-001 \\ 1 \\ 1 \end{pmatrix}, \varphi_4 = \begin{pmatrix} -3.774393965656190e+000 \\ 1.258992629081558e+000 \\ 1 \end{pmatrix},$$

$$\varphi_5 = \begin{pmatrix} 5.761724971880177e-001 \\ 8.143855574766981e-001 \\ 1 \end{pmatrix}, \varphi_6 = \begin{pmatrix} -6.183271240558999e-001 \\ 7.488290032764442e-001 \\ 1 \end{pmatrix}.$$

The algorithm 3.1 *should theorically give the optimal solution* $\mathbf{M} = \mathbf{M_a}, \mathbf{C} = \mathbf{C_a}$ *and* $\mathbf{K} = \mathbf{K_a}$. *The numerical result of the relative errors computed by algorithm* 3.1 *are estimated*

$\frac{\|M-M_a\|}{\|M\|} \simeq 5.489413060963724e-015$, $\frac{\|C-C_a\|}{\|C\|} \simeq 2.181617735324220e-015$,

$\frac{\|K-K_a\|}{\|K\|} \simeq 3.486665612122003e-015$.

5. CONCLUSION

One common procedure to improve the discrepancy between a mathematical model and the corresponding real- world system is to modify the model parameters in such a way to achieve a good correspondence between the analytical solution and the real data. In this paper we have considered a model updating self- adjoint quadratic pencils using all measured natural frequencies and mode shapes. The model updating problem is cast as a generalized inverse eigenvalues problem with prescribed eigenpairs. We have used a parametric representation of the solution to the IQEP in which symmetry is required of the matrices involved. The example which is given is used to demonstrate that algorithm is reliable. Also, the efficiency of the algorithm will be preserved for large n.

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References

[1] M. Baruch, Optimization procedure to correct stiffness and flexibility matrices using vibration data, AIAA Journal, Vol. 16, Number 11(1978), 1208-1210.

[2] M. Baruch and I.Y. Bar-Itzack, Optimal weighted orthogonalization of measured modes, AIAA Journal, Vol. 16, Number 4 (1973), 346-351.

[3] A. Berman, Comment on 'optimal weighted orthogonalization of measured modes', AIAA Journal, Vol.17, Number 8 (1979), 927-928.

[4] A. Berman and E. J. Nagy, Improvement of a large analytical model using test data, AIAA Journal, Vol. 21, Number 8 (1983), 1168-1173.

[5] M.T. Chu, B.N. Datta, W.W. Lin and S.F .Xu, The Spill-Over Phenomenon in Quadratic Model Updating, AIAA Journal.

[6] B.N. Datta, Finite element model updating, eigenstructure assignment and eigenvalue embedding techniques for vibrating systems, Mechanical systems and signal processing, Special Volume on Vibration Control, Vol. 16 (2002), 83-96.

[7] B.N. Datta, S. Elhay, Y.M. Ram and D.R. Sarkissian, Partial eigen structure assignment for the quadratic pencil. Journal of sound and Vibration, Vol. 230 (2000), 101-110.

[8] B.N. Datta and D. R. Sarkissian, Theory and computations of some inverse eigenvalue problems for the quadratic pencil, In Structured Matrices in Mathematics, Computer Science, and Engineering I, Contemp. Math. 280, AMS, Providence, RI (2001), 221-240.

[9] M.I. Friswell, D.J. Inman and D.F. Pilkey, The direct updating of damping and stiffness matrices, AIAA Journal, Vol. 36, Number 3 (1998), 491-493.

[10] M.I. Friswell and J.E. Mottershead, Finite element model updating in structural dynamics, Kluwer Academics Publishers, 1995.

[11] Y.C. Kuo, W.W. Lin and S.F. Xu, A new model updating method for quadratic eigenvalues problems, AIAA Journal (2006).

[12] M. Mohseni Moghadam, A. Tajaddini, Inverse eigenvalue problem with non-Simple eigenvalues for damped vibration systems, J. of Informatics and Mathematical Sciences, Vol. 1, Number 2, 3 (2009), 91-97.

[13] M. Mohseni Moghadam, A. Tajaddini,The model updating of mass and stiffness matrices, Iranian J. of Sciences and Technology, Transaction A, Number 32 (A4) (2008).

[14] J.E. Mottershead and M.I. Friswell, Model updating in structural dynamics: A survey, Journal of sound and Vibration, Vol. 167, Number 2 (1993), 347-375.

[15] F. Tisseur ,K. Meebergen,The quadratic eigenvalue problem, SIAM REVIEW,vol. 43, Number 2 (2001), 235-286.

[16] F. S. Wei, Structural dynamics model identification using vibration test data, 7th IMAC, Las Vegas, Nevada (1989), 562-567.

[17] F.S. Wei, Mass and stiffness interaction effects in analytical model identification, AIAA Journal, Vol. 28, Number 9, 1686-1688, (1990).

[18] F.S. Wei, Structural dynamic model improvment using vibration test data, AIAA Journal, Vol. 28, Number 1 (1990), 175-177.